An implicit differential equation with MAPLE and IRONDEL

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Abstract. In this paper, we review different approaches available to study an implicit differential equation. We illustrate on an example the results obtained by

1. elementary numerical methods
2. analytical methods
3. differential algebra point of view
4. local analysis in the complex plane
5. complex graphical analysis

1 Introduction

In a previous paper[9], we explained how to organize many works done about differential equations in order to build a library called IRONDEL, for *Integration and Reduction of Ordinary NonLinear Differential Equations Library.*
In the scalar case, we study equations of the type $f(x, X, y, y', \ldots, y^{(d)}) = 0$, where f is a scalar function of $d+2+r$ variables: x and X are the independant variables, x is scalar and X is a vector of dimension r, y is the scalar dependant variable and $y', \ldots$ are its derivatives with respect to x up to order d. The different cases of such equations, and the corresponding methods to *solve* them can be visualized on the Figure 1.

The aim of this paper is to illustrate the right branch of the diagram. For this purpose, we will test various methods on the following equation :

$$y'(x)^2 + y(x)^3 + y(x) = 0. \tag{1}$$

This equation seems rather simple, it is a first order implicit one. It appears in [16] to prove that the field of complex transseries is not differentially algebraically closed. A differential field F is said to be *differentially algebraically closed*, if for any pair (P, Q) of differential polynomials over F, such that the order of P is strictly larger than the order of Q, there exists a root of P in F, which is not a root of Q. The proof of the previous affirmation lies on the fact that the only transseries solutions of the equation (1) are the three constant functions $y = 0$, $y = i$ and $y = -i$. This is the starting point of this work, and the pretext to compare in the same frame, global results of differential algebra, local series expansions, and graphical results obtained in the complex plane.

2 Numerical results

The simplest manner to grasp such an equation is to plot some solutions. This is possible with the MAPLE command DEplot:

```
> with(DEtools):
> ode:=y(x)+y(x)^3+diff(y(x),x)^2;
```

$$ode := \mathrm{y}(x) + \mathrm{y}(x)^3 + \left(\frac{\partial}{\partial x}\mathrm{y}(x)\right)^2$$

```
> DEplot(ode,y(x),x=-1..1,[[y(0)=-1]]);

Error, (in DEtools/convertsys) unable to convert to an explicit
first-order system
```

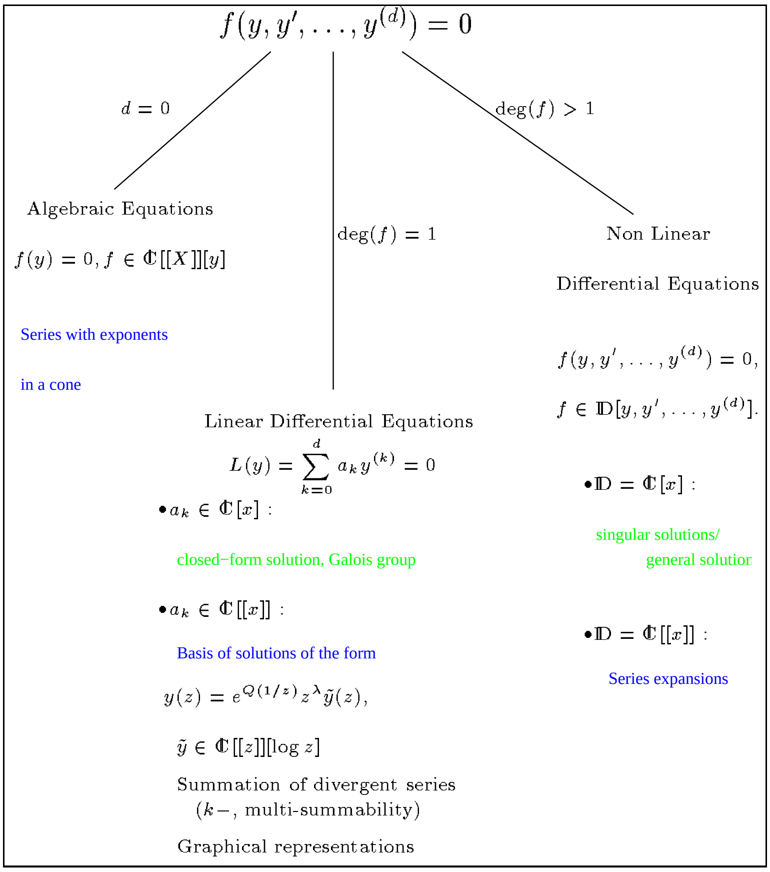


Fig. 1. IRONDEL, scalar case

```
> ode1:=diff(y(x),x)=sqrt(-y(x)-y(x)^3);
```

$$ode1 := \frac{\partial}{\partial x}\,\mathrm{y}(x) = \sqrt{-\mathrm{y}(x) - \mathrm{y}(x)^3}$$

```
> DEplot(ode1,y(x),x=-1..1,[[y(0)=-1],[y(0)=0],[y(0)=-1/2]],
> linecolor=[black,blue,red]);
```

(see fig. 2)

```
> DEplot(ode1,y(x),x=-1..2,[[y(0)=-1/2]]);

Error, (in DEtools/DEplot/drawlines) Stopping integration due to sqrt
of a negative number
```

But we see that we encounter the following restrictions:

- we have to "replace" the implicit equation by an explicit one, which is of course not equivalent; considering the *other* explicit equation $y'(x) = -\sqrt{-y(x) - y(x)^3}$, we obtain solutions, symetric from the first one with respect to the y axis;
- it seems to be impossible to continue the solution passing through the point $(0, -1/2)$ beyond $x = 1$;
- it seems that the constant solution $y(x) = 0$ is the unique one passing through the point $(0, 0)$.

