Ghost Imaging

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- General topic: correlation-based imaging with noise sources.
- Particular application: Ghost imaging.

Scalar wave equation and Green's function

- In this talk, we consider the scalar wave model in $\mathbb{R}^d$, $d = 2, 3$:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u}{\partial t^2} - \Delta_{\vec{\boldsymbol{x}}} u = n(t, \vec{\boldsymbol{x}})$$

$n(t, \vec{\boldsymbol{x}})$: source.
$c(\vec{\boldsymbol{x}})$: propagation speed (parameter of the medium), assumed to be constant outside a domain with compact support.

In the Fourier domain:

$$\hat{u}(\omega, \vec{\boldsymbol{x}}) = \int u(t, \vec{\boldsymbol{x}}) e^{i\omega t} dt$$

we have

$$\hat{u}(\omega, \vec{\boldsymbol{x}}) = \int \hat{G}(\omega, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) \hat{n}(\omega, \vec{\boldsymbol{y}}) d\vec{\boldsymbol{y}}$$

where the time-harmonic Green's function $\hat{G}(\omega, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}})$ is the solution of the Helmholtz equation

$$\Delta_{\vec{\boldsymbol{x}}} \hat{G} + \frac{\omega^2}{c^2(\vec{\boldsymbol{x}})} \hat{G} = -\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}),$$

with the Sommerfeld radiation condition ($c(\vec{\boldsymbol{x}}) = c_0$ at infinity):

$$\lim_{|\vec{\boldsymbol{x}}| \to \infty} |\vec{\boldsymbol{x}}|^{\frac{d-1}{2}} \left(\frac{\vec{\boldsymbol{x}}}{|\vec{\boldsymbol{x}}|} \cdot \nabla_{\vec{\boldsymbol{x}}} - i \frac{\omega}{c_0} \right) \hat{G}(\omega, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) = 0$$

Green's function estimation with ambient noise sources (1/3)

$$\frac{1}{c^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u}{\partial t^2} - \Delta_{\vec{\boldsymbol{x}}} u = n(t, \vec{\boldsymbol{x}})$$

- Sources $n(t, \vec{\boldsymbol{x}})$: random process, stationary in time, with mean zero and covariance

$$\big\langle n(t_1, \vec{\boldsymbol{y}}_1) n(t_2, \vec{\boldsymbol{y}}_2) \big\rangle = F(t_2 - t_1) K(\vec{\boldsymbol{y}}_1) \delta(\vec{\boldsymbol{y}}_1 - \vec{\boldsymbol{y}}_2)$$

$\langle \cdot \rangle$: statistical average.
The function $\hat{F}$ is the power spectral density of the sources.
The function K characterizes the spatial support of the sources.

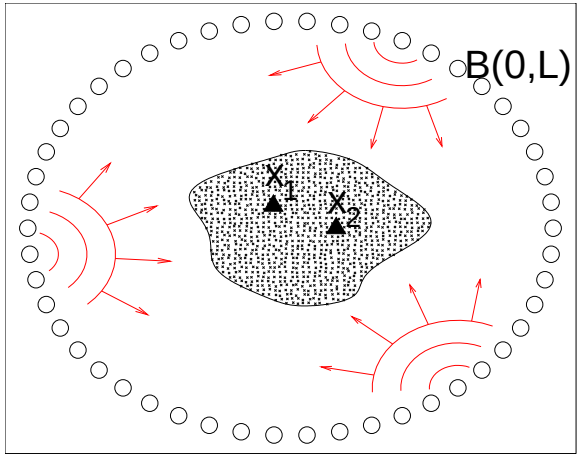
- The empirical cross correlation:

$$C_T(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = \frac{1}{T}\int_0^T u(t, \vec{\boldsymbol{x}}_1) u(t+\tau, \vec{\boldsymbol{x}}_2) dt$$

converges in probability as $T \to \infty$ to the statistical cross correlation $C^{(1)}$ given by

$$\begin{aligned} C^{(1)}(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) &= \big\langle u(0, \vec{\boldsymbol{x}}_1) u(\tau, \vec{\boldsymbol{x}}_2) \big\rangle \\ &= \frac{1}{2\pi}\int d\vec{\boldsymbol{y}} \int d\omega \overline{\hat{G}}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{y}}) \hat{G}(\omega, \vec{\boldsymbol{x}}_2, \vec{\boldsymbol{y}}) K(\vec{\boldsymbol{y}}) \hat{F}(\omega) e^{-i\omega\tau} \end{aligned}$$

Green's function estimation with ambient noise sources (2/3)



Cross correlation with noise sources distributed on a <u>closed surface</u> $\partial B(\mathbf{0}, L)$:

$$C^{(1)}(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = \frac{1}{2\pi} \int d\omega \int_{\partial B(\mathbf{0},L)} d\sigma(\vec{\boldsymbol{y}}) \overline{\hat{G}}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{y}}) \hat{G}(\omega, \vec{\boldsymbol{x}}_2, \vec{\boldsymbol{y}}) \hat{F}(\omega) e^{-i\omega\tau}$$

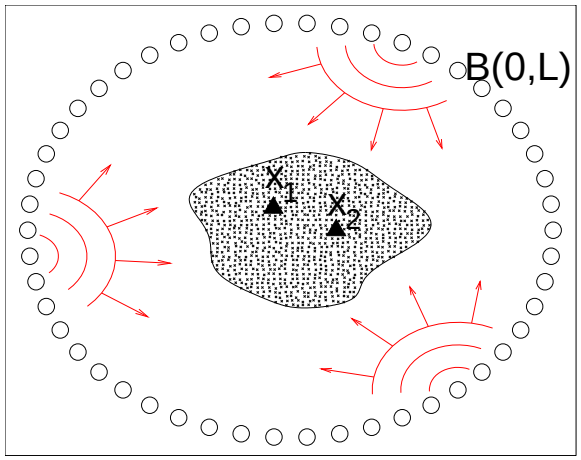
By Helmholtz-Kirchhoff identity,

$$\hat{G}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) - \overline{\hat{G}}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = \frac{2i\omega}{c_0} \int_{\partial B(\mathbf{0},L)} d\sigma(\vec{\boldsymbol{y}}) \overline{\hat{G}}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{y}}) \hat{G}(\omega, \vec{\boldsymbol{x}}_2, \vec{\boldsymbol{y}})$$

we have

$$C^{(1)}(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = \frac{c_0}{4\pi} \int \frac{\hat{F}(\omega)}{\omega} \mathrm{Im}\big(\hat{G}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2)\big) e^{-i\omega\tau} d\omega$$

Green's function estimation with ambient noise sources (3/3)



$$\partial_\tau C^{(1)}(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = -\frac{ic_0}{4\pi} \int \hat{F}(\omega) \mathrm{Im}\big(\hat{G}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2)\big) e^{-i\omega\tau} d\omega$$

$$= -\frac{c_0}{2}\Big(F *_\tau G(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) - F *_\tau G(-\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2)\Big)$$

- The cross correlation is related to the Green's function.

$\hookrightarrow$ the passive sensors can be transformed into virtual sources.

