Multi-material shape optimization via a level set method

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29th January, 2015
Multi-phase optimization is about finding the optimal repartition of two, or several, materials with conflicting properties within a fixed set. This problem has multiple applications in industrial design:

- **At the macroscopic level**: Repartition of several materials within a given structure to combine their respective assets.

- **At the microscopic level**: Mixture of several phases to achieve new materials with unique features (e.g. design of materials with negative Poisson’s ratio, or negative coefficient of thermal expansion...).

Design of a material with negative CTE \([M_i]\).
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A shape is a bounded domain $\Omega \subset \mathbb{R}^d$, which is

- fixed on a part $\Gamma_D$ of its boundary,
- submitted to surface loads $g$, applied on $\Gamma_N \subset \partial \Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$.

The displacement vector field $u_\Omega : \Omega \to \mathbb{R}^d$ is governed by the linear elasticity system:

$$
\begin{cases}
-\text{div}(Ae(u_\Omega)) &= 0 \quad \text{in } \Omega \\
u_\Omega &= 0 \quad \text{on } \Gamma_D \\
Ae(u_\Omega)n &= g \quad \text{on } \Gamma_N \\
Ae(u_\Omega)n &= 0 \quad \text{on } \Gamma := \partial \Omega \setminus (\Gamma_D \cup \Gamma_N)
\end{cases}
$$

where $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the strain tensor, and $A$ is the Hooke’s law of the material.
Goal: Starting from an initial structure $\Omega_0$, find a new one $\Omega$ that minimizes a certain functional of the domain $J(\Omega)$.

Examples:

- The work of the external loads $g$ or compliance $C(\Omega)$ of domain $\Omega$:
  \[ C(\Omega) = \int_{\Omega} A e(u_\Omega) : e(u_\Omega) \, dx = \int_{\Gamma_N} g \cdot u_\Omega \, ds \]

- A least-square error between $u_\Omega$ and a target displacement $u_0 \in H^1(\Omega)^d$ (useful when designing micro-mechanisms):
  \[ D(\Omega) = \left( \int_{\Omega} k(x) |u_\Omega - u_0|^\alpha \, dx \right)^{\frac{1}{\alpha}} \]
  where $\alpha$ is a fixed parameter, and $k(x)$ is a weight factor.

A volume constraint may be enforced with a fixed penalty parameter $\ell$:

\[
\text{Minimize } J(\Omega) := C(\Omega) + \ell \operatorname{Vol}(\Omega), \text{ or } D(\Omega) + \ell \operatorname{Vol}(\Omega).
\]
Differentiation with respect to the domain: Hadamard’s method

Hadamard’s boundary variation method describes variations of a reference, Lipschitz domain $\Omega$ of the form:

$$\Omega \rightarrow \Omega_\theta := (I + \theta)(\Omega),$$

for ‘small’ $\theta \in \mathcal{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$.

Definition 1.

Given a smooth domain $\Omega$, a functional $F(\Omega)$ of the domain is **shape differentiable** at $\Omega$ if the function

$$\mathcal{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \ni \theta \mapsto F(\Omega_\theta)$$

is Fréchet-differentiable at 0, i.e. the following expansion holds around 0:

$$F(\Omega_\theta) = F(\Omega) + F'(\Omega)(\theta) + o(\|\theta\|_{\mathcal{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)}).$$
Techniques close to optimal control theory make it possible to compute shape gradients; in the case of ‘many’ functionals of the domain $J(\Omega)$, the shape derivative has the particular structure:

$$J'(\Omega)(\theta) = \int_{\Gamma} v_{\Omega} \cdot \theta \cdot n \, ds,$$

where $v_{\Omega}$ is a scalar field depending on $u_{\Omega}$, and possibly on an adjoint state $p_{\Omega}$.

