Solution for Math 4428 Final Exam  
May 10, 2007  
2 hours closed book.

1. (a) The density of the demand is given by \( f_k(z) = \frac{1}{\beta(k)} H(b(k) - z) \). As the number of books to order has to be less than the maximum possible demand, if we plug \( f_k(z) \) into the expression giving \( u^*(k) \) we get

\[
\frac{u^*(k)}{b(k)} = \frac{p - c_0}{p + c_1}
\]

thus

\[
u^*(k) = \frac{p - c_0}{p + c_1} b(k)
\]

(b) Then \( J(u^*(k)) = \frac{p + c_1}{\beta(k)} \frac{1}{2}(u^*(k))^2 - k = \frac{(p - c_0)^2}{2(p + c_1)} b(k) - k \).

Let us define \( \beta = \frac{(p - c_0)^2}{2(p + c_1)} \).

(c) We first look for critical points of \( J(u^*(k)) \). These correspond to \( k \) such that

\[
0 = \frac{d}{dk} J(u^*(k)) = \beta b'(k) - 1 = \beta \frac{2}{3} \alpha (k - k_0)^{-\frac{3}{2}} - 1
\]

Thus

\[
k^* = k_0 + \left( \frac{2\beta}{3\alpha} \right)^{\frac{3}{2}}
\]

(d) Application: \( c_0 = 10, c_1 = 1, p = 15, \alpha = .1 \). Thus \( \beta = \frac{25}{32} \). Then

\[
k^* = 50 + \left( \frac{50}{3(0.32)} \right)^{\frac{3}{2}} \approx 191.3
\]

Thus \( b(k^*) \approx 85 \), and \( J(u^*(k^*)) \approx \frac{25}{32} b(k^*) - 191.3 \approx 20.6 \). Thus the optimal strategy for the bookseller is to order 85 books and spend $191.3 on advertising. He will earn $161.9 doing that.

2. For the two separate queues and cashiers, each of these have a rate of arrival \( \lambda_1 = \lambda/2 \) and a rate of service \( \mu_1 = \mu \). Thus \( W_1 = \frac{\lambda}{\mu - \lambda/2} = \frac{2}{2\mu - \lambda} \). For the single queue with two cashiers, \( \lambda_2 = \lambda \) and \( \mu_2 = \mu \) thus \( W_2 = \frac{4\mu}{(2\mu - \lambda)(2\mu + \lambda)} \). Then \( W_1/W_2 = \frac{\lambda + \lambda}{2\mu} = 1 + \frac{\lambda}{2\mu} > 1 \). The single queue solution is better. Application: \( W_1 = \frac{1}{3} \text{hr} = 12 \text{ min} \), whereas \( W_2 = 6.9 \text{ min} \).

3. Done in class.