- If the medium is homogeneous

$$\hat{G}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = \frac{1}{4\pi|\vec{\boldsymbol{x}}_1 - \vec{\boldsymbol{x}}_2|} \exp\left(i\frac{\omega}{c_0}|\vec{\boldsymbol{x}}_1 - \vec{\boldsymbol{x}}_2|\right)$$

then

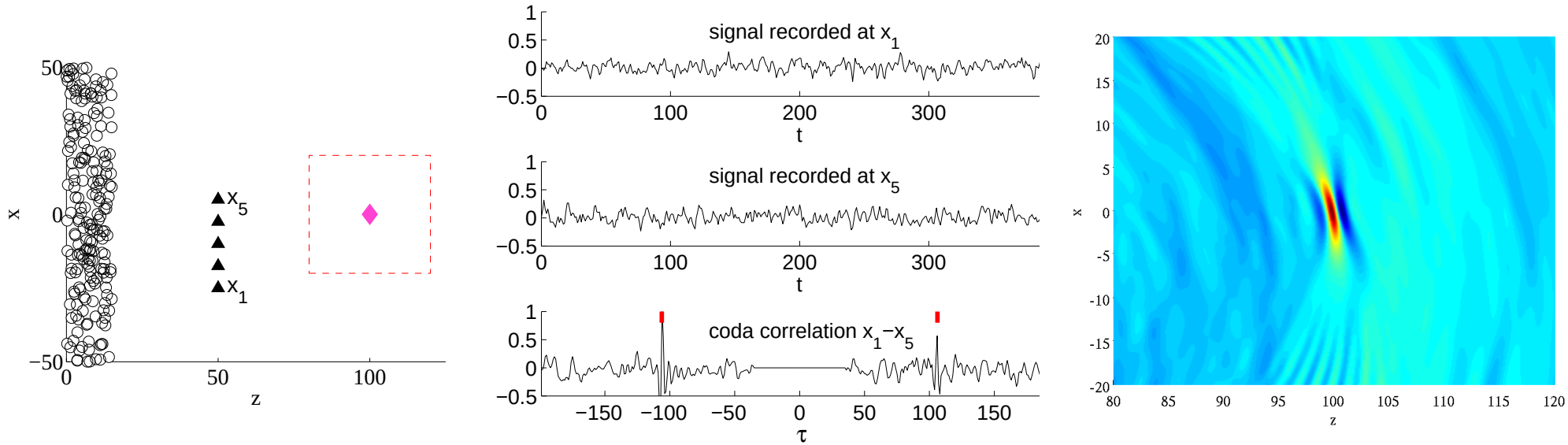
$$\hat{C}^{(1)}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = -\frac{c_0}{2}\frac{\hat{F}(\omega)}{\omega}\mathrm{sinc}\left(\frac{\omega}{c_0}|\vec{\boldsymbol{x}}_1 - \vec{\boldsymbol{x}}_2|\right)$$

Reflector imaging with a passive receiver array

- Ambient noise sources (○) emit stationary random signals.
- The signals $(u(t,\vec{\boldsymbol{x}}_r))_{r=1,\ldots,N_{\rm r}}$ are recorded by the receivers $(\vec{\boldsymbol{x}}_r)_{r=1,\ldots,N_{\rm r}}$ (▲).
- The reflector (♦) is imaged by migration of the cross correlation matrix [1]:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{r,r'=1}^{N_{\rm r}} C_T\Big(\frac{|\vec{\boldsymbol{x}}_{r'} - \vec{\boldsymbol{y}}^S|}{c_0} + \frac{|\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{y}}^S|}{c_0}, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}\Big)$$

$$\text{with} \quad C_T(\tau, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}) = \frac{1}{T}\int_0^T u(t+\tau, \vec{\boldsymbol{x}}_{r'})u(t, \vec{\boldsymbol{x}}_r)dt$$



Good image provided the ambient noise illumination is long (in time) and diversified (in angle) [1].

[1] J. Garnier and G. Papanicolaou, *SIAM J. Imaging Sciences* **2**, 396 (2009).

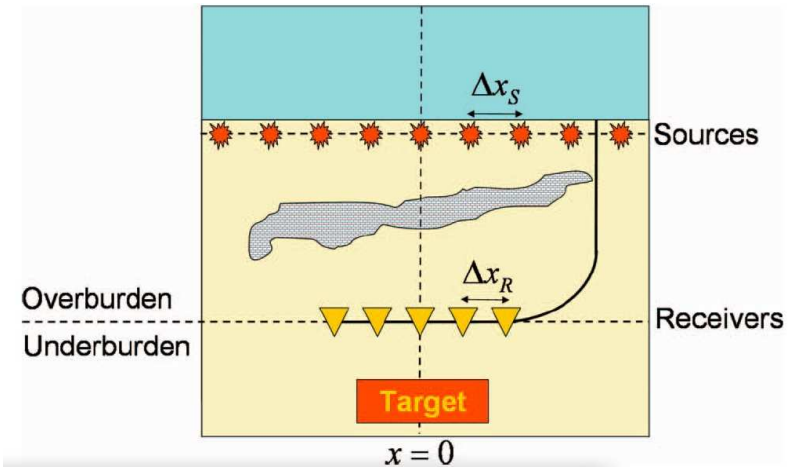
A successful application: seismic exploration below an overburden

Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), t \in \mathbb{R}, r = 1, \ldots, N_{\rm r}, s = 1, \ldots, N_{\rm s}\}$

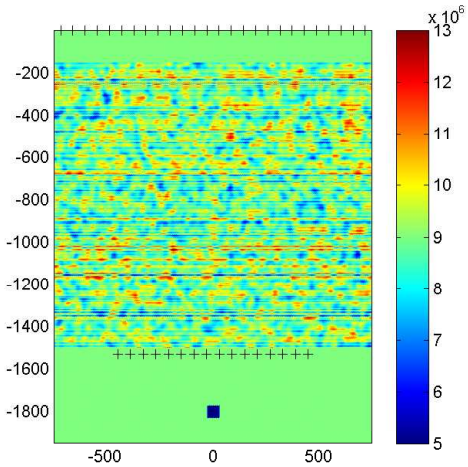
Correlations: $\{C(t, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}), t \in \mathbb{R}, r, r' = 1, \ldots, N_{\rm r}\}$

$C(t, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}) = \sum_{s=1}^{N_s} \int_{-\infty}^{\infty} u(t' + t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) u(t', \vec{\boldsymbol{x}}_{r'}; \vec{\boldsymbol{x}}_s) dt'$

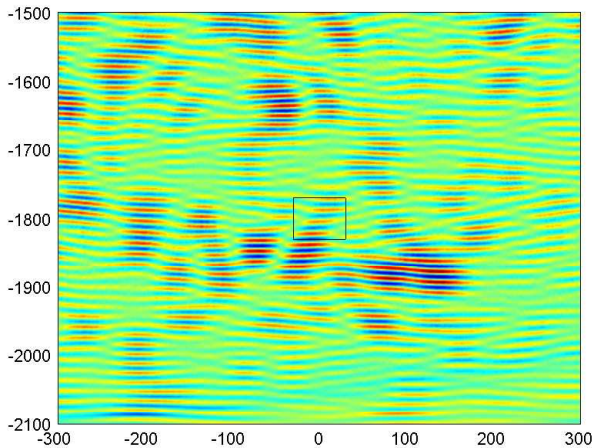
Imaging by migration of correlations [1]



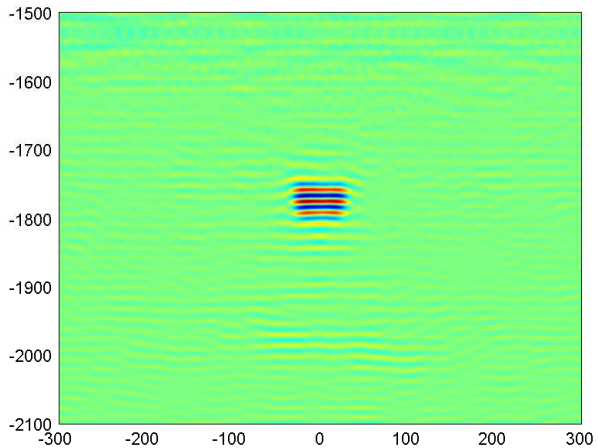
From Bakulin and Calvert (2009)