**Example:** If $J(\Omega) = C(\Omega) = \int_{\Gamma_N} g \cdot u_{\Omega} \, ds$ is the compliance, $v_{\Omega} = -Ae(u_{\Omega}) : e(u_{\Omega})$. 
The generic numerical algorithm

This shape gradient provides a natural descent direction for functional $J$: for instance, defining $\theta$ as

$$\theta = -\nu_\Omega n$$

yields, for $t > 0$ sufficiently small (to be found numerically):

$$J((I + t\theta)(\Omega)) = J(\Omega) - t \int_{\Gamma} \nu_\Omega^2 ds + o(t) < J(\Omega)$$

Gradient algorithm: For $n = 0, ...$ convergence,

1. Compute the solution $u_{\Omega^n}$ (and $p_{\Omega^n}$) of the elasticity system on $\Omega^n$.
2. Compute the shape gradient $J'(\Omega^n)$ thanks to the previous formula, and infer a descent direction $\theta^n$ for the cost functional.
3. Advect the shape $\Omega^n$ according to $\theta^n$, so as to get $\Omega^{n+1} := (I + \theta^n)(\Omega^n)$.
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The multi-material shape optimization setting (I)

- A fixed working domain $D \subset \mathbb{R}^d$ is occupied by two complementary phases $\Omega^0$ and $\Omega^1$, filled with elastic materials with Hooke’s laws $A_0$, $A_1$.
- The structure $D$ is clamped on a region $\Gamma_D \subset \partial D$, surface loads are applied on $\Gamma_N \subset \partial D$, as well as body forces $f$.
- The total, discontinuous Hooke’s law in $D$ is:

$$A_{\Omega^0} := A_0 \chi_0 + A_1 \chi_1,$$

where $\chi_i$ is the characteristic function of the phase $\Omega^i$. 
The multi-material shape optimization setting (II)

• The displacement

\[ u_{\Omega^0} \in H^1_{\Gamma_D}(D)^d := \{ u \in H^1(D)^d, u = 0 \text{ on } \Gamma_D \} \]

of the total structure \( D \) satisfies:

\[
\begin{align*}
-\text{div}(A_{\Omega^0} e(u)) &= f \text{ in } D \\
u &= 0 \text{ on } \Gamma_D \\
A_1 e(u) n &= g \text{ on } \Gamma_N
\end{align*}
\]

• **Goal:** Minimize a functional of the mixture of the form:

\[
J(\Omega^0) = \int_D j(x, u_{\Omega^0}) \, dx + \int_{\Gamma_N} k(x, u_{\Omega^0}) \, ds,
\]

under constraints (e.g. on the volume of one of the phases).

• **Example:** The compliance of the total structure \( D \):

\[
C(\Omega^0) = \int_D A_{\Omega^0} e(u_{\Omega^0}) : e(u_{\Omega^0}) \, dx = \int_D f \cdot u_{\Omega^0} \, dx + \int_{\Gamma_N} g \cdot u_{\Omega^0} \, ds.
\]
• The material properties are different from either side of $\Gamma \Rightarrow$ some quantities are discontinuous across $\Gamma$.

• If $\alpha$ is a discontinuous quantity, with values $\alpha^0$, $\alpha^1$ in $\overline{\Omega^0}$, $\overline{\Omega^1}$ respectively, $[\alpha] := \alpha^1 - \alpha^0$ is the jump of $\alpha$ across $\Gamma$.

• If $\mathcal{M}$ is a tensor-valued function, denote as:

$$\forall x \in \Gamma, \quad \mathcal{M} = \begin{pmatrix} \mathcal{M}_{\tau\tau}(x) & \mathcal{M}_{\tau n}(x) \\ \mathcal{M}_{n\tau}(x) & \mathcal{M}_{nn}(x) \end{pmatrix}$$

its representation in a local basis $(\tau, n)$ of $\mathbb{R}^d$.

• **Difficulty**: The strain tensor $e \equiv e(u_{\Omega^0})$ has continuous components $e_{\tau\tau}$, but discontinuous components $e_{\tau n}$, $e_{n\tau}$, $e_{nn}$. The stress tensor $\sigma \equiv \sigma(u_{\Omega^0})$ has continuous components $\sigma_{n\tau}$, $\sigma_{\tau n}$ and $\sigma_{nn}$, but $\sigma_{\tau\tau}$ is discontinuous.
Theorem 1 ([AlJouVG]).