3 Analytical solutions

Again, we consider the explicit first order equation

$$y'(x) = f(x, y(x)) = \sqrt{-y(x) - y(x)^3}. \tag{2}$$

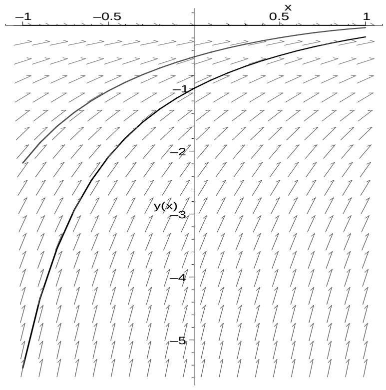


Fig. 2. Numerical solutions

The function f is defined and continuous on $\mathbb{R}\times\mathbb{R}^-$. We search solutions as differentiable functions $\varphi : I \to \mathbb{R}$ such that

$$\forall x \in \mathbb{R}, \varphi(x) \leq 0, \quad \text{and} \quad \varphi'(x) = \sqrt{-\varphi(x) - \varphi(x)^3}.$$

The function f is $\mathcal{C}^1$ on $\mathbb{R} \times \mathbb{R}^{*-}$. There is unicity for the Cauchy problem at each point $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}^{*-}$.
If forall $x \in I, y(x) \neq 0$, we separate the variables:

$$\frac{dy}{\sqrt{-y(1+y^2)}} = dx,$$

and putting $z = \sqrt{-y}$, we obtain:

$$\frac{-2dz}{\sqrt{1+z^4}} = dx,$$

which can be integrated in terms of power series:

$$x + c_1 = \phi(X), \text{with } X = -2z,$$

(c_1 is an arbitrary constant and ϕ has radius of convergence 2).
Let ψ be the reciprocal series of ϕ, then

$$X = -2z = -2\sqrt{-y} = \psi(x + c_1), \quad \psi(x + c_1) \leq 0$$

and

$${}^1y_{c_1}(x) = \frac{-\psi(x + c_1)^2}{4}, \quad x \leq -c_1.$$

The solution satisfying $y(0) = -1/2$ corresponds to $c_1 = \phi(-\sqrt{2})$ and can be plotted.

```
> with(powseries):
> a:=inverse(powsqrt(1+X^4/16)):
> phi:=powint(a):
> tpsform(phi,X,10);
```

$$X - \frac{1}{160} X^5 + \frac{1}{6144} X^9 + \mathrm{O}(X^{10})$$

```
> psi:=reversion(phi):
> tpsform(psi,X,10);
```

$$X + \frac{1}{160} X^5 + \frac{1}{30720} X^9 + \mathrm{O}(X^{10})$$

```
> c0:=evalseries(phi,-sqrt(2),20);
```

$$c0 := -1.382098746$$

```
> functionpsi:=proc(x) evalseries(psi,x,20)
> end proc :
> plot(-functionpsi(x+c0)^2/4, x=-1..-c0);
```

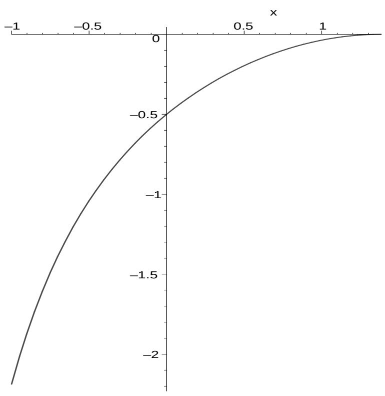


Fig. 3. the graph of ${}^1y_{c_0}$, $c_0 = \phi(-\sqrt{2})$

Consider now the equation

$$y'(x) = -\sqrt{-y(x) - y(x)^3}. \tag{3}$$

The solutions are

$${}^2y_{c_2}(x) = \frac{-\psi(-x + c_2)^2}{4}, \quad c_2 \leq x.$$

We can merge ${}^1y_c$ and ${}^2y_{-c}$ to obtain a solution of the initial equation (1)

$$y_c(x) = \frac{-\psi(x + c)^2}{4}$$

defined for $|x + c| < R$, where R is the radius of convergence of ψ.
Satisfaying $y(0) = -1/2$, we obtain two solutions corresponding to $c = \phi(\pm\sqrt{2})$.
Of course, with other values of c, we obtain other solutions, whose graphical representations are simply translated one from the others : on Figure 4, are highlighted the non null solution y_0 passing through the point $(0, 0)$, corresponding to $c = 0$, and the two solutions passing through the point $(0, -1/2)$, i.e. y_{c_0} and y_{-c_0}. For the graphical plot, all these series are truncated at order 15.
For comparison with further results, we can expand the solution y_0:

```
> sol0:=multconst(multiply(psi,psi),-1/4):tpsform(sol0,x,15);
```

$$-\frac{1}{4}x^2 - \frac{1}{320}x^6 - \frac{1}{38400}x^{10} - \frac{1}{5324800}x^{14} + \mathrm{O}(x^{15}) \tag{4}$$

To these (analytical) solutions, we have to add the “particular” solution $y(x) = 0$. Then, for all c, c', we can merge the solution $y_c, x \leq -c$ with the null function between $-c$ and $-c'$, and continue with $y_{c'}$, for $x \geq -c'$. Doing so, we obtain an infinity of $\mathcal{C}^1$ solutions passing through each point $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}^{*-}$.