geometric set-up



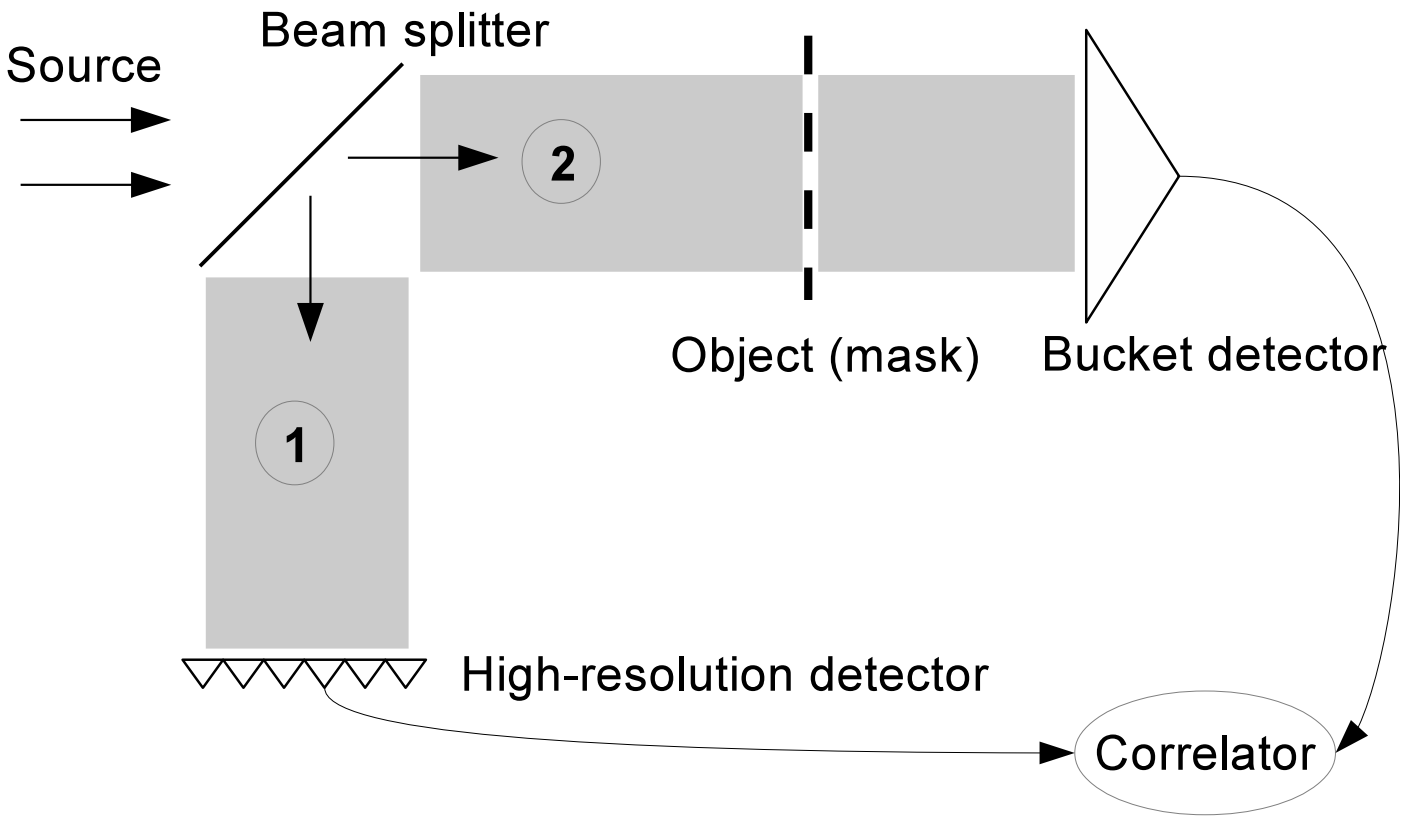
data migration



correlation migration

[1] J. Garnier and G. Papanicolaou, *Inverse Problems* **28**, 075002 (2012); *SIIMS* **7**, 1210 (2014).

Ghost imaging



- Noise source.
- without object in path 1; a high-resolution detector measures the spatially-resolved and time-resolved intensity $I_1(t, \boldsymbol{x})$.
- with object (mask) in path 2; a single-pixel detector measures the spatially-integrated and time-resolved intensity $I_2(t)$.

Experimental results: the correlation of I_1 and I_2 is an image of the object [1,2] !

[1] A. Valencia et al., *PRL* **94**, 063601 (2005); [2] J. H. Shapiro et al., *Quantum Inf. Process* **1** 949 (2012).

Ghost imaging

- Wave equation in paths 1 and 2:

$$\frac{1}{c_j^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u_j}{\partial t^2} - \Delta_{\vec{\boldsymbol{x}}} u_j = n(t,\boldsymbol{x})\delta(z), \qquad \vec{\boldsymbol{x}} = (\boldsymbol{x}, z) \in \mathbb{R}^2 \times \mathbb{R}, \qquad j = 1, 2$$

- Noise source:

$$\big\langle n(t,\boldsymbol{x})n(t,\boldsymbol{x}')\big\rangle = F(t-t')K(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{x}')$$

with $\hat{F}(\omega)$ centered on ω_0.

- Observations:

$$\begin{aligned} I_1(t,\boldsymbol{x}) &= |u_1(t,(\boldsymbol{x},L))|^2 \\ I_2(t) &= \int_{\mathbb{R}^2} I_2(t,\boldsymbol{x}')d\boldsymbol{x}' = \int_{\mathbb{R}^2} |u_2(t,(\boldsymbol{x}',L+L_0))|^2 d\boldsymbol{x}' \end{aligned}$$

- Correlation:

$$C_T(\boldsymbol{x}) = \frac{1}{T}\int_0^T I_1(t,\boldsymbol{x})I_2(t)dt - \Big(\frac{1}{T}\int_0^T I_1(t,\boldsymbol{x})dt\Big)\Big(\frac{1}{T}\int_0^T I_2(t)dt\Big)$$

- Let us first carry out the analysis in homogeneous media.

- Wave representation in path 1:

$$\hat{u}_1(\omega, \boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{G}_1\big(\omega, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big)\hat{n}(\omega, \boldsymbol{y})d\boldsymbol{y}$$

with $\hat{G}_1\big(\omega, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big) = \hat{G}_0\big(\omega, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big)$ (no object).

- Wave representation in path 2:

$$\hat{u}_2(\omega, \boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{G}_2\big(\omega, (\boldsymbol{x}, L + L_0), (\boldsymbol{y}, 0)\big)\hat{n}(\omega, \boldsymbol{y})d\boldsymbol{y}$$

with
$\hat{G}_2\big(\omega, (\boldsymbol{x}, L + L_0), (\boldsymbol{y}, 0)\big) = \int_{\mathbb{R}^2} \hat{G}_0\big(\omega, (\boldsymbol{x}, L + L_0), (\boldsymbol{r}, L)\big)\mathcal{T}(\boldsymbol{r})\hat{G}_0\big(\omega, (\boldsymbol{r}, L), (\boldsymbol{y}, 0)\big)d\boldsymbol{r}$
(mask $\mathcal{T}(\boldsymbol{r})$ in the plane $z = L$).

- Wave representation in path 1:

$$\hat{u}_1(\omega, \boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{G}_1\big(\omega, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big)\hat{n}(\omega, \boldsymbol{y})d\boldsymbol{y}$$

with $\hat{G}_1\big(\omega, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big) = \hat{G}_0\big(\omega, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big)$ (no object).

- Wave representation in path 2:

$$\hat{u}_2(\omega, \boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{G}_2\big(\omega, (\boldsymbol{x}, L + L_0), (\boldsymbol{y}, 0)\big)\hat{n}(\omega, \boldsymbol{y})d\boldsymbol{y}$$

with
$\hat{G}_2\big(\omega, (\boldsymbol{x}, L + L_0), (\boldsymbol{y}, 0)\big) = \int_{\mathbb{R}^2} \hat{G}_0\big(\omega, (\boldsymbol{x}, L + L_0), (\boldsymbol{r}, L)\big)\mathcal{T}(\boldsymbol{r})\hat{G}_0\big(\omega, (\boldsymbol{r}, L), (\boldsymbol{y}, 0)\big)d\boldsymbol{r}$
(mask $\mathcal{T}(\boldsymbol{r})$ in the plane $z = L$).

