A deterministic approximation method in shape optimization under random uncertainties

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Mechanical systems rely on data, e.g. the loads, the properties of a constituent material, or the geometry of the system itself.

In concrete situations, such data are plagued with uncertainties because:

- they may be available only through (error-prone) measurements,
- they may be altered with time (wear) and conditions of the ambient medium.

The performances of structures are very sensitive to small perturbations of data.

⇒ Need to somehow anticipate uncertainties when designing and optimizing shapes.
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   - Foreword
   - The main ideas in an abstract framework

2 Applications in shape optimization
   - Shape optimization of elastic structures
   - Shape optimization under random loads
   - Shape optimization under uncertainties on the elastic material
   - Shape optimization under geometric uncertainties
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• $\mathcal{U}_{ad} \subset \mathcal{H}$ is a set of admissible designs $h$ (e.g. the thickness of a plate, the geometry of a shape).

• $(\mathcal{P}, \| \cdot \|)$ is a Banach space of data $f$ (forces, parameters of a material).

• The performances of a design $h$ are evaluated in terms of a cost $C \equiv C(f, u_{h,f})$, which involves a state $u_{h,f}$, solution to a physical system:

$$A(h)u_{h,f} = b(f),$$

where $f$ acts on the right-hand side for simplicity.

• The data are uncertain, and read:

$$f = f_0 + \hat{f}(\omega),$$

where $f_0$ is a mean value, and $\omega$ is an event, in an abstract probability space $(\mathcal{O}, \mathcal{F}, \mathbb{P})$. 
There are two different settings to deal with uncertainties:

- **Worst-case approach**: When only a maximum bound $||\hat{f}||_P \leq m$ is available on perturbations, one considers the worst-case functional:

  $$J_{wc}(h) = \sup_{||\hat{f}||_P \leq m} C(f_0 + \hat{f}, u_h, f_0 + \hat{f}).$$

  *Main drawback:* Pessimistic approach, which may yield designs with unnecessarily bad nominal performances.

- **Probabilistic approach**: When information is available on the moments of the uncertainties, one may try to minimize the mean value:

  $$M(h) = \int_{\mathcal{O}} C(f_0 + \hat{f}(\omega), u_h, f_0 + \hat{f}(\omega)) \mathbb{P}(d\omega),$$

  or a failure probability:

  $$\mathcal{P}(h) = \mathbb{P} \left( \{ \omega \in \mathcal{O}, C(f_0 + \hat{f}(\omega), u_h, f_0 + \hat{f}(\omega)) > \alpha \} \right).$$
Working hypotheses:

- Perturbations are **small**: depending on the context, this may mean:
  - $\hat{f} \in L^\infty(O, \mathcal{P})$: all the realizations $\hat{f}(\omega) \in \mathcal{P}$ are small.
  - $\hat{f} \in L^p(O, \mathcal{P})$, for $p < \infty$: $\hat{f}$ may have unprobably large realizations.

- Perturbations are **finite-dimensional**:

  \[
  \hat{f}(\omega) = \sum_{i=1}^{N} f_i \xi_i(\omega),
  \]

  where $f_i \in \mathcal{P}$, and the $\xi_i$ are normalized, uncorrelated random variables:

  \[
  \int_{O} \xi_i(\omega) \mathbb{P}(d\omega) = 0, \quad \int_{O} \xi_i(\omega)\xi_j(\omega) \mathbb{P}(d\omega) = \delta_{i,j}.
  \]

  **Example**: $\hat{f}$ is obtained as a truncated Karhunen-Loève expansion.
The main ideas in an abstract framework (IV)

**Strategy:**

- Calculate approximate functionals $\tilde{M}(h)$ and $\tilde{P}(h)$, which are
  - **deterministic:** no random variable or probabilistic integral is involved.
  - **consistent** with their exact counterparts, i.e. the differences $|M(h) - \tilde{M}(h)|$ and $|P(h) - \tilde{P}(h)|$ are ‘small’.