A rapprochment can bee seen between this “by-hand” method and the result of the MAPLE command *dsolve*:

```
> dsolve(ode,y(x));
```

$$\mathrm{y}(x) = 0,\ \mathrm{y}(x) = -I,\ \mathrm{y}(x) = I,\ x - \int^{\mathrm{y}(x)} \frac{1}{\sqrt{-\_a\,(1 + \_a^2)}}\, d\_a - \_C1 = 0,$$

$$x - \int^{\mathrm{y}(x)} -\frac{1}{\sqrt{-\_a\,(1 + \_a^2)}}\, d\_a - \_C1 = 0$$

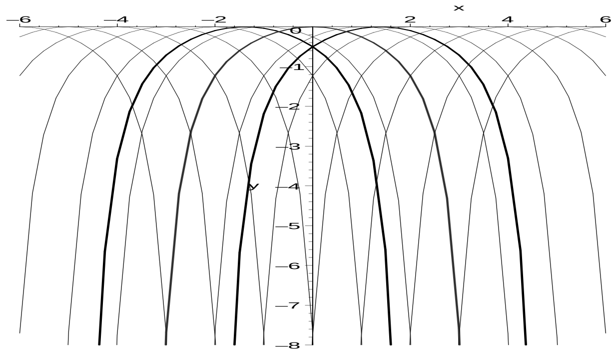


Fig. 4. the graph of y_0, y_{c_0}, y_{-c_0} and some other solutions

. . . even if the non null solution passing through $(0, 0)$ cannot be found by this way.

```
> dsolve({ode,y(0)=0},y(x));
```

$$\mathrm{y}(x) = 0$$

4 Differential Algebra

In this section, we want to know what kind of informations can be given by the algorithms based on differential algebra. In all generality, the package *diffalg* of MAPLE deals with systems of polynomial (ordinary or partial) differential equations and inequations.
For such a system S, the Rosenfeld-Gröbner algorithm will detect if the system leads to a contradiction and therefore has no solution. Otherwise it returns a finite set of *regular* subsystems of differential equations and inequations, say $S_1, \ldots, S_r$. The set of solutions is preserved: precisely, the set of inifinitely differentiable solutions of S is the union of the solutions of the S_i. This decomposition gives a description of the different types of solutions, but in general is not minimal (some solutions of a subsystem S_i can be particular solutions of an other S_j).
For a single differential equation, it is possible to obtain such a minimal decomposition. (For our particular equation, the two decompositions are the same, this means that the Rosenfeld-Gröbner decomposition is already minimal).

```
> with(diffalg):
> R := differential_ring (derivations = [x],
> ranking = [y],notation = diff);
```

$$R := \mathit{ODE\_ring}$$

```
> MR:=essential_components(ode,R);
```

$$\mathit{MR} := [\mathit{characterizable},\ \mathit{characterizable}]$$

```
> equations(MR[1]);inequations(MR[1]);
```

$$[(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2 + \mathrm{y}(x) + \mathrm{y}(x)^3]$$

$$[\frac{\partial}{\partial x}\,\mathrm{y}(x)]$$

This first system describes the general solution.

```
> equations(MR[2]);inequations(MR[2]);
```

$$[\mathrm{y}(x)\,(1 + \mathrm{y}(x)^2)]$$

$$[1 + 3\,\mathrm{y}(x)^2]$$

The second one describes the singular essential solutions, the three constant functions $y(x) = 0, y(x) = \pm i$.
In the next section (paragraph 5.2), we will see how this preparing work can be followed by the expansion of the general solution in terms of power series.

5 Series solutions

Now we want to test the possibility of computing the solutions in terms of power series.

5.1 With the option *series* of MAPLE

```
> Order:=5:sol:=dsolve(ode,y(x),'series');
```

$$sol := \mathrm{y}(x) = \mathrm{y}(0) + \mathrm{RootOf}(\mathrm{y}(0) + \_Z^2 + \mathrm{y}(0)^3,\ label = \_L2)\, x + (-\frac{3}{4}\,\mathrm{y}(0)^2 - \frac{1}{4})\, x^2 - \frac{1}{2}\,\mathrm{y}(0)\,\mathrm{RootOf}(\mathrm{y}(0) + \_Z^2 + \mathrm{y}(0)^3,\ label = \_L2)\, x^3 + (\frac{5}{16}\,\mathrm{y}(0)^3 + \frac{3}{16}\,\mathrm{y}(0))\, x^4 + \mathrm{O}(x^5)$$

For all values of $y(0)$ not equal to $0, \pm i$, we find by this way two series solution. For example, for $y(0) = -1/2$:

```
> allvalues(subs(y(0)=-1/2,sol));
```

$$\mathrm{y}(x) = -\frac{1}{2} + \frac{\sqrt{10}}{4}\,x - \frac{7}{16}\,x^2 + \frac{\sqrt{10}}{16}\,x^3 - \frac{17}{128}\,x^4 + \mathrm{O}(x^5),$$
$$\mathrm{y}(x) = -\frac{1}{2} - \frac{\sqrt{10}}{4}\,x - \frac{7}{16}\,x^2 - \frac{\sqrt{10}}{16}\,x^3 - \frac{17}{128}\,x^4 + \mathrm{O}(x^5)$$

For $y(0) = 0$, the null solution is missed:

```
> allvalues(subs(y(0)=0,sol));
```

$$\mathrm{y}(x) = -\frac{1}{4}\,x^2 + \mathrm{O}(x^5),\ \mathrm{y}(x) = -\frac{1}{4}\,x^2 + \mathrm{O}(x^5)$$