- Result *with $n(t, \boldsymbol{x})$ Gaussian distributed*:

$$C_T(\boldsymbol{x}) \stackrel{T\to\infty}{\longrightarrow} C^{(1)}(\boldsymbol{x})$$

with

$$\begin{aligned} C^{(1)}(\boldsymbol{x}) \quad = \quad & \frac{1}{2\pi^2} \iint d\omega d\omega' \int_{\mathbb{R}^2} d\boldsymbol{y} \int_{\mathbb{R}^2} d\boldsymbol{y}' K(\boldsymbol{y})K(\boldsymbol{y}')\hat{F}(\omega)\hat{F}(\omega') \\ & \times \hat{G}_1\big(\omega, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big)\overline{\hat{G}_1}\big(\omega', (\boldsymbol{x}, L), (\boldsymbol{y}', 0)\big) \\ & \times \int_{\mathbb{R}^2} d\boldsymbol{x}' \overline{\hat{G}_2}\big(\omega, (\boldsymbol{x}', L + L_0), (\boldsymbol{y}, 0)\big)\hat{G}_2\big(\omega', (\boldsymbol{x}', L + L_0), (\boldsymbol{y}', 0)\big) \end{aligned}$$

- Wave representation in path 1:

$$\hat{u}_1(\omega, \boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{G}_1\big(\omega, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big)\hat{n}(\omega, \boldsymbol{y})d\boldsymbol{y}$$

with $\hat{G}_1\big(\omega, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big) = \hat{G}_0\big(\omega, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big)$ (no object).

- Wave representation in path 2:

$$\hat{u}_2(\omega, \boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{G}_2\big(\omega, (\boldsymbol{x}, L + L_0), (\boldsymbol{y}, 0)\big)\hat{n}(\omega, \boldsymbol{y})d\boldsymbol{y}$$

with
$\hat{G}_2\big(\omega, (\boldsymbol{x}, L + L_0), (\boldsymbol{y}, 0)\big) = \int_{\mathbb{R}^2} \hat{G}_0\big(\omega, (\boldsymbol{x}, L + L_0), (\boldsymbol{r}, L)\big)\mathcal{T}(\boldsymbol{r})\hat{G}_0\big(\omega, (\boldsymbol{r}, L), (\boldsymbol{y}, 0)\big)d\boldsymbol{r}$
(mask $\mathcal{T}(\boldsymbol{r})$ in the plane $z = L$).

- Result *with* $n(t, \boldsymbol{x})$ *Gaussian distributed*:

$$C_T(\boldsymbol{x}) \stackrel{T\to\infty}{\longrightarrow} C^{(1)}(\boldsymbol{x})$$

with *quasi-monochromatic approximation* $\hat{F}$ *concentrated around* ω_0:

$$\begin{aligned} C^{(1)}(\boldsymbol{x}) \quad = \quad & 2\int_{\mathbb{R}^2} d\boldsymbol{y} \int_{\mathbb{R}^2} d\boldsymbol{y}' K(\boldsymbol{y})K(\boldsymbol{y}') \\ & \times \hat{G}_1\big(\omega_0, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big)\overline{\hat{G}_1}\big(\omega_0, (\boldsymbol{x}, L), (\boldsymbol{y}', 0)\big) \\ & \times \int_{\mathbb{R}^2} d\boldsymbol{x}' \overline{\hat{G}_2}\big(\omega_0, (\boldsymbol{x}', L + L_0), (\boldsymbol{y}, 0)\big)\hat{G}_2\big(\omega_0, (\boldsymbol{x}', L + L_0), (\boldsymbol{y}', 0)\big) \end{aligned}$$

$$
\begin{aligned}
C^{(1)}(\boldsymbol{x}) &= 2\int_{\mathbb{R}^2} d\boldsymbol{y}\int_{\mathbb{R}^2} d\boldsymbol{y}' K(\boldsymbol{y})K(\boldsymbol{y}') \\
&\quad \times \hat{G}_1\big(\omega_0,(\boldsymbol{x},L),(\boldsymbol{y},0)\big)\overline{\hat{G}_1}\big(\omega_0,(\boldsymbol{x},L),(\boldsymbol{y}',0)\big) \\
&\quad \times \int_{\mathbb{R}^2} d\boldsymbol{x}'\overline{\hat{G}_2}\big(\omega_0,(\boldsymbol{x}',L+L_0),(\boldsymbol{y},0)\big)\hat{G}_2\big(\omega_0,(\boldsymbol{x}',L+L_0),(\boldsymbol{y}',0)\big)
\end{aligned}
$$

With $\hat{G}_1\big(\omega_0,(\boldsymbol{x},L),(\boldsymbol{y},0)\big) = \hat{G}_0\big(\omega_0,(\boldsymbol{x},L),(\boldsymbol{y},0)\big)$ and
$\hat{G}_2\big(\omega_0,(\boldsymbol{x},L+L_0),(\boldsymbol{y},0)\big) = \int_{\mathbb{R}^2}\hat{G}_0\big(\omega_0,(\boldsymbol{x},L+L_0),(\boldsymbol{r},L)\big)\mathcal{T}(\boldsymbol{r})\hat{G}_0\big(\omega_0,(\boldsymbol{r},L),(\boldsymbol{y},0)\big)d\boldsymbol{r}$
we get

$$
\begin{aligned}
C^{(1)}(\boldsymbol{x}) &= 2\int_{\mathbb{R}^2} d\boldsymbol{r}\int_{\mathbb{R}^2} d\boldsymbol{r}'\overline{\mathcal{T}(\boldsymbol{r})}\mathcal{T}(\boldsymbol{r}') \\
&\quad \times \int_{\mathbb{R}^2} d\boldsymbol{y}K(\boldsymbol{y})\hat{G}_0\big(\omega_0,(\boldsymbol{x},L),(\boldsymbol{y},0)\big)\overline{\hat{G}_0}\big(\omega_0,(\boldsymbol{r},L),(\boldsymbol{y},0)\big) \\
&\quad \times \int_{\mathbb{R}^2} d\boldsymbol{y}'K(\boldsymbol{y}')\overline{\hat{G}_0}\big(\omega_0,(\boldsymbol{x},L),(\boldsymbol{y}',0)\big)\hat{G}_0\big(\omega_0,(\boldsymbol{r}',L),(\boldsymbol{y}',0)\big) \\
&\quad \times \int_{\mathbb{R}^2} d\boldsymbol{x}'\overline{\hat{G}_0}\big(\omega_0,(\boldsymbol{x}',L+L_0),(\boldsymbol{r},L)\big)\hat{G}_0\big(\omega_0,(\boldsymbol{x}',L+L_0),(\boldsymbol{r}',L)\big)
\end{aligned}
$$

$$
\begin{aligned}
C^{(1)}(\boldsymbol{x}) \quad = \quad & 2\int_{\mathbb{R}^2} d\boldsymbol{r} \int_{\mathbb{R}^2} d\boldsymbol{r}' \overline{\mathcal{T}(\boldsymbol{r})}\mathcal{T}(\boldsymbol{r}') \\
& \times \int_{\mathbb{R}^2} d\boldsymbol{y} K(\boldsymbol{y}) \hat{G}_0\big(\omega_0, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big) \overline{\hat{G}_0}\big(\omega_0, (\boldsymbol{r}, L), (\boldsymbol{y}, 0)\big) \\
& \times \int_{\mathbb{R}^2} d\boldsymbol{y}' K(\boldsymbol{y}') \overline{\hat{G}_0}\big(\omega_0, (\boldsymbol{x}, L), (\boldsymbol{y}', 0)\big) \hat{G}_0\big(\omega_0, (\boldsymbol{r}', L), (\boldsymbol{y}', 0)\big) \\
& \times \int_{\mathbb{R}^2} d\boldsymbol{x}' \overline{\hat{G}_0}\big(\omega_0, (\boldsymbol{x}', L + L_0), (\boldsymbol{r}, L)\big) \hat{G}_0\big(\omega_0, (\boldsymbol{x}', L + L_0), (\boldsymbol{r}', L)\big)
\end{aligned}
$$