- Calculate their derivatives $\tilde{M}'(h)(\hat{h})$ and $\tilde{P}'(h)(\hat{h})$,

- Minimize the approximate functionals $\tilde{M}(h)$ and $\tilde{P}(h)$ (under constraints), by using the expressions of their derivatives.
  - e.g. relying on a **steepest-descent algorithm**.
The main ideas in an abstract framework (V)

Use the smallness of perturbations to perform a first- or second-order Taylor expansion of the mappings $f \mapsto u_{h,f}$ and $f \mapsto C(f, u_{h,f})$ around $f_0$:

$$u_{h,f_0 + \hat{f}} \approx u_h + u_h^1(\hat{f}) + \frac{1}{2} u_h^2(\hat{f}, \hat{f}),$$

where

$$A(h)u_h^1(\hat{f}) = \frac{\partial b}{\partial f}(f_0)(\hat{f}), \quad \text{and} \quad A(h)u_h^2(\hat{f}, \hat{f}) = \frac{\partial^2 b}{\partial f^2}(f_0)(\hat{f}, \hat{f}).$$

$$C(f_0 + \hat{f}, u_{h,f_0 + \hat{f}}) \approx C(f_0, u_h) + L_h(\hat{f}) + \frac{1}{2} B_h(\hat{f}, \hat{f}),$$

where the linear and bilinear forms $L_h$ and $B_h$ read:

$$L_h(\hat{f}) = \frac{\partial C}{\partial f}(f_0, u_h)(\hat{f}) + \frac{\partial C}{\partial u}(f_0, u_h)(u_h^1(\hat{f})), \quad \text{and} \quad B_h(\hat{f}, \hat{f}) = \frac{\partial^2 C}{\partial f^2}(f_0, u_h)(\hat{f}, \hat{f}) + 2 \frac{\partial^2 C}{\partial f \partial u}(f_0, u_h)(\hat{f}, u_h^1(\hat{f}))$$

$$+ \frac{\partial^2 C}{\partial u^2}(f_0, u_h)(u_h^1(\hat{f}), u_h^1(\hat{f})) + \frac{\partial C}{\partial u}(f_0, u_h)(u_h^2(\hat{f}, \hat{f})).$$
Approximation of moment functionals

• Replacing the cost with its second-order expansion gives rise to the approximate mean-value functional:

\[ \tilde{\mathcal{M}}(h) = \mathcal{C}(f_0, u_h) + \int_{\mathcal{O}} \mathcal{L}_h(\hat{f}(\omega)) \, \mathbb{P}(d\omega) + \frac{1}{2} \int_{\mathcal{O}} \mathcal{B}_h(\hat{f}(\omega), \hat{f}(\omega)) \, \mathbb{P}(d\omega). \]

• Using the structure of perturbations \( \hat{f}(\omega) = \sum_{i=1}^{N} f_i \xi_i(\omega) \), it comes:

\[ \tilde{\mathcal{M}}(h) = \mathcal{C}(f_0, u_h) + \frac{1}{2} \sum_{i=1}^{N} \mathcal{B}_h(f_i, f_i), \]

a formula which involves the calculation of the \( N + 2 \) 'reduced states':

\[ u_h, \ u_{h,i} := u_h^1(f_i), \ (i = 1, \ldots, N), \ \text{and} \ u_h^2 := \sum_{i=1}^{N} u_h^2(f_i, f_i). \]

• This approach can be applied to other moments of \( \mathcal{C} \), e.g. its variance:

\[ \mathcal{V}(h) = \int_{\mathcal{O}} \left( \mathcal{C}(f_0 + \hat{f}(\omega), u_{h,f_0+\hat{f}(\omega)}) - \mathcal{M}(h) \right)^2 \mathbb{P}(d\omega). \]
Additional hypotheses: The random variables $\xi_i$ are:

- independent,
- Gaussian, i.e. their cumulative distribution function is:

$$\mathbb{P} \left( \{ \omega \in \mathcal{O}, \xi_i(\omega) < \alpha \} \right) = \Phi(\alpha) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{\xi^2}{2}} \, d\xi.$$  

The (exact) failure probability reads:

$$\mathcal{P}(h) = \frac{1}{(2\pi)^{N/2}} \int_{D(h)} e^{-\frac{||\xi||^2}{2}} \, d\xi,$$

where the failure region $D(h)$ is:

$$D(h) = \left\{ \xi \in \mathbb{R}^N, C \left( f_0 + \sum_{i=1}^{N} f_i \xi_i, u_h, f_0 + \sum_{i=1}^{N} f_i \xi_i \right) > \alpha \right\}.$$
Approximation of failure probabilities (II)

**Idea:** Approximate the failure region with:

\[ \tilde{D}(h) = \left\{ \xi \in \mathbb{R}^N, \ C(f_0, u_h) + \sum_{i=1}^{N} \mathcal{L}_h(f_i)\xi_i > \alpha \right\} \]

