1. The cost of heating a room at temperature $T_i$ is proportional to the yearly outgoing flux of heat across the walls enclosing the room. We assume that the outside temperature $T(t)$ increases from $T_c$ at $t = 0$ to $T_h$ at $t = 0.5$, and then decreases back to $T_c$ for $t = 1$, the time unit being the year. Thus $t = 0.5$ corresponds to 6 months. The temperature is supposed to be a linear function of time from $t = 0$ to $t = 0.5$ and from $t = 0.5$ to $t = 1$. Let $\Delta T(t)$ be the temperature difference between $T_i$ and $T(t)$ during the year. The heating is operating only when $\Delta T(t) > 0$. There is no cooling device.

Let $k$ and $d$ denote the wall conductivity and thickness, respectively. The cost of the material per unit wall thickness is inversely proportional to $k$, while the cost of the wall is proportional to the material cost and the wall thickness.

(i) (10 pts) We assume $T_c < T_i < T_h$. Can you justify this assumption?
(ii) (20 pts) Express the yearly outgoing heat flux in terms of $T_i$, $T_c$, $T_h$.
(iii) (10 pts) Show that the yearly heating cost is proportional to

$$\frac{k (T_i - T_c)^2}{d (T_h - T_c)}$$

(iv) (20 pts) Find a combination of conductivity and wall thickness that minimizes the total cost (one year of heating and initial construction).
(v) (10 pts) Could you propose a more realistic, yet explicit, function $T(t)$?

2. (30 pts) For $\alpha > 0$ we consider the following differential system

$$\begin{cases} \frac{dx}{dt} = x - \frac{1}{3}x^3 - y + \frac{1}{3}\alpha^3 \\ \frac{dy}{dt} = x - y \end{cases}$$

Find and classify the equilibrium point of this system.