Use reciprocity:

$$
\begin{aligned}
C^{(1)}(\boldsymbol{x}) \quad = \quad & 2\int_{\mathbb{R}^2} d\boldsymbol{r} \int_{\mathbb{R}^2} d\boldsymbol{r}' \overline{\mathcal{T}(\boldsymbol{r})}\mathcal{T}(\boldsymbol{r}') \\
& \times \int_{\mathbb{R}^2} d\boldsymbol{y} K(\boldsymbol{y}) \hat{G}_0\big(\omega_0, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big) \overline{\hat{G}_0}\big(\omega_0, (\boldsymbol{y}, 0), (\boldsymbol{r}, L)\big) \\
& \times \int_{\mathbb{R}^2} d\boldsymbol{y}' K(\boldsymbol{y}') \hat{G}_0\big(\omega_0, (\boldsymbol{r}', L), (\boldsymbol{y}', 0)\big) \overline{\hat{G}_0}\big(\omega_0, (\boldsymbol{y}', 0), (\boldsymbol{x}, L)\big) \\
& \times \int_{\mathbb{R}^2} d\boldsymbol{x}' \hat{G}_0\big(\omega_0, (\boldsymbol{r}', L), (\boldsymbol{x}', L + L_0)\big) \overline{\hat{G}_0}\big(\omega_0, (\boldsymbol{x}', L + L_0), (\boldsymbol{r}, L)\big)
\end{aligned}
$$

$$
\begin{aligned}
C^{(1)}(\boldsymbol{x}) \quad = \quad & 2\int_{\mathbb{R}^2} d\boldsymbol{r} \int_{\mathbb{R}^2} d\boldsymbol{r}' \overline{\mathcal{T}(\boldsymbol{r})}\mathcal{T}(\boldsymbol{r}') \\
& \times \int_{\mathbb{R}^2} d\boldsymbol{y} K(\boldsymbol{y}) \hat{G}_0\big(\omega_0, (\boldsymbol{x}, L), (\boldsymbol{y}, 0)\big) \overline{\hat{G}_0}\big(\omega_0, (\boldsymbol{y}, 0), (\boldsymbol{r}, L)\big) \\
& \times \int_{\mathbb{R}^2} d\boldsymbol{y}' K(\boldsymbol{y}') \hat{G}_0\big(\omega_0, (\boldsymbol{r}', L), (\boldsymbol{y}', 0)\big) \overline{\hat{G}_0}\big(\omega_0, (\boldsymbol{y}', 0), (\boldsymbol{x}, L)\big) \\
& \times \int_{\mathbb{R}^2} d\boldsymbol{x}' \hat{G}_0\big(\omega_0, (\boldsymbol{r}', L), (\boldsymbol{x}', L + L_0)\big) \overline{\hat{G}_0}\big(\omega_0, (\boldsymbol{x}', L + L_0), (\boldsymbol{r}, L)\big)
\end{aligned}
$$

Quick guess:

$$
\begin{aligned}
C^{(1)}(\boldsymbol{x}) \quad \simeq \quad & 2\int_{\mathbb{R}^2} d\boldsymbol{r} \int_{\mathbb{R}^2} d\boldsymbol{r}' \overline{\mathcal{T}(\boldsymbol{r})}\mathcal{T}(\boldsymbol{r}') \\
& \times h_K(\boldsymbol{r} - \boldsymbol{x}) \\
& \times h_K(\boldsymbol{r}' - \boldsymbol{x}) \\
& \times h_0(\boldsymbol{r} - \boldsymbol{r}') \\
\simeq \quad & C|\mathcal{T}(\boldsymbol{x})|^2
\end{aligned}
$$

- Resolution analysis:

$$\big\langle n(t,\boldsymbol{x})n(t',\boldsymbol{x}')\big\rangle = F(t-t')K(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{x}'), \qquad K(\boldsymbol{x}) = I_0 \exp\Big(-\frac{|\boldsymbol{x}|^2}{r_0^2}\Big)$$

Paraxial Green's function:

$$\hat{G}_0\big(\omega_0,(\boldsymbol{x},L),(\boldsymbol{y},0)\big) = \frac{1}{4\pi L}\exp\Big(i\frac{\omega_0}{c_0}L + i\frac{\omega_0}{c_0}\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2L}\Big)$$

- Result:

$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} h(\boldsymbol{x}-\boldsymbol{r})|\mathcal{T}(\boldsymbol{r})|^2 d\boldsymbol{r}$$

with

$$h(\boldsymbol{x}) = \frac{I_0^2 r_0^4}{2^7\pi^2 L^2}\exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\mathrm{gi0}}^2}\Big), \qquad \rho_{\mathrm{gi0}}^2 = \frac{c_0^2 L^2}{2\omega_0^2 r_0^2}$$

Resolution: $\rho_{\mathrm{gi0}} \sim \lambda_0 L/r_0$, $\lambda_0 = 2\pi c_0/\omega_0$ (Rayleigh resolution formula).

- Extension for partially coherent source (Gauss-Schell model):

$$\langle n(t,\boldsymbol{x})n(t',\boldsymbol{x}')\rangle = F(t-t')I_0 \exp\Big(-\frac{|\boldsymbol{x}+\boldsymbol{x}'|^2}{4r_0^2} - \frac{|\boldsymbol{x}-\boldsymbol{x}'|^2}{4\rho_0^2}\Big)$$

- Result:

$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} H(\boldsymbol{x},\boldsymbol{r})|\mathcal{T}(\boldsymbol{r})|^2 d\boldsymbol{r}$$

with

$$H(\boldsymbol{x},\boldsymbol{r}) = \frac{I_0^2 r_0^2 \rho_0^2 c_0^2}{32\omega_0^4 \rho_{\mathrm{gi1}}^2 R_{\mathrm{gi1}}^2} \exp\Big(-\frac{|\boldsymbol{x}-\boldsymbol{r}|^2}{4\rho_{\mathrm{gi1}}^2} - \frac{|\boldsymbol{x}+\boldsymbol{r}|^2}{4R_{\mathrm{gi1}}^2}\Big),$$

$$\rho_{\mathrm{gi1}}^2 = \frac{c_0^2 L^2}{2\omega_0^2 r_0^2} + \frac{\rho_0^2}{4} = \rho_{\mathrm{gi0}}^2 + \frac{\rho_0^2}{4}, \qquad R_{\mathrm{gi1}}^2 = \frac{c_0^2 L^2}{2\omega_0^2 \rho_0^2} + \frac{r_0^2}{4}$$

- Loss of resolution due to the partial coherence of the source: $\rho_{\mathrm{gi1}} > \rho_{\mathrm{gi0}}$.
- Fully incoherent case $\rho_0 \to 0$: cf previous case $\rho_{\mathrm{gi1}} = \rho_{\mathrm{gi0}}$.
- Fully coherent case $r_0 = \rho_0$: the kernel is $H(\boldsymbol{x},\boldsymbol{r}) = \frac{I_0^2 \rho_0^4 c_0^2}{32\omega_0^4 \rho_{\mathrm{gi1}}^2} \exp\Big(-\frac{|\boldsymbol{x}|^2}{2\rho_{\mathrm{gi1}}^2} - \frac{|\boldsymbol{r}|^2}{2\rho_{\mathrm{gi1}}^2}\Big)$,

which means there is no resolution at all.

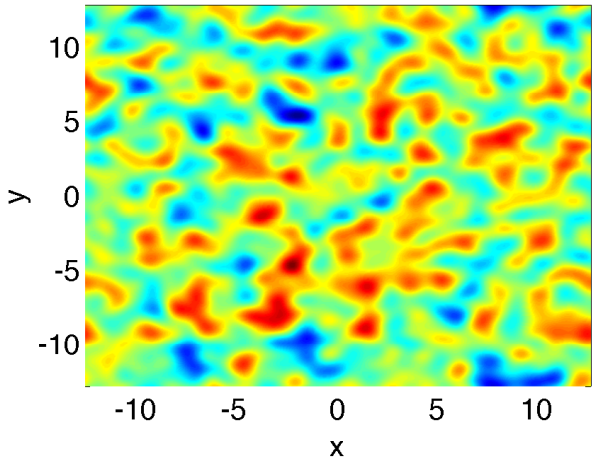
Wave propagation in a random medium

- Random medium model:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_0^2}\big(1+\mu(\vec{\boldsymbol{x}})\big)$$

c_0 is a reference speed,

$\mu(\vec{\boldsymbol{x}})$ is a zero-mean random process.



