An introduction to shape and topology optimization

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Part V

Topology optimization

- A glimpse at mathematical homogenization

Prologue : The direct method of the calculus of variations

Let E be a Banach space and $J : E \to \mathbb{R}$. Assume that J is continuous and that J 'tends to infinity at infinity'

$$\forall\, C > 0,\ \exists\, M > 0, \quad \text{s.t.} \quad |x| > M \quad \Rightarrow \quad J(x) > C$$

We consider the optimisation problem

$$\text{find } x_* \in A, \text{ s.t.} \quad J(x_*) \quad = \quad I \ := \inf_{x \in A} J(x)$$

where $A \subset E$ is the set of admissible candidates

Let $(x_n) \subset A$ be a minimizing sequence (such a sequence always exists)

As $J(x_n) \to I$, the sequence (x_n) is bounded in E and so there exists $x_* \in E$ and a subsequence (not renamed) such that

$$x_n \quad \rightharpoonup \quad x_* \quad \text{weakly in } E \text{ as } n \to \infty$$

Prologue (2)

If A is closed for the weak topology, then $u_* \in A$

If J is weakly lower semi-continuous then

$$J(x_*) \leq \liminf_{n\to\infty} J(x_n) = I$$

and we can conclude that x_* is indeed a minimum of the optimisation problem

The hypotheses required for this program are satisfied in particular when

- J is convex and coercive
- A is a (strongly) closed convex set

(and a converse statement is also true)

Prologue (3)

If A and J do not satisfy these conditions, one can seek a relaxation of the optimisation problem

$$\begin{aligned} A^* &= \left\{ x \in E, \quad \text{s.t.} \quad x = \text{weak-*lim} x_n, \quad (x_n) \subset A \right\} \\ J_*(x) &= \inf_{x_n \rightharpoonup x} J(x_n) \end{aligned}$$

and show that the relaxed problem $\min_{x \in A^*} J_*(x)$ has a solution

Non existence (1)

1. Non existence of minimizers

The direct method of the calculus of variations does not apply in general to shape optimisation problems

- In general the set of design parameters is not closed and convex (and sometimes not even a Banach space)

- In general the objective functional is not weakly lower semi-continuous

We illustrate these facts with the following optimisation problem

Non existence (2)

Let $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ be a fixed domain in which we want to find how to distribute 2 materials (= phases) with conductivities α, β with $0 < \alpha < \beta < \infty$

An admissible distribution (= design) is represented by the characteristic function χ of the phase α and the conductivity distribution is given by

$$a_\chi(x) \quad = \quad \alpha\chi(x) + \beta(1 - \chi(x)) \quad x \in \Omega$$

Let $\sigma_0 = |\sigma_0| e_1$ be a fixed vector in the direction $e_1 = (1, 0)$

The voltage potential u_χ resulting from the application of the current σ_0 on $\partial\Omega$ to the design χ is given by

$$\begin{cases} -\mathrm{div}(a_\chi \nabla u_\chi) & = & 0 & \text{in } \Omega \\ a_\chi \nabla u_\chi \cdot n & = & \sigma_0 \cdot n & \text{on } \partial\Omega \end{cases}$$

the variational formulation of which is

$$\forall\, v \in H^1(\Omega), \quad \int_\Omega a_\chi \nabla u_\chi \cdot \nabla v \quad = \quad \int_{\partial\Omega} \sigma_0 \cdot n\, v$$

Note that the constraint $\int_{\partial\Omega} \sigma_0 \cdot n = 0$ is satisfied, and that u_χ is only defined up to a constant

Non existence (3)

The objective function (the dissipated electrostatic energy) is defined by

$$\begin{aligned} J(\chi) &= \int_{\partial\Omega} \sigma_0 \cdot n u_\chi \, ds \, + \, \lambda \int_\Omega (1-\chi)\, dx \\ &= \int_\Omega a_\chi \nabla u_\chi \cdot \nabla u_\chi \, dx \, + \, \lambda \int_\Omega (1-\chi)\, dx \\ &= \min_{\sigma \in H_0} \int_\Omega a_\chi^{-1} \sigma \cdot \sigma \, + \, \lambda \int_\Omega (1-\chi)\, dx \end{aligned}$$

where H_0 is the space

$$H_0 = \left\{ \sigma \in L^2(\Omega), \, \left\{ \begin{array}{rcll} \mathrm{div}(\sigma) & = & 0 & \text{in } \Omega \\ \sigma \cdot n & = & \sigma_0 \cdot n & \text{on } \partial\Omega \end{array} \right. \right\}$$

and where $\lambda > 0$ is a Lagrange multiplier that constrains the amount of the phase β in the design

Non existence (4)

The optimisation problem is:

Find $\chi_* \in L^\infty(\Omega, \{0,1\})$ such that $\quad J(\chi_*) = \inf_{\chi \in L^\infty(\Omega,\{0,1\}} J(\chi)$

