Equations différentielles de Bessel et de Bessel modifiée

$$y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = 0 \qquad \textbf{Solutions :} \quad y(x) = a_0 J_\nu(x) + a_1 N_\nu(x), \qquad (a_0, a_1) \in \mathbb{R}^2$$

$$y'' + \frac{1}{x}y' - (1 + \frac{\nu^2}{x^2})y = 0 \qquad \textbf{Solutions :} \quad y(x) = a_0 I_\nu(x) + a_1 K_\nu(x), \qquad (a_0, a_1) \in \mathbb{R}^2$$

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{\Gamma(n+\nu+1)n!} \qquad I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{\Gamma(n+\nu+1)n!}$$

$$N_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)} \qquad K_\nu(x) = \frac{\pi}{2\sin(\nu\pi)}(I_{-\nu}(x) - I_\nu(x)) \quad \text{pour } \nu \notin \mathbb{Z}$$

$$N_n(x) = \lim_{\nu \to n} N_\nu(x) \qquad K_n(x) = \lim_{\nu \to n} K_\nu(x) \text{ pour } n \in \mathbb{Z}$$

Propriétés

$$J_{-n} = (-1)^n J_n \quad N_{-n} = (-1)^n N_n \quad I_{-n} = I_n \text{ pour } n \in \mathbb{N} \qquad K_{-\nu} = K_\nu \text{ pour } \nu \in \mathbb{R}.$$

Comportement en $x = 0$

$$J_0(0) = I_0(0) = 1 \qquad J_\nu(0) = I_\nu(0) = 0 \text{ pour } \nu > 0, \qquad \lim_{x\to 0} N_\nu(x) = \lim_{x \to 0} K_\nu(x) = \infty \text{ pour } \nu \geq 0$$

Expressions explicites

$$J_{\frac{1}{2}}(x) = N_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad J_{-\frac{1}{2}}(x) = -N_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x \quad I_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\operatorname{sh} x \quad I_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\operatorname{ch} x$$

Dérivation

$$\frac{d}{dx}[x^\nu Z_\nu(mx)] = \begin{cases} mx^\nu Z_{\nu-1}(mx) \text{ si } Z = J, N, I \\ -mx^\nu Z_{\nu-1}(mx) \text{ si } Z = K \end{cases} \qquad \frac{d}{dx}[x^{-\nu} Z_\nu(mx)] = \begin{cases} -mx^{-\nu} Z_{\nu+1}(mx) \text{ si } Z = J, N, K \\ mx^{-\nu} Z_{\nu+1}(mx) \text{ si } Z = I \end{cases}$$

$$\frac{d}{dx}[Z_\nu(mx)] = \begin{cases} mZ_{\nu-1}(mx) - \frac{\nu}{x}Z_\nu(mx) \text{ si } Z = J, N, I \\ -mZ_{\nu-1}(mx) - \frac{\nu}{x}Z_\nu(mx) \text{ si } Z = K \end{cases} \qquad \frac{d}{dx}[Z_\nu(mx)] = \begin{cases} -mZ_{\nu+1}(mx) + \frac{\nu}{x}Z_\nu(mx) \text{ si } Z = J, N, K \\ mZ_{\nu+1}(mx) + \frac{\nu}{x}Z_\nu(mx) \text{ si } Z = I \end{cases}$$

NB : Pour $\nu = 0$, $J_0' = -J_1$, $I_0' = I_1$, $N_0' = -N_1$, $K_0' = -K_1$.

Equations différentielles se ramenant au cas précédent

$$y'' + \frac{1}{x}y' + (m^2 - \frac{\nu^2}{x^2})y = 0 \qquad \textbf{Solutions :} \quad y(x) = a_0 J_\nu(mx) + a_1 N_\nu(mx), \qquad (a_0, a_1) \in \mathbb{R}^2$$

$$y'' + \frac{1}{x}y' - (m^2 + \frac{\nu^2}{x^2})y = 0 \qquad \textbf{Solutions :} \quad y(x) = a_0 I_\nu(mx) + a_1 K_\nu(mx), \qquad (a_0, a_1) \in \mathbb{R}^2$$

$$y'' + \frac{a}{x}y' + b^2 y = 0 \qquad \textbf{Solutions :} \quad y(x) = x^\nu \left[a_0 J_\nu(bx) + a_1 N_\nu(bx)\right], \quad \nu = \frac{1-a}{2}, \qquad (a_0, a_1) \in \mathbb{R}^2$$

$$y'' + \frac{a}{x}y' - b^2 y = 0 \qquad \textbf{Solutions :} \quad y(x) = x^\nu \left[a_0 I_\nu(bx) + a_1 K_\nu(bx)\right], \quad \nu = \frac{1-a}{2}, \qquad (a_0, a_1) \in \mathbb{R}^2$$

$$y'' + \frac{\alpha}{x}y' + \gamma^2 x^\beta y = 0 \quad (\beta \neq -2) \qquad \textbf{Solutions :} \quad \begin{cases} y(x) = x^{\frac{\nu}{\mu}}\left[a_0 J_\nu(\gamma\mu x^{\frac{1}{\mu}}) + a_1 N_\nu(\gamma\mu x^{\frac{1}{\mu}})\right], \\ \mu = \frac{2}{\beta+2}, \quad \nu = \frac{1-\alpha}{\beta+2}, \quad (a_0, a_1) \in \mathbb{R}^2 \end{cases}$$

$$y'' + \frac{\alpha}{x}y' - \gamma^2 x^\beta y = 0 \quad (\beta \neq -2) \qquad \textbf{Solutions :} \quad \begin{cases} y(x) = x^{\frac{\nu}{\mu}}\left[a_0 I_\nu(\gamma\mu x^{\frac{1}{\mu}}) + a_1 K_\nu(\gamma\mu x^{\frac{1}{\mu}})\right], \\ \mu = \frac{2}{\beta+2}, \quad \nu = \frac{1-\alpha}{\beta+2}, \quad (a_0, a_1) \in \mathbb{R}^2 \end{cases}$$

$$y'' + \frac{\alpha}{x}y' \pm \frac{\gamma^2}{x^2}y = 0 \qquad \textbf{Solutions :} \quad \begin{cases} y(x) = a_0 x^{r_1} + a_1 x^{r_2}, \text{ si } r^2 + (\alpha-1)r \pm \gamma^2 = 0 \\ \qquad \text{admet deux racines réelles } r_1, r_2 \\ y(x) = x^a\left[a_0\cos(b\ln x) + a_1\sin(b\ln x)\right], \text{ si } r^2 + (\alpha-1)r \pm \gamma^2 = 0 \\ \qquad \text{admet deux racines complexes conjuguées } a \pm ib \\ y(x) = x^{\frac{1-\alpha}{2}}\left[a_0 + a_1 \ln x\right], \text{ dans le cas d'une racine double.} \end{cases}$$