- The background Green's function (deterministic and known):

$$\Delta_{\vec{\boldsymbol{x}}}\hat{G}_0(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) + \frac{\omega^2}{c_0^2}\hat{G}_0(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) = -\delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})$$

The physical Green's function (random and unknown):

$$\Delta_{\vec{\boldsymbol{x}}}\hat{G}(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) + \frac{\omega^2}{c_0^2}\big(1+\mu(\vec{\boldsymbol{x}})\big)\hat{G}(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) = -\delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})$$

Note:

- the statistics of $(\hat{G}(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}))_{\vec{\boldsymbol{x}}}$ is a *nonlinear* function of the statistics of $(\mu(\vec{\boldsymbol{x}}))_{\vec{\boldsymbol{x}}}$.
- a detailed stochastic analysis is possible in different regimes of separation of scales (small wavelength, large propagation distance, large bandwidth, ...).

Wave propagation in a random medium: the paraxial regime

- Consider the time-harmonic form of the scalar wave equation ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$)

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}\big(1 + \mu(\boldsymbol{x}, z)\big)\hat{u} = 0.$$

Consider the paraxial regime $\lambda \ll l_c \ll L$. More precisely, in the scaled regime

$$\omega \to \frac{\omega}{\varepsilon^4}, \qquad \mu(\boldsymbol{x}, z) \to \varepsilon^3 \mu\big(\frac{\boldsymbol{x}}{\varepsilon^2}, \frac{z}{\varepsilon^2}\big),$$

the function $\hat{\phi}^\varepsilon$ defined by

$$\hat{u}^\varepsilon(\omega, \boldsymbol{x}, z) = e^{i\frac{\omega z}{\varepsilon^4 c_0}}\hat{\phi}^\varepsilon\big(\frac{\omega}{\varepsilon^4}, \frac{\boldsymbol{x}}{\varepsilon^2}, z\big)$$

satisfies

$$\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i\frac{\omega}{c_0}\partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2}\frac{1}{\varepsilon}\mu\big(\boldsymbol{x}, \frac{z}{\varepsilon^2}\big)\hat{\phi}^\varepsilon\right) = 0.$$

[1] J. Garnier and K. Sølna, Ann. Appl. Probab. **19** 318 (2009).

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- In the regime $\varepsilon \ll 1$, the forward-scattering approximation in direction z is valid and $\hat{\phi} = \lim_{\varepsilon \to 0} \hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

$$2i\frac{\omega}{c_0}\partial_z \hat{\phi} + \Delta_\perp \hat{\phi} + \frac{\omega^2}{c_0^2}\dot{B}(\boldsymbol{x}, z)\hat{\phi} = 0$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}')\, z \wedge z'$,
$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$.

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- In the regime $\varepsilon \ll 1$, the forward-scattering approximation in direction z is valid and $\hat{\phi} = \lim_{\varepsilon \to 0} \hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

$$d\hat{\phi} = \frac{ic_0}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}')\, z \wedge z'$,
$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$.

[1] J. Garnier and K. Sølna, Ann. Appl. Probab. **19** 318 (2009).

- We introduce the fundamental solution $\hat{\phi}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)$:

$$d\hat{\phi} = \frac{ic_0}{2\omega}\Delta_\perp\hat{\phi}dz + \frac{i\omega}{2c_0}\hat{\phi}\circ dB(\boldsymbol{x},z)$$

starting from $\hat{\phi}\big(\omega,(\boldsymbol{x},z=z_0),(\boldsymbol{x}_0,z_0)\big) = \delta(\boldsymbol{x}-\boldsymbol{x}_0)$.

- In a homogeneous medium ($\mu\equiv 0$, $B\equiv 0$) the fundamental solution is

$$\hat{\phi}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big) = \frac{\exp\left(\frac{i\omega|\boldsymbol{x}-\boldsymbol{x}_0|^2}{2c_0|z-z_0|}\right)}{2i\pi c_0\frac{|z-z_0|}{\omega}}.$$

- In a random medium:

$$\mathbb{E}\big[\hat{\phi}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\big] = \hat{\phi}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\exp\Big(-\frac{\gamma(\mathbf{0})\omega^2|z-z_0|}{8c_0^2}\Big),$$

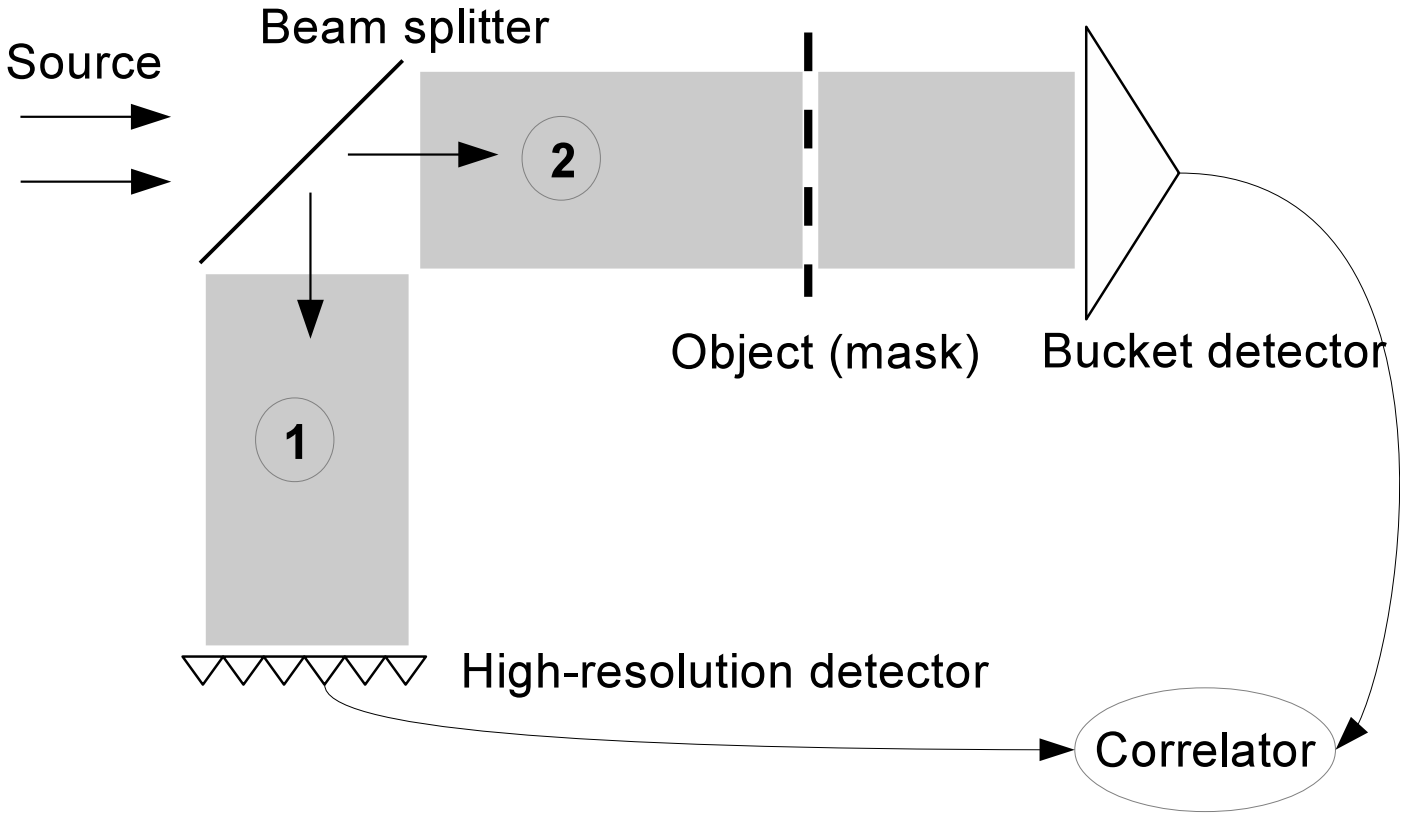
where $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty}\mathbb{E}[\mu(\mathbf{0},0)\mu(\boldsymbol{x},z)]dz \Longrightarrow$ Strong damping of the coherent wave, with the scattering mean free path $l_{\rm sca} = 8c_0^2/(\gamma(\mathbf{0})\omega^2)$.