The approximate failure probability

\[ \tilde{P}(h) = \frac{1}{(2\pi)^{N/2}} \int_{\tilde{D}(h)} e^{-\frac{||\xi||^2}{2}} \, d\xi \]

can be calculated in closed form as:

\[ \tilde{P}(h) = \Phi \left( \frac{-\alpha - C(f_0, u_h)}{\sqrt{\sum_{i=1}^{N} \mathcal{L}_h(f_i)^2}} \right) \]
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The usual linear elasticity setting

A shape is a bounded domain $\Omega \subset \mathbb{R}^d$, which is

- fixed on a part $\Gamma_D$ of its boundary,
- submitted to surface loads $g$, applied on $\Gamma_N \subset \partial \Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$.

The displacement vector field $u_\Omega \in H^1_{\Gamma_D}(\Omega)^d$ is governed by the linear elasticity system:

$$
\begin{cases}
-\text{div}(A e(u_\Omega)) = f & \text{in } \Omega \\
u_\Omega = 0 & \text{on } \Gamma_D \\
A e(u_\Omega)n = g & \text{on } \Gamma_N \\
A e(u_\Omega)n = 0 & \text{on } \Gamma := \partial \Omega \setminus (\Gamma_D \cup \Gamma_N)
\end{cases}
$$

where $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the strain tensor, and $A$ is the Hooke’s law of the material.
Hadamard’s boundary variation method describes variations of a reference, Lipschitz domain $\Omega$ of the form:

$$\Omega \to \Omega_\theta := (I + \theta)(\Omega),$$

for ‘small’ $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$.

**In practice:**

- We restrict to a set of admissible shapes:
  $$U_{ad} := \{ \Omega \subset \mathbb{R}^d \text{ is open, bounded and Lipschitz, } \Gamma_D \cup \Gamma_N \subset \partial \Omega \}. $$

- Deformations $\theta$ are assumed within the admissible set:
  $$\Theta_{ad} := \{ \theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \text{ such that } \theta = 0 \text{ on } \Gamma_D \cup \Gamma_N \}. $$
Differentiation with respect to the domain: Hadamard’s method (II)

Definition 1.

Given a smooth domain $\Omega$, a functional $J(\Omega)$ of the domain is **shape differentiable** at $\Omega$ if the function

$$W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \ni \theta \mapsto J(\Omega_{\theta})$$

is Fréchet-differentiable at 0, i.e. the following expansion holds around 0:

$$J(\Omega_{\theta}) = J(\Omega) + J'(\Omega)(\theta) + o \left( ||\theta||_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)} \right).$$

Shape derivatives can be computed using techniques from optimal control; in the case of ‘many’ functions of the domain $J(\Omega)$, they enjoy the **structure**:

$$J'(\Omega)(\theta) = \int_{\Gamma} v_{\Omega} \theta \cdot n \, ds,$$

where $v_{\Omega}$ is a scalar field depending on $u_{\Omega}$, and possibly on an **adjoint state** $p_{\Omega}$. 
This shape gradient provides a natural descent direction for functional $J$: for instance, defining $\theta$ as

$$\theta = -\nu_\Omega n$$

yields, for $t > 0$ sufficiently small (to be found numerically):

$$J((I + t\theta)(\Omega)) = J(\Omega) - t \int_\Gamma \nu_\Omega^2 ds + o(t) < J(\Omega)$$

**Gradient algorithm:** For $n = 0, \ldots$ convergence,

1. Compute the solution $u_{\Omega^n}$ (and $p_{\Omega^n}$) of the elasticity system on $\Omega^n$.
2. Compute the shape gradient $J'(\Omega^n)$ thanks to the previous formula, and infer a descent direction $\theta^n$ for the cost functional.
3. Advect the shape $\Omega^n$ according to $\theta^n$, so as to get $\Omega^{n+1} := (I + \theta^n)(\Omega^n)$.
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We consider uncertainties on the body forces \( f = L^2(\mathbb{R}^d)^d \):

\[
  f(x) = f_0(x) + \hat{f}(x, \omega), \quad \text{where} \quad \hat{f}(x, \omega) = \sum_{i=1}^{N} f_i(x) \xi_i(\omega) \in L^2(O, L^2(\mathbb{R}^d)^d).
\]