The functional \( J(\Omega^0) \) is shape differentiable, and its derivative reads:

\[
\forall \theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \quad J'(\Omega^0)(\theta) = -\int_{\Gamma} D(x, u_{\Omega^0}, p_{\Omega^0}) \theta \cdot n \, ds,
\]

where the integrand factor \( D(x, u, p) \) is defined as:

\[
D(x, u, p) = -\sigma(p)_{nn} : [e(u)_{nn}] - 2\sigma(u)_{n\tau} : [e(p)_{n\tau}] + [\sigma(u)_{\tau\tau}] : e(p)_{\tau\tau},
\]

and \( p_{\Omega^0} \in H^1_{\Gamma_D}(D)^d \) is an adjoint state, defined as the solution to:

\[
\begin{align*}
-\text{div} \left( A_{\Omega^0} e(p) \right) &= -j'(x, u_{\Omega^0}) \quad \text{in } D, \\
p &= 0 \quad \text{on } \Gamma_D, \\
(A_1 e(p)) n &= -k'(x, u_{\Omega^0}) \quad \text{on } \Gamma_N,
\end{align*}
\]
This formula is difficult to use in numerical practice, since it involves the jumps of discontinuous quantities across $\Gamma$.

**Potential remedies:**

- **Discrete approach:** [AlDaDelMi]

  Consider the shape derivative of the discretization $J_h(\Omega^0)$ of $J(\Omega^0)$ on the actual mesh, which features the numerical solution $u_h$ (resp. $p_h$) of the state (resp. adjoint) elasticity system.

- **Body-fitted approach:** [AlDaFr]

  The interface $\Gamma$ is explicitly discretized at each step of the process.
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Definition 2.

The signed distance function $d_{\Omega}$ to a bounded domain $\Omega \subset \mathbb{R}^d$ is defined as:

$$\forall x \in \mathbb{R}^d,$$

$$\begin{cases} 
- d(x, \partial \Omega) & \text{if } x \in \Omega \\
0 & \text{if } x \in \partial \Omega \\
d(x, \partial \Omega) & \text{if } x \in \overline{\Omega} 
\end{cases}$$

where $d(\cdot, \partial \Omega)$ stands for the usual Euclidean distance function to $\partial \Omega$. 

Graph of the signed distance function to a union of two disks (in black)
The smoothed-interface setting (I)

- The discontinuous tensor $A_{\Omega^0}$ is approximated by:
  \[
  \forall x \in D, \quad A_{\Omega^0,\varepsilon}(x) := A_0 + h_\varepsilon(d_{\Omega^0}(x))(A_1 - A_0),
  \]
  where $h_\varepsilon$ is a smooth approximation of the Heaviside function:
  \[
  h_\varepsilon(t) = \begin{cases} 
  0 & \text{if } t < -\varepsilon \\
  \frac{1}{2} \left(1 + \frac{t}{\varepsilon} + \frac{1}{\pi} \sin \left(\frac{\pi t}{\varepsilon}\right)\right) & \text{if } -\varepsilon \leq t \leq \varepsilon \\
  1 & \text{if } t > \varepsilon
  \end{cases}
  \]

- This accounts for a smooth interpolation of the material properties between the two phases over a tubular neighborhood of $\Gamma$ of fixed width $2\varepsilon$. 

![Diagram showing the smoothed-interface setting](image-url)
The smoothed-interface setting (II)

- The smoothed-interface problem is then that of minimizing:

\[ J_\varepsilon(\Omega^0) = \int_D j(x, u_{\Omega^0, \varepsilon}) \, dx + \int_{\Gamma_N} k(x, u_{\Omega^0, \varepsilon}) \, ds \]

(under constraints), where \( u_{\Omega^0, \varepsilon} \) arises as the solution to:

\[
\begin{cases}
-\text{div}(A_{\Omega^0, \varepsilon}e(u)) = f & \text{in } D \\
u = 0 & \text{on } \Gamma_D \\
A_1e(u)n = g & \text{on } \Gamma_N
\end{cases}
\]

- It is worth considering for at least two reasons:

  - It is an approximation of the sharp-interface problem, and is easier to handle numerically.
  - It has some interest on its own, especially when it comes to modelling interfaces: interfaces may involve complex and ill-understood processes, which are better described e.g. by non monotone transition regions.
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Definition 3.

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz, bounded open set;

- Let $x \in \mathbb{R}^d$; the set of projections $\Pi_{\partial \Omega}(x)$ of $x$ onto $\partial \Omega$ is:
  $$\Pi_{\partial \Omega}(x) = \{ y \in \partial \Omega, \ d(x, \partial \Omega) = |x - y| \} .$$

- When this set is a singleton, $p_{\partial \Omega}(x)$ is the projection of $x$ onto $\partial \Omega$.