$$\begin{aligned}&\mathbb{E}\big[\hat{\phi}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{\phi}\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\big]\\ &= \hat{\phi}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{\phi}_0\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\exp\Big(-\frac{\gamma_2(\boldsymbol{x}-\boldsymbol{x}')\omega^2|z-z_0|}{4c_0^2}\Big),\end{aligned}$$

where $\gamma_2(\boldsymbol{x}) = \int_0^1\gamma(\mathbf{0})-\gamma(\boldsymbol{x}s)ds$.

[1] J. Garnier and K. Sølna, Ann. Appl. Probab. **19** 318 (2009).

Ghost imaging in the random paraxial regime



The medium in paths 1 and 2 is heterogeneous (typically, turbulent atmosphere). They are two independent realizations of the same distribution.

- If the correlation function of the noise sources is

$$\big\langle n(t,\boldsymbol{x})n(t',\boldsymbol{x}')\big\rangle = F(t-t')I_0 \exp\Big(-\frac{|\boldsymbol{x}|^2}{r_0^2}\Big)\delta(\boldsymbol{x}-\boldsymbol{x}'),$$

if we denote the integrated correlation function of the medium fluctuations by

$$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0},0)\mu(\boldsymbol{x},z)]dz$$

- then

$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \mathcal{H}(\boldsymbol{x}-\boldsymbol{y})|\mathcal{T}(\boldsymbol{y})|^2 d\boldsymbol{y},$$

with

$$\mathcal{H}(\boldsymbol{x}) = \frac{I_0^2 r_0^4}{2^8\pi^3 L^4}\int_{\mathbb{R}^2} d\boldsymbol{\beta} \exp\Big(-\frac{|\boldsymbol{\beta}|^2}{2} - \frac{\gamma_2(r_0\boldsymbol{\beta})\omega_0^2 L}{2c_0^2} + i\frac{\omega_0 r_0 \boldsymbol{x}\cdot\boldsymbol{\beta}}{c_0 L}\Big).$$

and $\gamma_2(\boldsymbol{x}) = \int_0^1 \gamma(\mathbf{0}) - \gamma(\boldsymbol{x}s)ds$.

• If the medium is strongly scattering, in the sense that the propagation distance is larger than the scattering mean free path $L/l_{\rm sca} \gg 1$, with

$$l_{\rm sca} = \frac{8c_0^2}{\gamma(\mathbf{0})\omega_0^2},$$

• then

$$\mathcal{H}(\boldsymbol{x}) = \frac{I_0^2 r_0^4 \rho_{\rm gi0}^2}{2^7\pi^2 L^4 \rho_{\rm gi2}^2} \exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi2}^2}\Big),$$

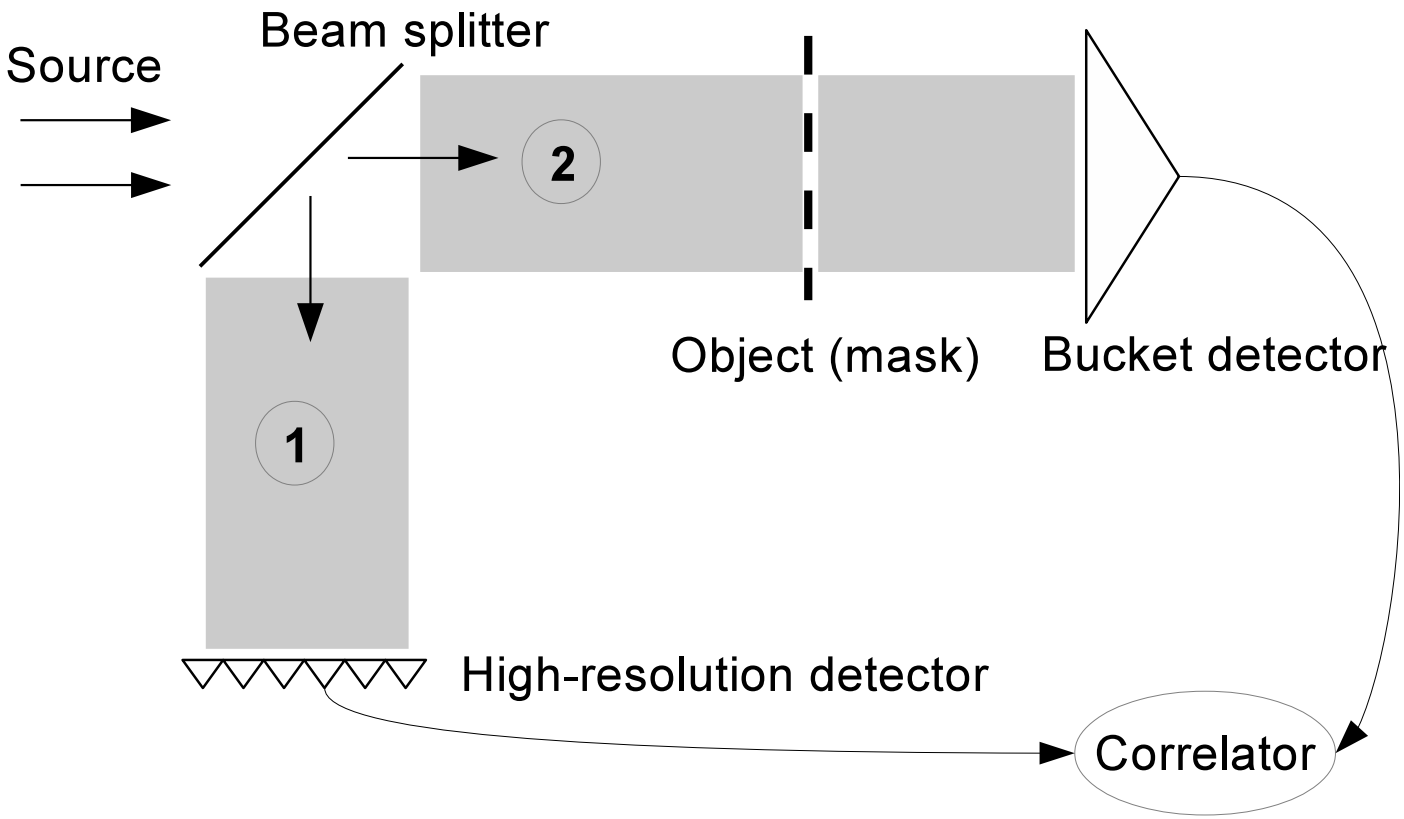
with

$$\rho_{\rm gi2}^2 = \frac{c_0^2 L^2}{2\omega_0^2 r_0^2} + \frac{4c_0^2 L^3}{3\omega_0^2 l_{\rm sca} l_{\rm cor}^2} = \rho_{\rm gi0}^2 + \frac{4c_0^2 L^3}{3\omega_0^2 l_{\rm sca} l_{\rm cor}^2},$$

and the correlation radius of the medium $l_{\rm cor}$ is defined by:
$\gamma(\boldsymbol{x}) = \gamma(\mathbf{0})[1 - |\boldsymbol{x}|^2/l_{\rm cor}^2 + o(|\boldsymbol{x}|^2/l_{\rm cor}^2)]$.

