**Exercise 1 A property of harmonic functions**

Let $\Omega$ an open set in $\mathbb{R}^n$, and a function $u : \Omega \rightarrow \mathbb{R}$. $u$ is an harmonic function iff $u \in C^2(\Omega)$ and $\Delta u = 0$ on $\Omega$.

1.1 Let $u$ a quadratic form: $u(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i<j} b_{ij} x_i x_j$. Which is the condition for $u$ to be harmonic?

1.2 Prove that, if $u$ is harmonic and if $u \in C^3(\Omega)$, then $\frac{\partial u}{\partial x_i}$ is also harmonic ($i = 1, \ldots, n$).

1.3 Prove that, if $u$ is harmonic and if $u \in C^{n+2}(\Omega)$, then its partial derivative up to order $n$ are also harmonic functions.

**Exercise 2 Laplace equation on a rectangle**

Let consider the problem:

$$\begin{cases}
\Delta u = 0 & \text{in } (0, L_x) \times (0, L_y) \\
u(0, y) = h(y), u(L_x, y) = u(x, 0) = u(x, L_y) = 0
\end{cases}$$

Using separation of variables, prove that $u(x, y) = \sum_{k \geq 1} \alpha_k \left( e^{\lambda_k x} - e^{\lambda_k (2L_x-x)} \right) \sin(\lambda_k y)$ where $\lambda_k = \frac{k\pi}{L_y}$ and $\alpha_k = \frac{2}{1 - e^{2\lambda_k L_x}} \int_0^{L_y} h(y) \sin(\lambda_k y) \, dy$.

**Exercise 3 Laplace equation on a disk**

Let $\Omega$ the open disk of center $(0, 0)$ and radius $R$. Let consider the problem:

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
u = g(\theta) & \text{on } \partial \Omega
\end{cases}$$

The Laplacian operator in polar coordinates reads $\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$.

3.1 Using a separation of variables, prove that any function $u(r, \theta) = \sum_{n \geq 0} (\alpha_n \cos n\theta + \beta_n \sin n\theta) \, r^n$ satisfies $\Delta u = 0$.

3.2 Assuming that $g \in C^1$, its Fourier series reads $g(\theta) = a_0 + \sum_{n \geq 1} (a_n \cos n\theta + b_n \sin n\theta)$.
with \( a_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) d\theta, a_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos n\theta \, d\theta, b_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin n\theta \, d\theta. \)

Prove that the solution of (1) is \( u(r, \theta) = a_0 + \sum_{n \geq 1} \left( a_n \cos n\theta + b_n \sin n\theta \right) \left( \frac{r}{R} \right)^n \)

3.3 Prove that the preceding expression can be transformed into

\[
u(r, \theta) = \frac{1}{\pi} \left[ \frac{1}{2} \int_0^{2\pi} g(\alpha) \, d\alpha + \sum_{n \geq 1} \frac{r^n}{R^n} \int_0^{2\pi} g(\alpha) \cos n(\theta - \alpha) \, d\alpha \right]
\]

3.4 Prove that the preceding expression can be transformed into

\[
u(r, \theta) = K(r, \theta) * g(\theta) = \frac{1}{2\pi} \int_0^{2\pi} K(r, \theta - \alpha) g(\alpha) \, d\alpha \quad \text{where} \quad K(r, \theta) = \frac{R^2 - r^2}{R^2 + r^2 - 2rR \cos \theta}
\]

**Exercise 4 1D and 2D Laplacian matrices**

4.1 Let first consider the ODE: \(-u''(x) = f(x), x \in (0, L), \) with \( u(0) = u(L) = 0. \) What are the eigenvalues and eigenvectors of the second order derivative operator on \( \{ u \in C^2(0, L), u(0) = u(L) = 0 \} \)?

4.2 Let now a standard second order finite difference discretization of this problem, with mesh step \( h = L/(N + 1). \) It reads

\[
\frac{1}{h^2} A_N \mathbf{U} = \mathbf{F} \quad \text{with} \quad A_N = \begin{bmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \ddots & \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & -1 \\
0 & \cdots & 0 & -1 & 2
\end{bmatrix}
\]

By analogy with the continuous case (or by a direct calculation), find the eigenvectors and eigenvalues of \( A_N \) (and make sure that it is consistent with the continuous case).