- The skeleton $\Sigma$ of $\partial \Omega$ is:
  $$\Sigma := \{ x \in \mathbb{R}^d, \ d^2_\Omega \text{ is not differentiable at } x \} .$$

- For $x \in \partial \Omega$, the ray emerging from $x$ is:
  $$\text{ray}_{\partial \Omega}(x) := p_{\partial \Omega}^{-1}(x).$$
$x$ has a unique projection over $\partial\Omega$, whereas $x'$ has two such points $y_1, y_2$. 
Proposition 2.

Let \( \Omega \subset \mathbb{R}^d \) be a Lipschitz, bounded open set;

- A point \( x \in \mathbb{R}^d \) has a unique projection point \( p_{\partial \Omega}(x) \) iff \( x \notin \Sigma \). In such a case, \( d_\Omega \) is differentiable at \( x \), and its gradient reads:

\[
\nabla d_\Omega(x) = \frac{x - p_{\partial \Omega}(x)}{d_\Omega(x)}.
\]

In particular, \( |\nabla d_\Omega(x)| = 1 \) wherever it makes sense.

- If \( \Omega \) is of class \( C^1 \), this last quantity equals \( \nabla d_\Omega(x) = n(p_{\partial \Omega}(x)) \).

- If \( \Omega \) is of class \( C^k \), \( k \geq 2 \), then \( d_\Omega \) is also of class \( C^k \) on a neighborhood of \( \partial \Omega \).
Some level sets of $d_\Omega$ are depicted in color; $d_\Omega$ is as smooth as the boundary $\partial \Omega$ on the shaded area (at least).
Lemma 3.

Let $\Omega \subset \mathbb{R}^d$ be a $C^1$ bounded domain, and $x \notin \Sigma$. The function $\theta \mapsto d_{\Omega \theta}(x)$, from $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ into $\mathbb{R}$ is Gâteaux-differentiable at $\theta = 0$, with derivative:

$$d'_{\Omega}(\theta)(x) = -\theta(p_{\partial \Omega}(x)) \cdot n(p_{\partial \Omega}(x)).$$
Remark: A more general formula holds, which encompasses the case \( x \in \Sigma \):

If \( x \in \Omega \),

\[
d_\Omega'(\theta)(x) = - \inf_{y \in \Pi_{\partial \Omega}(x)} \theta(y) \cdot n(y),
\]

If \( x \in c\overline{\Omega} \),

\[
d_\Omega'(\theta)(x) = - \sup_{y \in \Pi_{\partial \Omega}(x)} \theta(y) \cdot n(y).
\]
• **Formal clue:** Taking the shape derivative in
\[ |\nabla d_\Omega(x)|^2 = 1 \]
yields:
\[ \nabla d'_\Omega(\theta)(x) \cdot \nabla d_\Omega(x) = 0. \]
⇒ The shape derivative of \( d_\Omega \) is constant along the rays.

• **Rigorous proof:** Use of the definition:
\[ d^2_\Omega(x) = \min_{y \in \partial \Omega} |x - y|^2 \]
in combination to a theorem for differentiating a minimum value with respect to a parameter.
Lemma 4.

Let $\Omega$ be a $C^1$ bounded domain, enclosed in a large computational domain $D$, and $j : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be of class $C^1$; define the functional:

$$ J(\Omega) = \int_D j(x, d_\Omega(x)) \, dx. $$

Then $\theta \mapsto J(\Omega_\theta)$ is Gâteaux-differentiable at $\theta = 0$ with derivative:

$$ J'(\Omega)(\theta) = - \int_D \frac{\partial j}{\partial s}(x, d_\Omega(x)) \theta(p_{\partial \Omega}(x)) \cdot n(p_{\partial \Omega}(x)) \, dx. $$

This formula is awkward insofar it is not easily put under the form:

$$ J'(\Omega)(\theta) = \int_\Gamma \nu \theta \cdot n \, ds, $$

and does not lend itself to the inference of a ‘natural’ descent direction for $J$. 
Proposition 5.