But:

```
> dsolve({ode,y(0)=0},y(x),'series');
```

$$\{\mathrm{y}(x) = -\frac{1}{4}\,x^2 + \mathrm{O}(x^5),\ \mathrm{y}(x) = \mathrm{O}(x^5)\}$$

The phenomenom is the same for $y(0) = \pm i$.

```
> Order:=20:sol:=op(2,dsolve(ode,y(0)=-1/2,y(x),'series')):
> sol1:=convert(sol,polynom)):
> sol2:=seq(rem(sol1[i],x^5,x),i=1..2):
> plot({sol1,sol2},x=-3..3,y=-10..1,thickness=2);
```

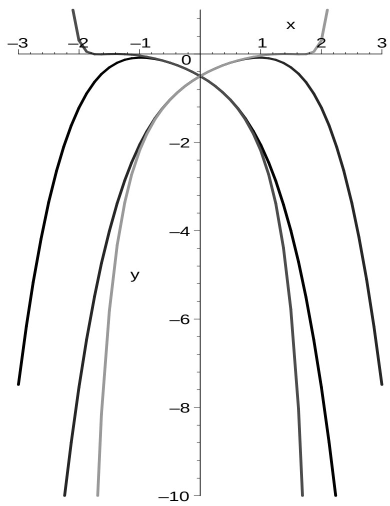


Fig. 5. two approximations of y_{c_0} and y_{-c_0}

The two solutions passing through $(0, -1/2)$ are represented by the series $sol_1[1]$ and $sol_1[2]$ (truncated at x^{20}). $sol_2[1]$ and $sol_2[2]$ are the same series truncated at order 5. We can plot these four polynoms (Figure 5) and observe that they give a good approximation of the solutions, at least over the range $[-1,1]$. Moreover, $sol_1[i]$ and $sol_2[i]$ enclose the corresponding solution (due to the parity of the order of truncature of alternating series).

5.2 With the package *diffalg*

The following lines are the continuation of paragraph 4.

```
> sol:=op(2,op(1,power_series_solution ([x=0], 5, MR[1], 'syst', 'params')));
```

$$sol = \_Cy + x\,\_Cy\_x - \frac{1}{4}\,\frac{x^2\,(\_Cy\_x + 3\,\_Cy^2\,\_Cy\_x)}{\_Cy\_x}$$
$$- \frac{1}{6}\,\frac{x^3\,(-3\,\_Cy^2\,\_Cy\_x - 3\,\_Cy\_x\,\_Cy^4)}{-\_Cy - \_Cy^3}$$
$$- \frac{1}{24}\,\frac{x^4\,(24\,\_Cy\_x\,\_Cy^4 + 15\,\_Cy^6\,\_Cy\_x + 9\,\_Cy^2\,\_Cy\_x)}{-2\,\_Cy\_x\,\_Cy - 2\,\_Cy\_x\,\_Cy^3}$$

```
> syst;params;
```

$$[\_Cy\_x^2 + \_Cy + \_Cy^3 = 0,\ \_Cy\_x \neq 0]$$
$$\{\_Cy\}$$

For the *general* solution, we can choose $y(0)$ (i.e. $\_Cy$), with respect to the equations of "syst". For all values of $y(0)$ not equal to $0, \pm i$, we find again the same two solutions as before.

```
> sol1:=subs(_Cy=-1/2,sol);
```

$$sol1 := -\frac{1}{2} + x\,\_Cy\_x - \frac{7}{16}\,x^2 + \frac{1}{4}\,x^3\,\_Cy\_x - \frac{17}{128}\,x^4$$

```
> subs(_Cy=-1/2,syst);
```

$$[\_Cy\_x^2 - \frac{5}{8} = 0,\ \_Cy\_x \neq 0]$$

```
> Cy_x:=[solve(%[1],_Cy_x)];
```

$$Cy\_x := [\frac{1}{4}\sqrt{10},\ -\frac{1}{4}\sqrt{10}]$$

```
> sol1:=seq(subs(_Cy_x=Cy_x[i],sol1),i=1..2);
```

$$sol1 := -\frac{1}{2} + \frac{1}{4}\,x\sqrt{10} - \frac{7}{16}\,x^2 + \frac{1}{16}\,x^3\sqrt{10} - \frac{17}{128}\,x^4,$$
$$-\frac{1}{2} - \frac{1}{4}\,x\sqrt{10} - \frac{7}{16}\,x^2 - \frac{1}{16}\,x^3\sqrt{10} - \frac{17}{128}\,x^4$$

Neverthless, it is impossible to expand the general solution passing through the point $(0,0)$...

```
> subs(_Cy=0,sol);

Error, division by zero
```

Indeed, for $\_Cy = 0$, the system "syst" has no solution in $y'(0)$ (i.e. $\_Cy\_x$).

```
> subs(_Cy=0,syst);
```

$$[0 = 0,\ 1 \neq 0]$$

5.3 Newton algorithm

Here we want to perform the local analysis of the studied equation near $x = 0$ in the complex plane. This means : find formal solutions as series in a large sense, power series, Laurent series, Puiseux series, generalized series with logarithmic or exponential terms.