Thm : Let $\lambda^- := \frac{|\sigma_0|^2(\beta-\alpha)}{\beta^2} \qquad \lambda^+ := \frac{|\sigma_0|^2(\beta-\alpha)}{\alpha^2}$

1. If $\lambda \leq \lambda^-$ then $\chi \equiv 0$ is the unique minimizer
2. If $\lambda \geq \lambda^+$ then $\chi \equiv 1$ is the unique minimizer
3. If $\lambda^- < \lambda < \lambda^+$ then the optimization problem does not have a minimizer

Non existence (5)

Lemma : Consider $\phi\ : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\phi(a,\sigma) = \frac{1}{a}|\sigma|^2$
Then ϕ is convex and

$$\phi(a,\sigma) \quad = \quad \phi(a_0,\sigma_0) + D\phi(a_0,\sigma_0)\cdot\begin{pmatrix} a-a_0 \\ \sigma-\sigma_0 \end{pmatrix} + \phi(a,\sigma - \frac{a}{a_0}\sigma_0) \qquad (1)$$

Proof : We compute

$$\frac{\partial\phi}{\partial a}(a,\sigma) \;=\; \frac{-1}{a^2}|\sigma|^2 \qquad \frac{\partial\phi}{\partial\sigma_i}(a,\sigma) \;=\; \frac{2\sigma_i}{a}$$

and thus

$$D\phi(a_0,\sigma_0)\cdot\begin{pmatrix} a-a_0 \\ \sigma-\sigma_0 \end{pmatrix} \quad = \quad \frac{-(a-a_0)}{a_0^2}|\sigma_0|^2 + \frac{2\sigma_0\cdot(\sigma-\sigma_0)}{a_0}$$

It is then easy to check that (1) holds

Non existence (6)

Proof of the Thm :

Set $\theta = \frac{1}{|\Omega|}\int_\Omega \chi(x)dx$ and $a_\theta = \frac{1}{|\Omega|}\int_\Omega a_\chi(x)dx = \theta\alpha + (1-\theta)\beta$

Let $\sigma \in H_0$. Then, for i =1,2

$$0 = \int_\Omega \mathrm{div}(\sigma - \sigma_0)x_i = \int_\Omega (\sigma - \sigma_0)\cdot x_i + \int_{\partial\Omega} (\sigma - \sigma_0)\cdot n\, x_i\, ds = \int_\Omega (\sigma_i - \sigma_{0,i})$$

so that

$$\frac{1}{|\Omega|}\int_\Omega \sigma\, dx = \sigma_0 \qquad (2)$$

From the Lemma, we have for any $x \in \Omega$

$$\begin{aligned} a_\chi^{-1}\sigma(x)\cdot\sigma(x) &= \phi(a_\theta,\sigma_0) + D\phi(a_\theta,\sigma_0)\cdot\begin{pmatrix} a_\chi(x) - a_\theta \\ \sigma(x) - \sigma_0 \end{pmatrix} \\ &\quad + \phi(a_\chi(x), \sigma(x) - \frac{a_\chi(x)}{a_\theta}\sigma_0) \\ &\geq a_\theta^{-1}|\sigma_0|^2 + D\phi(a_\theta,\sigma_0)\cdot\begin{pmatrix} a_\chi(x) - a_\theta \\ \sigma(x) - \sigma_0 \end{pmatrix} \end{aligned}$$

Non existence (7)

Integrating obtains

$$\begin{aligned} \int_\Omega a_\chi^{-1}\sigma\cdot\sigma + \lambda(1-\chi)\,dx &\geq |\Omega| a_\theta^{-1}|\sigma_0|^2 + \lambda(1-\theta) \\ &= |\Omega|\frac{|\sigma_0|^2}{a_\theta} + \lambda(1-\theta) \end{aligned}$$

It follows that

$$\begin{aligned} \inf_\chi J(\chi) &= \inf_\chi \inf_{\sigma\in H_0} \int_\Omega a_\chi^{-1}\sigma\cdot\sigma + \lambda(1-\chi) \\ &\geq |\Omega| \inf_\theta \left(\frac{|\sigma_0|^2}{a_\theta} + \lambda(1-\theta)\right) =: \inf_\theta F(\theta) \end{aligned}$$

F is a strictly convex expression of θ and an easy computation shows that

$$\inf_\theta F(\theta) =: I_\lambda = |\Omega| \begin{cases} \frac{|\sigma_0|^2}{\alpha} & \text{if } \lambda \geq \lambda^+ \\ \frac{|\sigma_0|^2}{\beta} + \lambda & \text{if } \lambda \leq \lambda^- \\ 2|\sigma_0|\sqrt{\frac{\lambda}{\beta-\alpha}} - \frac{\alpha\lambda}{\beta-\alpha} & \text{if } \lambda^- < \lambda < \lambda^+ \end{cases}$$

Non existence (8)

Can the lower bound I_λ be attained ?