Let $\Omega \subset D$ be a bounded domain of class $C^2$, and let $\varphi \in L^1(D)$. Then,

$$\int_D \varphi(x)dx = \int_{\partial\Omega} \left( \int_{\text{ray}_{\partial\Omega}(y) \cap D} \varphi(z) \prod_{i=1}^{d-1} (1 + d_\Omega(z) \kappa_i(y))dz \right) dy,$$

where $z$ denotes a point in the ray emerging from $y \in \partial\Omega$ and $dz$ is the line integration along that ray.

Hint of proof:

Apply the coarea formula to the mapping:

$$p_{\partial\Omega} : D \setminus \Sigma \to \partial\Omega$$

to recast the integration over $D \approx D \setminus \Sigma$ as an integration over $\partial\Omega$ composed with an integration over the pre-images $p_{\partial\Omega}^{-1}(x) = \text{ray}_{\partial\Omega}(x)$, $x \in \partial\Omega$. 
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Theorem 6.

The objective function

\[ J_\varepsilon(\Omega^0) = \int_D j(x, u_{\Omega^0,\varepsilon}) \, dx + \int_{\Gamma_N} k(x, u_{\Omega^0,\varepsilon}) \, ds, \]

is s.t. \( \theta \mapsto J_\varepsilon(\Omega^0_\theta) \) admits a \textit{Gâteaux}-derivative at \( \theta = 0 \), which is

\[ \forall \theta \in W^{1,\infty}(D, \mathbb{R}^d), \quad J'_\varepsilon(\Omega^0)(\theta) = -\int_{\Gamma} j(x) \theta(x) \cdot n(x) ds(x). \]

Here, \( n \) is the outer unit normal to \( \Omega^0 \) and \( j \) is the scalar function defined by

\[ j(x) = \int_{\text{ray}_{\Gamma(x) \cap D}} h'_\varepsilon \left( d_{\Omega^0}(z) \right) (A_1 - A_0) e(u)(z) : e(p)(z) \prod_{i=1}^{d-1} (1 + d_{\Omega^0}(z) \kappa_i(x)) dz, \]

where \( u \equiv u_{\Omega^0,\varepsilon} \) and the \textit{adjoint state} \( p \equiv p_{\Omega^0,\varepsilon} \) is the solution to:

\[ \begin{cases} 
-\text{div} \left( A_{\Omega^0,\varepsilon} e(p) \right) & = -j'(x, u_{\Omega^0,\varepsilon}) \quad \text{in } D, \\
p & = 0 \quad \text{on } \Gamma_D, \\
(A_1 e(p)) n & = -k'(x, u_{\Omega^0,\varepsilon}) \quad \text{on } \Gamma_N,
\end{cases} \]
Shape derivative of the smoothed-interface functional (II)

Sketch of (formal) proof: For functions $v, q \in H^1_{\Gamma_D}(D)^d$, define the Lagrangian functional $\mathcal{L}(\Omega^0, v, q)$ as:

$$
\mathcal{L}(\Omega^0, v, q) = \int_D j(x, v) \, dx + \int_{\Gamma_N} k(x, v) \, ds \\
+ \int_D A_{\Omega^0,\varepsilon}(x) e(v) : e(q) \, dx - \int_D f \cdot q \, dx - \int_{\Gamma_N} g \cdot q \, ds.
$$

By definition,

$$
\forall q \in H^1_{\Gamma_D}(D)^d, \quad J_\varepsilon(\Omega^0) = \mathcal{L}(\Omega^0, u_{\Omega^0,\varepsilon}, q).
$$

Let us search for the critical points $(u, p)$ of $\mathcal{L}(\Omega^0, \cdot, \cdot)$.

- Expressing $\frac{\partial \mathcal{L}}{\partial p}(\Omega^0, u, p) = 0$ yields $u = u_{\Omega^0,\varepsilon}$.
- Expressing $\frac{\partial \mathcal{L}}{\partial u}(\Omega^0, u, p) = 0$ yields $p = p_{\Omega^0,\varepsilon}$. 
Thus, for any $q \in H^1_{\Gamma_D}(D)^d$, assuming that $u_{\Omega^0,\varepsilon}$ is differentiable with respect to the domain,

$$J'_\varepsilon(\Omega^0)(\theta) = \frac{\partial L}{\partial \Omega}(\Omega^0, u_{\Omega^0,\varepsilon}, q) + \frac{\partial L}{\partial u}(\Omega^0, u_{\Omega^0,\varepsilon}, q)(u'_{\Omega^0,\varepsilon}(\theta)).$$