Survey of existing algorithms. For this purpose, we will use a general process, first introduced by Newton in [12] in order to solve scalar algebraic equations (the left branch of the figure 1, in the particular case $r = 1$).
The initial algorithm has been extended to linear differential equations (the middle branch of the figure 1) [7].
In the non linear differential case, a similar process is presented in [11, 8]. Sufficient conditions for the existence of Puiseux series solutions can be found in [6, 5] in the case of quasi-linear differential equations.
All these algorithms are based on a polygon in two dimensions, the *Newton polygon*, which vertices and edges give the "beginnings" of solutions.
In order to treat multi-variate algebraic equations, we had to deal with a generalization of the Newton polygon, the *Newton polyhedron* (in $\mathbb{R}^{r+1}$), and a generalization of the Puiseux series, the series with exponents in a cone [3].
These tools have been used again in [2] to "solve" partial differential equations and we have at our disposal an implemented version of the corresponding algorithm. We can test it on our example, even if it is only an ordinary (not partial) differential equation.

```
> PSSPDE(ode,[x,y(x)],[[1]],5);
```

$$\textit{We have found , 2, possible beginning of solutions}$$

$$\textit{The , 1, th. solution is}$$

$$-\frac{4}{x^2} + \frac{x^2}{20} - \frac{x^6}{4800} + \frac{x^{10}}{2496000} - \frac{x^{14}}{1357824000}, +\ldots\textit{terms of higher order}$$

$$\textit{The , 2, th. solution is}$$

$$-\frac{1}{4}x^2 - \frac{1}{320}x^6 - \frac{1}{38400}x^{10} - \frac{1}{5324800}x^{14} - \frac{41}{32587776000}x^{18}, +\ldots\textit{terms of higher order}$$

The first remark, which has to be done, is the following: in the differential case, except for quasi-linear equations, or with extra by-hand computations, the beginnings of solutions found by this algorithm are not guaranteed to be actual solutions, in the sense that we are not sure that the following terms of the series can be computed indefinitely.

Bruno's approach. The same techniques have been developped by Bruno in [4].
The equation (1) can be treated by the method exposed in Chapter 6, devoted to systems of equations of a very general type, containing n_1 parameters, n_2 independant variables and n_3 dependant variables (with partial derivatives). For a "simple" differential polynomial $f(x, y)$, the method consists in searching the solutions of the equation $f(x, y) = 0$ of the form $y = g(x) = \sum g_k x^k$, $k \in \mathbb{R}$, and more precisely the first approximation to (or truncation of) such a solution with respect to an arbitrary (non null) "vector" order $T = (t_1)$[1]. That is to say, the monomial $g_k x^k$, where k is maximal if $t_1 > 0$, and minimal if $t_1 < 0$. By substituing $x = b\tau^{t_1}(1 + o(\tau)), \tau \to \infty$ in $g(x)$, we obtain $y = \tau^{t_1 k}(b^k g_k + o(\tau))$.
Since we are looking for solutions in the neighborhood of $x = 0$, we restrict ourselves to $t_1 = -1$. Then it is proved that the the first approximation $x = b\tau^{-1}, y = b^k g_k \tau^{-k}$ is solution of a "truncated equation" with respect to the order $P = (-1, -k)$.
All truncated equations can be found with the help of the Newton polygon. The polynomial $f(x, y)$ is a finite sum of differential monomials, which are products of a constant, of powers of x and y, and derivatives $\frac{d^k y}{dx^k}$. To each differential monomial is associated its vector power in the following way : to a constant is associated $Q = (0, 0)$, to $x^k y^l$ is associated $Q = (k, l)$, to $\frac{d^k y}{dx^k}$ is associated $Q = (-k, 1)$. When the monomials are multiplied, their vector powers Q are summed. The set S of vector powers of its monomials is called the support of the polynomial f and its Newton polygon is defined as the convex hull of this set.
For the example (1), we put $S = \{Q_1 = (-2, 2), Q_2 = (0, 3), Q_3 = (0, 1)\}$. The polyhedron Γ is the triangle with vertices $\Gamma_k^{(0)} = Q_k$, $k = 1, 2, 3$. The truncated equations with respect to P are:

[1] In the general case, this vector has dimension $l_1 + n_2$, with $l_1 \leq n_1$. Such a vector is usefull in higher dimension to "order" the monomials.

- associated to the edge $\Gamma_1^{(1)}$: $y'^2 + y^3 = 0$. Its solutions with $P \in Vect\{(-1,2)\}$ (the normal cone associated to this edge) have the form $y = bx^{-2}$, i.e. $(-2b)^2 x^{-6} + b^3 x^{-6} = 0$, whence $b = -4$.
- associated to the edge $\Gamma_3^{(1)}$: $y'^2 + y = 0$. Its solutions with $P \in Vect\{(-1,-2)\}$ have the form $y = bx^2$, i.e. $(2b)^2 + b = 0$, whence $b = \frac{-1}{4}$.
- associated to the vertex $\Gamma_1^{(0)}$: $y'^2 = 0$. Its solutions $y = c$ have order $P = (-1, 0)$, which belongs to $U_1^{(0)}$ (the normal cone associated to this vertex).
- associated to the two others vertices, only the trivial solution $y = 0$.

Thus we have found by this way three possible asymptotics for the solutions.
But we note again the following remark: This "means that for some class of solutions, the first approximation to a solution to the system of equations is the solution to the corresponding first approximation to the system of equations. Apparently, this property should be considered as characteristic with the definition of the first approximation. The reverse statement, generally speaking, is not correct, i.e. not every solution to the first approximation (the truncation) of a system of equations is the first approximation to some solution of the system."([4], p 280-281).

Introduction of a regular case. For this reason, we defined in [1] the notion of "regular case", already present for algebraic equations or linear differential equations: under some conditions, we assure that the solution can be completed indefinitely (by a quadratic Newton iteration). More precisely, we recall the following result.