What is the condition number of \( A_N \)?

4.3 Let consider now the 2D Poisson problem \(-\Delta u = f\) on \( \Omega = (0, L_x) \times (0, L_y), \) with zero Dirichlet boundary condition. We consider a regular discretization grid, with mesh steps \( h_x = L_x/(N_x + 1) \) and \( h_y = L_y/(N_y + 1) \) in the \( x \) and \( y \) directions respectively (\( N_x \) and \( N_y \) are integers).

4.3.1 Write the linear system \( A_{2D} \mathbf{U} = \mathbf{F} \) corresponding to the usual 5-point second order finite difference scheme for the Laplacian (with the unknowns ordered as \( U_{i+(j-1)N_x} = u_{ij} \)).

The tensor product (also called Kronecker product) of 2 matrices \( \mathbf{A} \) and \( \mathbf{B} \) being defined, with obvious notations, as the block matrix

\[
\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix}
a_{11} \mathbf{B} & a_{12} \mathbf{B} & \cdots & a_{1p} \mathbf{B} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} \mathbf{B} & a_{n2} \mathbf{B} & \cdots & a_{np} \mathbf{B}
\end{bmatrix},
\]
prove that
\[ \mathbf{A}_{2D} = \frac{1}{h_x^2} \mathbf{I}_{N_y} \otimes \mathbf{A}_{N_x} + \frac{1}{h_y^2} \mathbf{A}_{N_y} \otimes \mathbf{I}_{N_x} \]
where \( \mathbf{I}_n \) is the identity matrix of size \( n \).

4.3.2 Prove that the eigenvalues and eigenvectors of \( \mathbf{A}_{2D} \) are
\[ \lambda_{k,l} = \frac{4}{h^2_x} \sin^2 \left( \frac{k\pi}{2(N_x + 1)} \right) + \frac{4}{h^2_y} \sin^2 \left( \frac{l\pi}{2(N_y + 1)} \right) \quad (1 \leq k \leq N_x, 1 \leq l \leq N_y) \]
and
\[ X_{k,l}(i,j) = \sin \frac{k\pi i}{N_x + 1} \sin \frac{l\pi j}{N_y + 1} \quad (1 \leq k \leq N_x, 1 \leq l \leq N_y) \]

4.3.3 What is the condition number of \( \mathbf{A}_{2D} \)?

4.3.4 Let consider the alternative so called “red-black” order of unknowns, as illustrated below for a \( 4 \times 4 \) grid:

![Diagram of red-black ordering](image)

What about the corresponding new form for \( \mathbf{A}_{2D} \)?

**Exercise 5 9-point 2D Laplacian - Introduction of an artificial error**

Let consider the Poisson problem \( \Delta u = f \) in \( \Omega \subset \mathbb{R}^2 \). On a regular 2D grid, let consider the usual 5-point scheme for the Laplacian
\[ \Delta_5 u_{ij} = \frac{1}{h^2} \left[ u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{ij} \right] \]
and the alternative 9-point scheme
\[ \Delta_9 u_{ij} = \frac{1}{6h^2} \left[ 4u_{i-1,j} + 4u_{i+1,j} + 4u_{i,j-1} + 4u_{i,j+1} + u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1} - 20u_{ij} \right] \]
5.1 What is the dominant error term for each scheme? Is the 9-point scheme more accurate than the 5-point one?

5.2 Using the expression of $\Delta(\Delta u)$, prove that the 9-point scheme is actually fourth order accurate if $f$ is an harmonic function (i.e. satisfies $\Delta f = 0$).

5.3 More generally, prove that solving $\Delta_9 u_{ij} = f_{ij}$ and defining

$$f_{ij} = f(x_i, y_j) + \frac{h^2}{12} \Delta_5 f(x_i, y_j)$$

(if $f$ is sufficiently smooth) instead of $f_{ij} = f(x_i, y_j)$ leads to a fourth order accurate method. This method corresponds to deliberately introducing a $O(h^2)$ error intro the right-hand side of the equation that is chosen to cancel the $O(h^2)$ part of the local truncation error.

5.4 If $f$ is known only at the grid points (but is known to be sufficiently smooth), prove that we can achieve the same fourth order accuracy by using

$$f_{ij} = f(x_i, y_j) + \frac{h^2}{12} \Delta_5 f(x_i, y_j)$$