Math 4428 Midterm Exam

March 5, 2007

50 minutes closed book.

One double sided sheet of letter size paper allowed as document.

1. The cost of heating a room at temperature T_i is proportional to the yearly outgoing flux of heat across the walls enclosing the room. We assume that the outside temperature $T(t)$ increases from T_c at $t = 0$ to T_h at $t = 0.5$, and then decreases back to T_c for $t = 1$, the time unit being the year. Thus $t = 0.5$ corresponds to 6 months. The temperature is supposed to be a linear function of time from $t = 0$ to $t = 0.5$ and from $t = 0.5$ to $t = 1$. Let $\Delta T(t)$ be the temperature difference between T_i and $T(t)$ during the year. The heating is operating only when $\Delta T(t) > 0$. There is no cooling device.

   Let k and d denote the wall conductivity and thickness, respectively. The cost of the material per unit wall thickness is inversely proportional to k, while the cost of the wall is proportional to the material cost and the wall thickness.

   (i) (10 pts) We assume $T_c < T_i < T_h$. Can you justify this assumption ?
   It is realistic to heat a room to a temperature that is more than the lower outside temperature of the year, and less than the higher one. For example the outside temperature ranges between -25 and 90, and a typical heating temperature is around 68.

   (ii) (20 pts) Express the yearly outgoing heat flux in terms of T_i, T_c, T_h. *As $T(t)$ is a linear function of t from $t = 0$ to $t = 0.5$, and $T(0) = T_c$ and $T(0.5) = T_h$, we have*

   $$T(t) = T_c + \frac{T_h - T_c}{0.5} t$$

   on that time interval. Likewise, between $t = 0.5$ and $t = 1$ we have

   $$T(t) = T_h + \frac{T_c - T_h}{0.5}(t - 0.5).$$

   Now the heat flux is proportional to the difference between T_i and $T(t)$, if $T_i > T(t)$. As $T(t)$ is increasing from T_c to T_h and decreasing back to T_c, and $Tc < T_i < T_h$, there is two times between which $\Delta T(t) < 0$ and no heating will occur. Let t_1 be the smallest of these times, by the symetry of $T(t)$, the other time is $t_2 = 1 - t_1$. We compute t_1 such that

   $$T_i = T_c + \frac{T_h - T_c}{0.5} t$$

   which gives $t_1 = \frac{1}{2}\frac{T_i - T_c}{T_h - T_c}$. The total outgoing flux of heat during one year is thus

   $$\Phi = \int_0^{t_1} k \frac{\Delta T(t)}{d} dt + \int_{t_2}^{1} k \frac{\Delta T(t)}{d} dt$$

   which by symetry of $\Delta T(t)$ is

   $$\Phi = 2 \int_0^{t_1} k \frac{\Delta T(t)}{d} dt$$

We could use some simple quadrature formula to compute that integral, or compute it by

$$\Phi = 2\frac{k}{\delta}\int_0^{t_1} [T_i - T_c - 2(T_h - T_c)t]\, dt = 2\frac{k}{\delta}\left[(T_i - T_c)t_1 - (T_h - T_c)t_1^2\right]$$
$$= 2\frac{k}{\delta}\left[\frac{(T_i - T_c)^2}{2(T_h - T_c)} - \frac{(T_h - T_c)(T_i - T_c)^2}{4(T_h - T_c)^2}\right] = \frac{k}{2\delta}\frac{(T_i - T_c)^2}{(T_h - T_c)}.$$

(iii) (10 pts) Show that the yearly heating cost is proportional to

$$\frac{k}{d}\frac{(T_i - T_c)^2}{T_h - T_c}$$

The yearly cost is said to be proportional to the yearly outgoing flux, and we found that he was proportional to that quantity.

(iv) (20 pts) Find a combination of conductivity and wall thickness that minimizes the total cost (one year of heating and initial construction). *The total cost encompasses the heating costs and the building cost which is supposed to be proportianal to $\frac{d}{k}$. Thus the total cost could be written as*

$$C = \alpha\frac{k}{d}\frac{(T_i - T_c)^2}{T_h - T_c} + \beta\frac{d}{k} = \alpha X\frac{(T_i - T_c)^2}{T_h - T_c} + \beta\frac{1}{X}$$

where $X = \frac{k}{d}$. Looking for a critical point we compute the derivative of C

$$C' = \alpha\frac{(T_i - T_c)^2}{T_h - T_c} - \beta\frac{1}{X^2}$$

and find it is zero for

$$\frac{k}{d} = \sqrt{\frac{\beta}{\alpha}\frac{T_h - T_c}{(T_i - T_c)^2}}$$

which gives the optimal ratio between k and d. Indeed if we compute C'' we easily find that it is always positive, thus this critical point is indeed a minimum.

(v) (10 pts) Could you propose a more realistic, yet explicit, function $T(t)$? *We could propose a smoother one, which would be more natural. For example,*

$$T(t) = T_c + (T_h - T_c)\sin(\pi t) \quad \textit{or } T(t) = \frac{T_h + T_c}{2} + \frac{T_h - T_c}{2}\sin(2\pi(t - \frac{1}{4}))$$

where the second solution is smoother than the first one, in $t = 0$.

2. (30 pts) For $\alpha > 0$ we consider the following differential system

$$\begin{cases} \dfrac{dx}{dt} = x - \dfrac{1}{3}x^3 - y + \dfrac{1}{3}\alpha^3 \\ \dfrac{dy}{dt} = x - y \end{cases}$$

Find and classify the equilibrium point of this system. *First we find where this equilibrium point is by cancelling both right hand side of the two differential* *equalities. Thus this equilibrium point should verifiy $x = y$ and $x^3 = \alpha^3$ which gives (α, α) as solution. Now we compute the matrice G by differentiating the vector field:*

$$G = \begin{pmatrix} 1 - x^2 & -1 \\ 1 & -1 \end{pmatrix}$$

thus for $x = \alpha$ and $y = \alpha$ we find

$$G = \begin{pmatrix} 1 - \alpha^2 & -1 \\ 1 & -1 \end{pmatrix}$$

We compute $\operatorname{tr} G = -\alpha^2 < 0$ and $\det G = \alpha^2$. We know that in that case, it is a stable node (sink). Moreover if $(\operatorname{tr} G)^2 - 4 \det G = \alpha^4 - 4\alpha^2$ is negative, this is a spiral sink. This occurs if $\alpha < 2$.