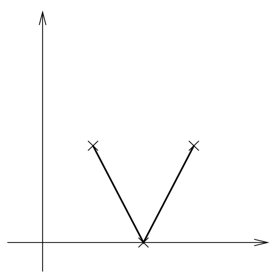
Consider $f(x, y, \dot{y}) \in \mathbb{C}[[x]][y, \dot{y}]$ ($\dot{y}$ denotes the Euler derivative $x\dfrac{dy}{dx}$).
Suppose that there exists $s \in \mathbb{N}$ and $y_s \in \mathbb{C}[x]$ such that $f(x, y_s, \dot{y}_s) = 0 \bmod x^{2^s}$ and that one of the following conditions is satisfied:

1. $\dfrac{\partial f}{\partial y}(x, y_s, \dot{y}_s) \neq 0 \bmod x$ and $\dfrac{\partial f}{\partial \dot{y}}(x, y_s, \dot{y}_s) = 0 \bmod x$,
2. $\dfrac{\partial f}{\partial y}(x, y_s, \dot{y}_s) = 0 \bmod x$ and $\dfrac{\partial f}{\partial \dot{y}}(x, y_s, \dot{y}_s) \neq 0 \bmod x$,
3. $\dfrac{\partial f}{\partial y}(x, y_s, \dot{y}_s) \neq 0 \bmod x$ and $\dfrac{\partial f}{\partial \dot{y}}(x, y_s, \dot{y}_s) \neq 0 \bmod x$, and $\dfrac{a_0}{b_0} \notin \mathbb{Z}^{*-}$,
   where $a_0 = \dfrac{\partial f}{\partial y}(0, y_s, \dot{y}_s), b_0 = \dfrac{\partial f}{\partial \dot{y}}(0, y_s, \dot{y}_s)$.

We will say that y_s satisfies the regular case for f.
Then there exists $y \in \mathbb{C}[[x]]$ solution of the differential equation $f(x, y, \dot{y}) = 0$ such that $y = y_s \bmod x^{2^s}$.

We can now detail the study of our example.
First of all, we rewrite the equation in terms of the Euler derivation: $f(x, y, \dot{y}) = \dot{y}^2 + x^2 y(1 + y^2)$.

In this context, we associate to each differential monomial $x^k y^l \dot{y}^m$ the point $Q = (l + m, k)$. The Newton polygon is then the lower convex hull of the set of points associated to the monomials appearing in f (or the convex hull of the set $\bigcup Q + \{0\} \times \mathbb{R}^+$). [2]



For our example, we obtain:

[2] There is no fundamental difference beetween this polygon and the polygon defined in Bruno's book. The way to associate a point in $\mathbb{R}^2$ to a differential monomial is the same, modulo the definition of $\dot{y}$ and the

Associated to the vertex $(0,2)$ and to the equation $f(x,y,\dot{y})$ mod $x=0$, i.e. $\dot{y}=0$, we find an infinity of beginnings of solutions $y=c, c\in\mathbb{C}$.
If we consider $s=0$ and $y_0=c$, the conditions of regular case are not satisfied.
Then we make the change of unknown $y=c+u$, and we have to solve the new equation satisfied by u, with the additional condition $O(u)>0$:

$$f_1(x,u,\dot{u}) = \dot{u}^2 + x^2(c+c^3+(1+3c^2)u+3cu^2+u^3).$$

Its Newton polygon depends essentially on the nullity of $c+c^3$.

- if $c+c^3=0$, $f_1(x,u,\dot{u})=\dot{u}^2+x^2((1+3c^2)u+3cu^2+u^3)=0$. We can find two types of suitable solutions:
  1. the solution $u=0$ leads to the three constant (exact) solutions $y=0,\pm i$.
  2. considering the edge of slope -2, we put $u=x^2v$ and we obtain $f_2(x,v,\dot{v})=x^{-4}f_1(x, x^2v, x^2(\dot{v}+2v))=(\dot{v}+2v)^2+(1+3c^2)v+3cx^2v^2+x^4v^3$. For this new equation $f_2(x,v,\dot{v})=0$, the value $v_0=-\frac{1+3c^2}{4}$ verifies the "regular case", because $\frac{\partial f_2}{\partial v}(x,v_0,\dot{v}_0)=-(1+3c^2)=\frac{\partial f_2}{\partial \dot{v}}(x,v_0,\dot{v}_0)$ mod x.
  So we obtain three solutions in $\mathbb{C}[[x]]$: $c-\frac{1+3c^2}{4}x^2+\ldots$, $c=0,\pm i$.
  For $c=0$, of course, this series corresponds to the edge of slope -2 of the Newton polygon of the initial equation f and is the expansion of the solution y_0 of the previous paragraphs.
- if $c+c^3\neq 0$, we consider the edge of slope -1 of the Newton polygon of f_1, put $u=xv$ and $f_2(x,v,\dot{v})=x^{-2}f_1(x,xv,x(v+\dot{v}))$. The constant values $v_0=\pm\sqrt{-c-c^3}$ are in the regular case for f_2 and lead to two solutions in $\mathbb{C}[[x]]$.
  If we substitute $c=0$ or $\pm i$ in these solutions, we find again the series of the above paragraph.

