Solution for Math 4428 Final Exam

May 10, 2007

2 hours closed book.

1. (a) The density of the demand is given by $f_k(z) = \frac{1}{b(k)}H(b(k) - z)$. As the number of books to order has to be less than the maximum possible demand, if we plug $f_k(z)$ into the expression giving $u^*(k)$ we get

$$\frac{u^*(k)}{b(k)} = \frac{p - c_0}{p + c_1}$$

thus

$$u^*(k) = \frac{p - c_0}{p + c_1} b(k)$$

(b) Then $J(u^*(k)) = \frac{p+c_1}{b(k)} \frac{1}{2}(u^*(k))^2 - k = \frac{(p-c_0)^2}{2(p+c_1)} b(k) - k$.
Let us define $\beta = \frac{(p-c_0)^2}{2(p+c_1)}$.

(c) We first look for critical points of $J(u^*(k))$. These correspond to k such that

$$0 = \frac{d}{dk} J(u^*(k)) = \beta b'(k) - 1 = \beta \frac{2}{3} \frac{1}{\alpha} (k - k_0)^{-\frac{1}{3}} - 1$$

Thus

$$k^* = k_0 + \left(\frac{2\beta}{3\alpha} \right)^3$$

(d) Application: $c_0 = 10$, $c_1 = 1$, $p = 15$, $\alpha = .1$. Thus $\beta = \frac{25}{32}$. Then $k^* = 50 + \left(\frac{50}{3(.1)32} \right)^3 \approx \$191.3$. Then $b(k^*) = 10 * (k^* - 50)^{\frac{2}{3}} \approx 271.3$, and $u^*(k^*) = \frac{5}{16} b(k^*) \approx 85$, and $J(u^*(k^*)) = \frac{25}{32} b(k^*) - 191.3 \approx \$20.6$. Thus the optimal strategy for the bookseller is to order 85 books and spend \$191.3 on advertising. He will earn \$161.9 doing that.

2. For the two separate queues and cashiers, each of these have a rate of arrival $\lambda_1 = \lambda/2$ and a rate of service $\mu_1 = \mu$. Thus $W_1 = \frac{1}{\mu - \lambda/2} = \frac{2}{2\mu - \lambda}$. For the single queue with two cashiers, $\lambda_2 = \lambda$ and $\mu_2 = \mu$ thus $W_2 = \frac{4\mu}{4\mu^2 - \lambda^2} = \frac{4\mu}{(2\mu-\lambda)(2\mu+\lambda)}$. Then $W_1/W_2 = \frac{2\mu+\lambda}{2\mu} = 1 + \frac{\lambda}{2\mu} > 1$. The single queue solution is better. Application: $W_1 = \frac{1}{5}$hr $= 12$ min, whereas $W_2 = 6.9$ min.

3. Done in class.