$\hookrightarrow$ Scattering only slightly reduces the resolution !
This imaging method is robust with respect to medium noise.

Ghost imaging in the random paraxial regime



The medium in paths 1 and 2 is heterogeneous (typically, turbulent atmosphere). Imagine that they are the *same realization* (just for the beauty of the analysis).

• If the correlation function of the noise sources is

$$\big\langle n(t,\boldsymbol{x})n(t',\boldsymbol{x}')\big\rangle = F(t-t')I_0 \exp\Big(-\frac{|\boldsymbol{x}|^2}{r_0^2}\Big)\delta(\boldsymbol{x}-\boldsymbol{x}'),$$

if we denote the integrated correlation function of the medium fluctuations by

$$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0},0)\mu(\boldsymbol{x},z)]dz,$$

if $L/l_{\rm sca} \gg 1$, with

$$l_{\rm sca} = \frac{8c_0^2}{\gamma(\mathbf{0})\omega_0^2},$$

if $\gamma(\boldsymbol{x}) = \gamma(\mathbf{0})[1 - |\boldsymbol{x}|^2/l_{\rm cor}^2 + o(|\boldsymbol{x}|^2/l_{\rm cor}^2)]$,

• then

$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \mathcal{H}(\boldsymbol{x}-\boldsymbol{y})|\mathcal{T}(\boldsymbol{y})|^2 d\boldsymbol{y},$$

with

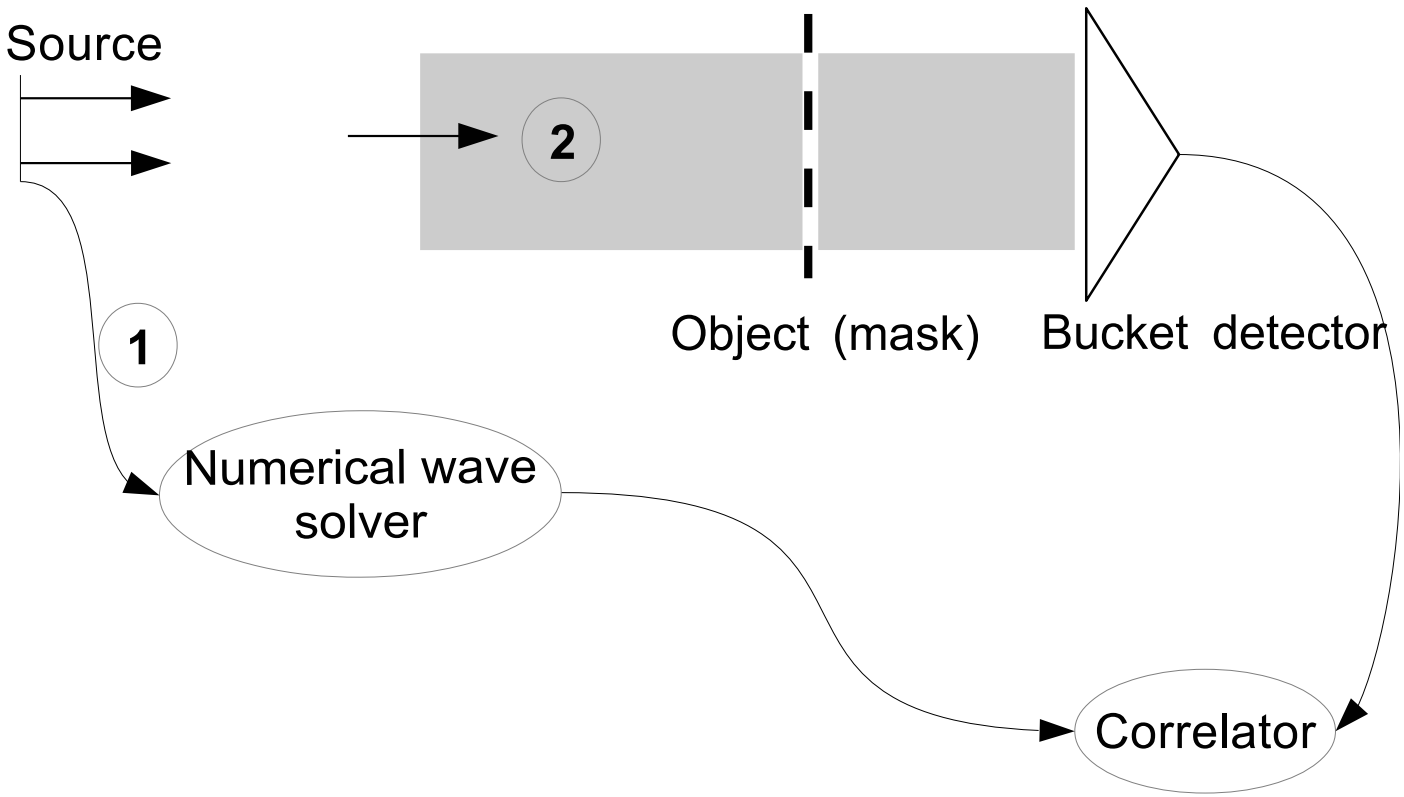
$$\mathcal{H}(\boldsymbol{x}) = \frac{I_0^2 r_0^4}{2^7\pi^2 L^4} \exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi3}^2}\Big),$$

with the radius

$$\frac{1}{\rho_{\rm gi3}^2} = \frac{1}{\rho_{\rm gi0}^2} + \frac{16L}{l_{\rm sca} l_{\rm cor}^2},$$

$\hookrightarrow$ the radius of the convolution kernel is reduced by scattering and can even be smaller than the Rayleigh resolution formula: enhanced resolution compared to the homogeneous case (similar phenomenon observed in time-reversal experiments) !

Virtual ghost imaging



- The medium in path 2 is randomly heterogeneous.
- There is no other measurement than the integrated transmitted intensity $I_2(t)$.
- The realization of the source is known (use of a Spatial Light Modulator) and the medium is taken to be homogeneous in the "virtual path 1" $\rightarrow$ one can *compute* the field (and therefore its intensity $I_1(t, \boldsymbol{x})$) in the "virtual" output plane of path 1.

- If the correlation function of the noise sources is

$$\big\langle n(t,\boldsymbol{x})n(t',\boldsymbol{x}')\big\rangle = F(t-t')I_0 \exp\Big(-\frac{|\boldsymbol{x}|^2}{r_0^2}\Big)\delta(\boldsymbol{x}-\boldsymbol{x}'),$$

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if $L/l_{\rm sca} \gg 1$, with

$$l_{\rm sca} = \frac{8c_0^2}{\gamma(\mathbf{0})\omega_0^2},$$

if $\gamma(\boldsymbol{x}) = \gamma(\mathbf{0})[1 - |\boldsymbol{x}|^2/l_{\rm cor}^2 + o(|\boldsymbol{x}|^2/l_{\rm cor}^2)]$,

- then

$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \mathcal{H}(\boldsymbol{x}-\boldsymbol{y})|\mathcal{T}(\boldsymbol{y})|^2 d\boldsymbol{y},$$

with

$$\mathcal{H}(\boldsymbol{x}) = \frac{I_0^2 r_0^4 \rho_{\rm gi0}^2}{2^7\pi^2 L^4 \rho_{\rm gi4}^2} \exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi4}^2}\Big),$$

with the radius

$$\rho_{\rm gi4}^2 = \frac{c_0^2 L^2}{2\omega_0^2 r_0^2} + \frac{2c_0^2 L^3}{3\omega_0^2 l_{\rm sca} l_{\rm cor}^2} = \rho_{\rm gi0}^2 + \frac{2c_0^2 L^3}{3\omega_0^2 l_{\rm sca} l_{\rm cor}^2}.$$

$\hookrightarrow$ a *one-pixel camera* can give a high-resolution image of the object in scattering media!

Conclusion

- Imaging with noise sources and/or through scattering media is possible using correlation-based techniques.
- First application: seismic interferometry, virtual source imaging.
- Second application: ghost imaging.
- Many other applications !