Considering the edge of slope 2 of the Newton polygon of f, we put $y=x^{-2}u$, and the new equation satisfied by u ($x^4 f(x,y,\dot{y})$) is in the regular case for $u_2=-4$. This solution in $\mathbb{C}((x))$ is in fact $\frac{1}{y_0(x)}$. This result is coherent with the fact that for all non null solution $y(x)$ of $f(x,y,\dot{y})=0$, $\frac{1}{y(x)}$ is also solution.

It is now time to do a comparison beetween the three previous approaches. Bruno's approach has the big advantage to treat a very general case (systems of equations with partial derivatives and parameters). The theory presented in [2] is an instance of these techniques in the case of one partial differential equation. The same mathematical objects are introduced with different names: a "strongly convex" cone is a "forward" one, the series with exponants in a cone C are the analogous of the series of class $\mathcal{C}(C)$ (up to ramification), the "characteristic equation" associated to an edge or a vertex of the Newton polyhedron is the same as the "truncated equation", and so on. Nevertheless, the algorithm implemented in PSSPDE is not complete with respect to the theory, because it takes into account only the edges of the Newton polygon. So it is not surprising that the two beginnings of solution found on example (1) are corresponding to the two edges, of slope 2 and -2, and that the solutions beginning with $c\in\mathbb{C}^*$ are missed.
By the other hand, the "regular case" is efficient to precise the algebraic nature of the series solutions. In the future, we hope to apply a slightly different version to obtain also series with logarithmic or exponential terms.

6 Complex graphical analysis

In this section, the aim is to represent the solutions y_0 and $\frac{1}{y_0}$ in the complex domain, that is as a complex function of the complex variable. For this purpose, we will use the methods already developped for the solutions of linear differential equations (the middle branch of figure 1). The first implementation of such a graphical toolbox is described in [14]. A new version of the 2D functionalities is now available, written in MAPLE and very easy to use.

inversion of coordinates (x and y). So (up to a translation), the two figures are symetrical with respect to the first diagonal. This inversion is done by us in reference to the first "Newton diagram" [12] and to the linear differential case. The consideration only of the lower part of the convex hull is related to the study near $x=0$ and the to the type of equation (polynomial in y and $\dot{y}$ with series coefficients). It corresponds to eliminate from Bruno's polygon the edges whose normal cone contains only vectors with a positive first coordinate.

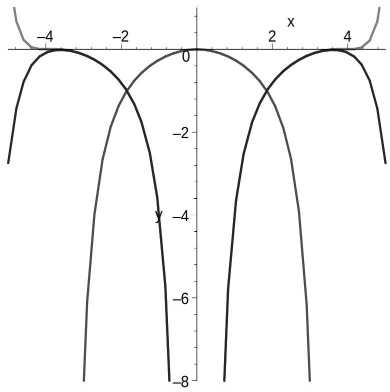


Fig. 6. the graph of y_0 and two approximations of $\frac{1}{y_0}$ (order 14 and 18)

- Graphical part
  The principle of representation is to plot the image under the considered function f of a circle or a circular arc around the singularity, in general 0. The color is used to associate a point in the domain and its image: each point $f(x)$ is plotted with a color corresponding to the argument of x.
  As the considered functions are in general multi-valued, points in the domain are represented with their Euler coordinates, whereas the image points are computed in cartesian coordinates. So we have to manipulate two types of colored points : $(\rho, \theta, color)$ and $(x, y, color)$. Two functions *draw_rhotheta* and *draw_xy* are associated to plot lists of such points. The function *create_circle(*ρ, θ^-, θ^+*, nbpoints)* creates an arc of circle around 0, of radius ρ, as a list of *nbpoints* of modulus ρ and argument between θ^- and θ^+.
- Numerical part
  The original work was dedicated to the advanced study of summation methods of divergent series, but so, was limited to the series obtained by the software DESIR. This new version is more oriented "user-friendly": the unique method of summation of divergent series integrated now is the most basic one (summation at least term for Gevrey series [13]), but it presents many advantages for a novice user. For a convergent power series, the summation method consists in summing the truncated series at the order defined by the global variable of MAPLE Order.
  The function *compute_series(series, list_of_points)* computes the image of the list of complex (colored) points *list_of_points* by the "series" *series* and returns the list of image points.
  Two types of "series" are acceptable as first argument: Frobenius series, computed by DESIR, and characterized by the DESIR type *logseries* and an other internal data type, named *Puiseuxseries*, which is a list $[ram, \lambda, s]$, where s is a MAPLE powerseries, ram is a rational and λ an algebraic number. This represents the series $x^{\frac{\lambda}{ram}} s(x^{\frac{1}{ram}})$.
  Of course, it is possible to convert any series of the type $x^\lambda u(x)$, where λ is an algebraic number and u a Puiseux series into this internal data structure (the conversion is done by the function *convert/Puiseuxseries*).
- Example
  Let us see what can be obtained on our example.
  The series $y_0(x) = \frac{-\psi(x)^2}{4}$ is convergent (its radius of convergence can be numerically estimated > 3.5). We can obtain the corresponding internal data structure of type *Puiseux series*, either from the powerseries $sol0$ (4), either from the MAPLE series computed by *dsolve* with the option 'series'.

```
> solc:=`convert/Puiseuxseries`(sol0);
```

$$solc := [1, 0, sol0]$$

```
> Order:53:dsolve({ode,y(0)=0},y(x),'series'):

> sol1:=`convert/Puiseuxseries`(op(2,op(2,%)),1/4,1/2);
```

$$sol1 := [\frac{1}{4}, \frac{1}{2}, \textbf{proc}(powparm) \ldots \textbf{end proc}]$$

```
> Order:=12:
> C1:=create_circle(0.1,0,Pi,30):seq1:=compute_series(sol1,C1):
> C2:=create_circle(0.5,0,Pi,50):seq2:=compute_series(sol1,C2):
> C3:=create_circle(1.,0,Pi,50):seq3:=compute_series(sol1,C3):
> C4:=create_circle(2.,0,Pi,80):seq4:=compute_series(sol1,C4):
> C5:=create_circle(3.,0,Pi,150):seq5:=compute_series(sol1,C5):
> draw_rhotheta([op(C1),op(C2),op(C3),op(C4),op(C5)],"");
```