The cost function is of the form:

\[
  C(f, \Omega) = \int_{\Omega} j(f, u_{\Omega,f}) \, dx,
\]

where \( j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) is smooth, satisfies growth conditions, and \( u_{\Omega,f} \in H^1_{\Gamma_D}(\Omega)^d \) is the solution \( u \) of:

\[
  \begin{cases}
    -\text{div}(Ae(u)) &= f \quad \text{in} \ \Omega \\
    u &= 0 \quad \text{on} \ \Gamma_D \\
    Ae(u)n &= 0 \quad \text{on} \ \Gamma_N \\
    Ae(u)n &= 0 \quad \text{on} \ \Gamma
  \end{cases}
\]
Shape optimization under random loads (II)

The approximate mean value functional reads:

\[
\tilde{M}(\Omega) = \int_{\Omega} j(f_0, u_\Omega) \, dx + \frac{1}{2} \sum_{i=1}^{N} \int_{\Omega} \nabla^2_f j(f_0, u_\Omega)(f_i, f_i) \, dx \\
+ \sum_{i=1}^{N} \int_{\Omega} \nabla_f \nabla_u j(f_0, u_\Omega)(f_i, u^1_{\Omega,i}) \, dx + \frac{1}{2} \sum_{i=1}^{N} \int_{\Omega} \nabla^2_u j(f_0, u_\Omega)(u^1_{\Omega,i}, u^1_{\Omega,i}) \, dx,
\]

the \( u^1_{\Omega,i} \) being the solutions of:

\[
\begin{cases}
-\text{div}(Ae(u)) = f_i & \text{in } \Omega \\
u = 0 & \text{on } \Gamma_D \\
Ae(u)n = 0 & \text{on } \Gamma_N \\
Ae(u)n = 0 & \text{on } \Gamma
\end{cases}
\]

**Proposition 1.**

*Under the additional assumption that \( \hat{f} \in L^3(\mathcal{O}, L^3(\mathbb{R}^d)^d) \), there exists a constant \( C > 0 \) (depending on \( \Omega \)) such that:*

\[
|\tilde{M}(\Omega) - M(\Omega)| \leq C \|\hat{f}\|_{L^3(\mathcal{O}, L^3(\mathbb{R}^d)^d)}^3.
\]
The cost function is the compliance of shapes:

\[ C(f, \Omega) = \int_\Omega f \cdot u_{\Omega,f} \, dx = \int_\Omega Ae(u_{\Omega,f}) : e(u_{\Omega,f}) \, dx. \]

Two load scenarios \( f_1, f_2 = (0, -m) \) are supported in the blue spots.

The considered objective function is:

\[ L(\Omega) = \tilde{M}(\Omega) + \delta \sqrt{\tilde{V}(\Omega)}. \]

A constraint \( \text{Vol}(\Omega) = V_T \) is enforced by an augmented Lagrangian algorithm.

(Left) The bridge test case, (right) optimal shape in the unperturbed situation.
Optimal shapes for $\delta = 0$ and $m = 1, 2, 5, 10$. 

Optimization of a bridge under random loads (II)
Optimization of a bridge under random loads (III)

Optimal shapes for $\delta = 3$ and $m = 1, 2, 5, 10$. 

*Optimal shapes for $\delta = 3$ and $m = 1, 2, 5, 10$.***
Comparison with the worst-case approach

Optimal shapes for the linearized worst-case design approach with $m = 1, 2, 5, 10$.

**Observation:** The optimal shapes for the probabilistic functionals show systematically better nominal performances than their worst-case counterparts.
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Optimization under material uncertainties

- Perturbations over the Young's modulus $E$ of the material are considered:

$$E = E_0 + \hat{E}(x, \omega), \quad \text{where} \quad \hat{E}(x, \omega) = \sum_{i=1}^{N} E_i(x) \xi_i(\omega) \in L^\infty(O, L^\infty(R^d)).$$

- The cost function is of the form $C(\Omega, E) = \int_\Omega j(u_{\Omega,E}) \, dx$, where:

$$\begin{cases}
-\text{div}(A(E)e(u_{\Omega})) &= 0 \quad \text{in } \Omega \\
u_{\Omega} &= 0 \quad \text{on } \Gamma_D \\
A(E)e(u_{\Omega})n &= g \quad \text{on } \Gamma_N \\
A(E)e(u_{\Omega})n &= 0 \quad \text{on } \Gamma
\end{cases}.$$

