Math 4428 Final Exam

May 10, 2007

2 hours closed book. Calculators allowed.

1. A bookseller wants to find how much he should spend on advertising for the new book he intends to sell, and how many copies he sould order. As a former student of M4428, he knows that he should order $u^*(k)$ copies defined by

$$\int_0^{u^*(k)} f_k(z)dz = \frac{p - c_0}{p + c_1}$$

where f_k is the density function of the random demand Z for an adversting level k, p is the price at which he sells a book, c_0 is the price at which he buys a book, and c_1 is the storage cost per book. He assumes that the demand is equally likely to fall between 0 and $b(k)$, with

$$b(k) = \begin{cases} 0 & \text{if } k \leq k_0 \\ \frac{1}{\alpha}(k - k_0)^{\frac{2}{3}} & \text{if } k > k_0 \end{cases}$$

where k is the amount of money spent on advertising (unknown), k_0 given fixed setup costs, and α the cost to send one brochure of advertising.

   (a) Express $u^*(k)$ in terms of p, c_0, c_1, $b(k)$.

   (b) Find the corresponding expected reward

$$J(u^*(k)) = (p + c_1) \int_0^{u^*(k)} z f_k(z)dz - k$$

   in terms of p, c_0, c_1, $b(k)$ and k.

   (c) Find a value of k which maximizes $J(u^*(k))$.

   (d) Application: The bookseller buys a book for \$10 per copy and sells them \$15 per book. Holding costs are \$1 per book. Concerning advertising, each sent brochure costs 10 cents, while fixed advertising costs are \$50. How much should he spend on advertising ? How many copies should he order ? What is its expected reward ?

2. We consider two cashiers in a supermarket and want to know if it is more efficient to have one unique waiting queue for these two, or separate queues. Assume customers arrive to checkout at an overall rate of λ per hour, and that each cashier can process μ customers per hour. We remind that we saw in class that for a single server and queue, with arrival rate λ_1 and service rate μ_1, the average waiting time was $W_1 = \frac{1}{\mu_1 - \lambda_1}$ while for a single queue with arrival rate λ_2 leading to two servers each with service time μ_2, it was $W_2 = \frac{4\mu_2}{4\mu_2^2 - \lambda_2^2}$. Concerning the waiting time, justify briefly if it is better to have one unique queue or two separate ones (assuming that the flow of customers splits equally on each queue if there are two). Application: $\lambda = 30$, $\mu = 20$, compute the average waiting time in each case.

3. Collecting weather data, it has been found that in Minneapolis, the day after a sunny day is 75% likely to be a sunny day, while the day after a rainy day is 50% likely to be a rainy day (we assume that a day is eiher sunny or rainy). Today is sunny.
We want to model this problem by a discrete elevator problem (i.e. Markov chain). Let n denote the number of the day, $n = 0$ corresponding to today. Let $X(n)$ be the state of the system at day n, which is 1 for sunny, 0 for rainy. At last, let $\pi_j(n) = P(X(n) = j)$, $j = 0, 1$ which defines a two vector $\pi(n)$.

   (a) Let U the event that $X(n+1) = j$, and U_i the event that $X(n) = i$, $i = 0, 1$. Express $P(U)$ in terms of the $P(U_i)$ and $r_{ij}^n = P(X(n+1) = j|X(n) = i)$.

   (b) Explain why $r_{ij}^n = r_{ij}^0$ and give these four coefficients.

   (c) Find A such that $\pi(n+1) = A\pi(n)$.

   (d) Deduce $\pi(n)$ in terms of A, n and $\pi(0)$.

   (e) What is the probability that it will rain in four days ?

   (f) Let $P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ with $P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$. Compute $D = P^{-1}AP$.

   (g) Prove that $A^n = PD^nP^{-1}$, and use it to compute A^n in terms of n.

   (h) At stationnary state (i.e. for large time) what is the probability of the occurence of a sunny day ?