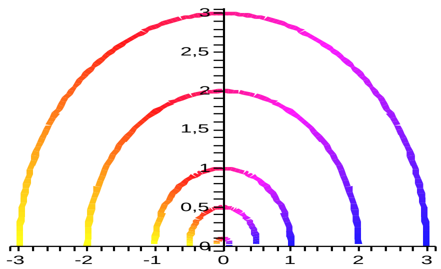


Fig. 7. domain (5 arcs of circle)

We have chosen to compute the solution only on demi-circles, because this is suffisant to obtain the "whole" image. This is due to the fact that the solution depends on x^2. In other words, the image of the corresponding complete circles would be the following sets, each point obtained two times.

```
> draw_xy([op(seq1),op(seq2),op(seq3),op(seq4),op(seq5)],"");
```

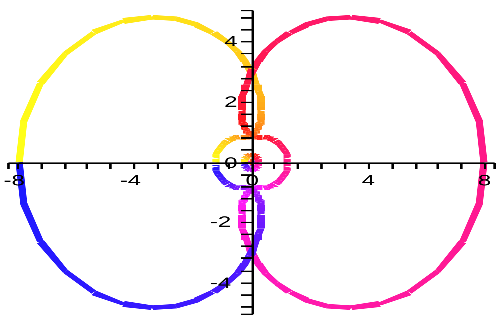


Fig. 8. co-domain (image of the five arcs of circle)

On the last figure, we represent the image of the two arcs of circle C_4 and C_5 by the two solutions y_0 and $1/y_0$. We don't plot the whole image, but only a zoom of the region near the origin.

```
> Order:=53:solinf:=series(1/sol0,x):
> sol2:=`convert/Puiseuxseries`(solinf,1/4,-1/2);
```

$$sol2 := [\frac{1}{4}, \frac{-1}{2}, \textbf{proc}(powparm)\ \ldots\ \textbf{end proc}]$$

```
> Order:=12:seqq4:=compute_series(sol2,C4):
> seqq5:=compute_series(sol2,C5):
> draw_xy([op(seq4),op(seq5),op(seqq4),op(seqq5)],"",-1.5,1.5,-1.5,1.5);
```

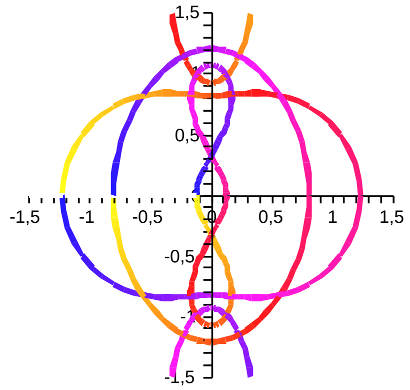


Fig. 9. example of zoom

Of course, this example is not the best one to illustrate the capability to sum various series (divergent, with logarithmic terms) but it shows some graphical possibilities (several curves on the same graph, zoom,...). Of course too, a "paper" version is black and white, but a colored version is available at the following address: http://www-lmc.imag.fr/lmc-mosaic/Francoise.Jung/.

7 Conclusion

It is now clear that the equation (1) has power series solutions that are not the three constants $0, \pm i$. And so, the question is : why such a power series is not a transseries ? Even though we can find such short definition of transseries : "The transseries are a generalization of the usual formal power series, allowing the recursive introduction of exponential and logarithmic variables." ([15], p18). To give a complete answer to this question, we have to enter the world of transseries (grid-based transseries, well-ordered transeries, and the last one, but also the most complicated, the complex transseries [16]), first introduced by J. Ecalle [10], then defined from an algebraic point of view by J. van der Hoeven [15].
If we restrict our attention to the real field of grid-based transseries $\mathbb{R}[[x]]$, the answer is as follows: let X be the monomial group $(1/x)^{\mathbb{R}}$, ordered with the same ordering as on $\mathbb{R}$. $\mathbb{R}[[X]]$ is the set of grid-based series, that is the set of mappings φ from X to $\mathbb{R}$, with grid-based support:

$$\exists m_1, \ldots, m_k, n \in X, m_i > 0, supp(\varphi) = \{m \in X, \varphi(m) \neq 0\} \subset m_1^{\mathbb{N}} \ldots m_k^{\mathbb{N}} n.$$

With this ordering on X, the support of the series *sol0* is an infinite sequence of strictly decreasing monomials and cannot belong to $\mathbb{R}[[X]]$. The field $\mathbb{R}[[x]]$ is then built by inductive insertion of exponentials and logarithms from the *basic set* $\mathbb{R}[[X]]$ and doesn't contain *sol0*.
It seems that the choice of ordering on X is arbitrary and related with the specific purpose of studying the asymptotic when x tends to infinity. Of course, if we choose the inverse ordering, the solution *sol0* would belong to the corresponding field of "transseries", but it would be easy to find an other equation having no "transseries" solutions, which are not solution of a lower order equation...
On the way, we show an example of application of the regular case, and graphical representations of complex functions with simple MAPLE instructions.

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