- Minimization of the approximate mean value of $C$:

$$\tilde{M}(\Omega) = \int_\Omega j(u_{\Omega}) \, dx + \frac{1}{2} \sum_{i=1}^{N} \int_\Omega \nabla^2 j(u_{\Omega})(u^1_{\Omega,i}, u^1_{\Omega,i}) \, dx + \frac{1}{2} \int_\Omega \nabla j(u_{\Omega}) \cdot u^2_{\Omega} \, dx,$$

where the $u^1_{\Omega,i}, i = 1, \ldots, N$, and $u^2_{\Omega}$ are the reduced states.
The cost function is

\[ C(\Omega, E) = \int_{\Omega} k(x) |u_{\Omega,E} - u_0|^2 \, dx, \]

where \( k \) is a localization factor, and \( u_0 \) is a target displacement, cooked so that the jaws close.

\[ Setting\ of\ the\ gripping\ mechanism\ example. \]
The perturbations $\hat{E}(x, \omega)$ are known via their two-point correlation function:

$$\text{Cor}(\hat{E})(x, y) := \int_\mathcal{O} \hat{E}(x, \omega)\hat{E}(y, \omega) \, \mathbb{P}(d\omega) = \beta^2 e^{-\frac{|x-y|}{\ell}},$$

where $\beta$ is a scaling factor, and $\ell$ is a characteristic length.

A Karhunen-Loève expansion of $\hat{E}$ is performed, then truncated:

$$\hat{E}(x, \omega) \approx \sum_{i=1}^{N} \sqrt{\lambda_i} f_i(x) \xi_i(\omega),$$

where the $(\lambda_i, f_i)$ are the eigenpairs of the Hilbert-schmidt operator

$$L^2(D) \ni f \mapsto \int_D \text{Cor}(\hat{E})(x, y) f(y) \, dx \in L^2(D),$$

and the $\xi_i(\omega) = \int_D \hat{E}(x, \omega) f_i(x) \, dx$ are normalized and uncorrelated random variables.
Optimal shapes associated to values of $\beta = 0, 0.5, 1, 1.5, 2, 2.5$. 
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Perturbations of a shape $\Omega \in \mathcal{U}_{ad}$ are considered with the structure:

$$
\Omega \mapsto (I + \chi(x)\hat{\nu}(x, \omega)n_{\Omega}(x))(\Omega),
$$

where:

- $\chi$ is a cutoff function, vanishing on $\Gamma_D \cup \Gamma_N$,
- $n_{\Omega}$ is (an extension of) the normal vector to $\partial\Omega$,
- The scalar field $\hat{\nu} \in L^\infty(\mathcal{O}, C^2, \infty(\mathbb{R}^d))$ arises as
  $$
  \hat{\nu}(x, \omega) = \sum_{i=1}^{N} \nu_i(x)\xi_i(\omega).
  $$

Perturbation of $\Gamma$ by a vector field $\hat{\nu}$.
Optimization of a L-beam under geometric uncertainties

- The cost function is of the form:

\[ C(\Omega) = \int_{\Omega} j(\sigma(u_\Omega)) \, dx, \]

where \( \sigma(u) = Ae(u) \) is the stress tensor.

- The approximate variance functional reads:

\[ \tilde{V}(\Omega) = \sum_{i=1}^{N} a_{\Omega,i}^2 \text{ with } a_{\Omega,i} = \int_{\Gamma} \left( j(\sigma(u_\Omega)) + Ae(u_\Omega) : e(p_\Omega) - f \cdot p_\Omega \right) v_i \, ds, \]

and the adjoint state \( p_\Omega \in H_{\Gamma_D}^{1}(\Omega)^d \) is the solution of:

\[
\begin{cases}
-\text{div}(Ae(p)) = \text{div}(A \frac{\partial j}{\partial \sigma}(\sigma(u_\Omega))) & \text{in } \Omega, \\
p = 0 & \text{on } \Gamma_D, \\
Ae(p)n = -A \frac{\partial j}{\partial \sigma}(\sigma(u_\Omega))n & \text{on } \Gamma \cup \Gamma_N.
\end{cases}
\]
• Perturbations occur on a subregion $D_p \subset D$; their correlation function is:

$$\text{Cor}(\hat{v})(x, \omega) = \beta^2 e^{-\frac{|x-y|}{\ell}}.$$

• The cost function is $C(\Omega) = \int_{\Omega} \|\sigma(u_\Omega)\|^5 \, dx$, and the objective $C(\Omega) + \delta \sqrt{\tilde{V}(\Omega)}$ is minimized under a volume constraint.

Details of the L-shaped beam test-case.
Optimal shapes in the minimization of the stress-based criterion, where the parameter $\delta$ equals (from the left to the right) 0, 0.5, 2.
Thank you for your attention!