Now choosing $q = p_{\Omega^0,\varepsilon}$, and using $\frac{\partial L}{\partial u}(\Omega^0, u, p) = 0$ yield:

$$J'_\varepsilon(\Omega^0)(\theta) = \frac{\partial L}{\partial \Omega}(\Omega^0, u_{\Omega^0,\varepsilon}, p_{\Omega^0,\varepsilon}),$$

which can be calculated thanks to Lemma 4.
The formula of Theorem 6 can be given consistent and convenient approximations in two important limits in applications:

- **Jacobian-free formula**: If the interface $\Gamma$ is approximately plane, that is $d_{\Omega_0^\kappa} \approx 0$, we obtain:

  $$J'_\varepsilon(\Omega^0)(\theta) = - \int_{\Gamma} j(x) \theta(x) \cdot n(x) ds(x),$$

  with

  $$j(x) \approx \int_{\text{ray}_{\Gamma(x)} \cap D} h'_\varepsilon(d_{\Omega_0}(z)) (A_1 - A_0) e(u)(z) : e(p)(z) dz.$$  

- **Thin-interface formula**: If the transition layer is very thin, i.e. $\varepsilon$ is very small,

  $$J'_\varepsilon(\Omega^0)(\theta) \approx - \int_{\Gamma} (A_1 - A_0) e(u)(x) : e(p)(x) \theta(x) \cdot n(x) ds(x).$$
Theorem 7.

Assume that $\Omega^0$ is of class $C^2$. Then the smoothed-interface problem converges to its sharp-interface counterpart in the sense that:

$$J_\varepsilon(\Omega^0) \xrightarrow{\varepsilon \to 0} J(\Omega^0),$$

and, for any deformation field $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$,

$$J_\varepsilon'(\Omega^0)(\theta) \xrightarrow{\varepsilon \to 0} J'(\Omega^0)(\theta).$$
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The Level Set Method

A paradigm: [OSe] the motion of an evolving domain is best described in an *implicit* way.

A bounded domain $\Omega \subset \mathbb{R}^d$ is equivalently defined by a function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that:

$$
\phi(x) < 0 \quad \text{if} \quad x \in \Omega \quad ; \quad \phi(x) = 0 \quad \text{if} \quad x \in \partial \Omega \quad ; \quad \phi(x) > 0 \quad \text{if} \quad x \in c\Omega
$$

A bounded domain $\Omega \subset \mathbb{R}^2$ (left); graph of an associated level set function (right).
Surface evolution equations in the level set framework

The motion of an evolving domain \( \Omega(t) \subset \mathbb{R}^d \) along a velocity field \( v(t, x) \in \mathbb{R}^d \) translates in terms of an associated ‘level set function’ \( \phi(t,.) \) into the level set advection equation:

\[ \forall t, \forall x \in \mathbb{R}^d, \quad \frac{\partial \phi}{\partial t}(t, x) + v(t, x) \cdot \nabla \phi(t, x) = 0 \]

In many applications, the velocity \( v(t, x) \) is normal to the boundary \( \partial \Omega(t) \):

\[ v(t, x) := V(t, x) \frac{\nabla \phi(t, x)}{|\nabla \phi(t, x)|} \]

Then the evolution equation rewrites as a Hamilton-Jacobi equation:

\[ \forall t, \forall x \in \mathbb{R}^d, \quad \frac{\partial \phi}{\partial t}(t, x) + V(t, x)|\nabla \phi(t, x)| = 0 \]
The shapes $\Omega^n$ under evolution are embedded in a working domain $D$ equipped with a fixed mesh.

The successive shapes $\Omega^n$ are accounted for in the level set framework, i.e. via a function $\phi^n : D \to \mathbb{R}$ which implicitly defines them.

This approach is very versatile and does not require a mesh of the shapes at each iteration.

Shape accounted for with a level set description
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Two-phase long cantilever (I)

- Optimization of the repartition of two materials with the same Poisson ratio $\nu_0 = \nu_1 = 0.3$, but different Young’s moduli $E_0 = 0.5$, $E_1 = 1$.
- The thickness parameter $\varepsilon$ is set to $2\Delta_x$.
- The compliance of the total structure $D$ is minimized.
- A constraint is imposed on the volume of the stronger phase: $V_T = 0.7|D|$, owing to an augmented Lagrangian algorithm.
Initial shape, optimized shape using the ‘true’ formula, optimized shape using the ‘Jacobian-free’ formula, optimized shape using the ‘thin-interface’ formula.
Two-phase long cantilever (III)

Convergence histories in the three cases of interest.
Two-phase long cantilever (IV)

Use of a larger thickness parameter $\varepsilon = 8\Delta x$ for the transition zone.

*Initial shape, optimized shape using the ‘true’ formula, optimized shape using the ‘Jacobian-free’ formula, optimized shape using the ‘thin-interface’ formula.*
Extension to more than 2 (e.g. 3, 4) phases

Two subdomains \( \mathcal{O}^0, \mathcal{O}^1 \subset D \), and the 4 phases derived by combining them.

- One subdomain \( \mathcal{O}_0 \subset D \) accounts for two phases \( \Omega^0 = \mathcal{O}^0, \Omega^1 = \overline{c \mathcal{O}^0} \).
- Combining 2 subdomains \( \mathcal{O}^0, \mathcal{O}^1 \subset D \), one can represent up to 4 phases:
  \[
  \Omega^0 = \mathcal{O}^0 \cap \mathcal{O}^1, \quad \Omega^1 = \overline{c \mathcal{O}^0} \cap \mathcal{O}^1, \quad \Omega^2 = \ldots
  \]
- The previous framework can be easily extended to deal with multiple phases:

  \[\Rightarrow\] Using \( m \) different level set functions allows to account for up to \( 2^m \) distinct phases.
Multiple-phase short cantilever

- **Two phases and void:**

  The Young’s moduli of the different phases are:

  \[ E_0 = 0.5, \quad E_2 = 1, \quad E_1 = E_3 = 1e^{-3}. \]

  (Phases 1 and 3 mimic void).

  Volume constraint:

  \[ V_0 = 0.2|D|, \quad V_2 = 0.1|D|. \]

- **Three phases and void:**

  The Young’s moduli are:

  \[ E_0 = 0.5, \quad E_1 = 0.25, \quad E_2 = 1, \quad E_3 = 1e^{-3}. \]

  Volume constraint:

  \[ V_0 = V_1 = V_2 = 0.1|D|. \]
Two-phase short cantilever

Short cantilever using two phases and void; (left) initialization, (right) optimal shape.
Three-phase short cantilever

Short cantilever with three phases and void; (left) initialization, (right) optimal shape.
Two-phase L-Beam

- Phase 0 has Young’s modulus $E_0 = 1$.
- Phases 1 and 3 mimic void ($E_1 = E_3 = 1e^{-3}$).
- Phase 2 has different Young’s moduli depending on the considered example.
- A constraint on the volumes of phases 0 and 2 is imposed:
  \[ V^0_T = V^2_T = 0.25|D|. \]
Optimal designs for the two-phase L-Beam problem with (from left to right) $E_2 = 0.2, 0.5, 0.8$. 
An example using non monotone interfaces

- Work carried out by G. Allaire, Y. Bréchet, R. Estevez, G. Michailidis, G. Parry and N. Vermaak [VerMi].

- Optimization of the repartition of two materials with the same Poisson ratio $\nu_0 = \nu_1 = 0.3$, but different Young’s moduli $E_0 = 0.1$, $E_1 = 1$.

- The compliance of the total structure $D$ is minimized, under a constraint $V_1^1 = 0.5|D|$ on the volume of the stronger phase.

- The properties of the material inside the transition layer are non monotone.
An example using non monotone interfaces

(Top-left) Profile of the Young’s modulus in the transition layer, (top-right) final design, (bottom) iterations 1, 10, 25, 40.
An example using non monotone interfaces

(Top-left) Profile of the Young’s modulus in the transition layer, (top-right) final design, (bottom) iterations 1, 50, 75, 90.
An example using non monotone interfaces

(Top-left) Profile of the Young’s modulus in the transition layer, (top-right) final design, (bottom) iterations 1, 5, 50, 110.
Thank you for your attention!