This would require that (1) is an equality, so that the remainder

$$\phi\Big(a_\chi(x), \sigma_\chi - \frac{a_\chi}{a_\theta}\sigma_0\Big) \quad = \quad 0 \quad a.e. x \in \Omega$$

so that the optimal current σ_χ satisfies

$$\sigma_\chi(x) \quad = \quad \frac{a_\chi}{a_\theta}\sigma_0 \quad a.e. x \in \Omega \tag{3}$$

• If χ is constant in Ω, then either $\chi = 0$ or $\chi = 1$ and $\sigma_\chi = \sigma_0$
One checks that $J(\chi) = I_\lambda$ if $\chi = 1$ and $\lambda \geq \lambda^+$ or if $\chi = 0$ and $\lambda \leq \lambda^-$

• If χ is not constant, then $0 < \theta < 1$ and (3) yields

$$\sigma_\chi \quad = \quad \begin{cases} \frac{\alpha}{a_\theta}\sigma_0 & \text{when } \chi = 1 \\ \frac{\beta}{a_\theta}\sigma_0 & \text{when } \chi = 0 \end{cases}$$

In particular σ_χ cannot match the boundary condition $\sigma_\chi \cdot n = \sigma_0 \cdot n$

Non existence (9)

However, I_λ is indeed the correct value of the minimum when $\lambda^- < \lambda < \lambda^+$

In this case, $F(\theta)$ is minimal for θ_* given by

$$\theta_* \quad = \quad \frac{1}{\beta - \alpha}\left(\beta - |\sigma_0|\sqrt{\frac{\lambda}{\beta - \alpha}}\right), \qquad 0 < \theta_* < 1$$

Let $\quad g(x_2) = \begin{cases} 1 & \text{if } 0 \leq x_2 < \theta_* \\ 0 & \text{if } \theta_* \leq x_2 < 1 \end{cases}$ extended by periodicity to the whole $\mathbb{R}$,

and set $\quad \chi_p(x) = g(px_2)$

As $(\chi_p)_{p\geq 1}$ is a sequence of periodic functions, bounded in $L^\infty(\Omega)$ it converges weakly-* to its average

$$\forall\ v \in L^1(\Omega), \quad \int_\Omega \chi_p(x) v(x) \quad \to \quad \theta \int_\Omega v(x)$$

Non existence (10)

In addition, the solutions to

$$\begin{cases} \operatorname{div}\left(a_{\chi_p}\nabla u_p\right) & = & 0 & \text{in } \Omega \\ a_{\chi_p}\nabla u_p \cdot n & = & \sigma_0 \cdot n & \text{on } \partial\Omega \end{cases}$$

converge, weakly in H^1, to the solution to

$$\begin{cases} \operatorname{div}\left(A_*\nabla u_*\right) & = & 0 & \text{in } \Omega \\ A_*\nabla u_* \cdot n & = & \sigma_0 \cdot n & \text{on } \partial\Omega \end{cases}$$

where $$A_* = \begin{pmatrix} \theta\alpha + (1-\theta)\beta & 0 \\ 0 & \left[\theta\alpha^{-1} + (1-\theta)\beta^{-1}\right]^{-1} \end{pmatrix}$$

And the energies converge

$$\lim_{p\to\infty} \int_\Omega a_{\chi_p}\nabla u_p \cdot \nabla u_p = \int_\Omega A_*\nabla u_* \cdot \nabla u_* = \min_{\sigma\in H_0} \int_\Omega A_*^{-1}\sigma \cdot \sigma$$

Non existence (11)

Since A_* is constant in Ω, u_* can be easily computed

$$\sigma_* \quad = \quad A_* \nabla u_* = \sigma_0 \qquad \text{in } \Omega$$

from which we obtain

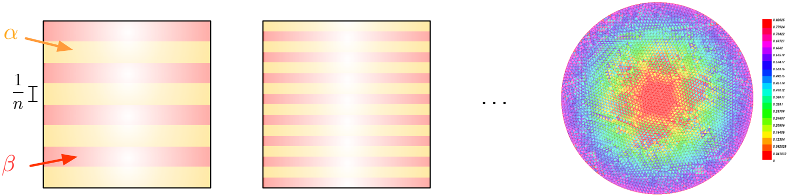
$$\begin{aligned} \int_\Omega A_*^{-1} \sigma_* \cdot \sigma_* \quad &= \quad \int_\Omega \begin{pmatrix} \theta\alpha + (1-\theta)\beta & 0 \\ 0 & [\theta\alpha^{-1} + (1-\theta)\beta^{-1}]^{-1} \end{pmatrix}^{-1} \begin{pmatrix} \sigma_0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sigma_0 \\ 0 \end{pmatrix} \\ &= \quad \frac{|\sigma_0|^2}{a_{\theta_*}} \end{aligned}$$

and we see that

$$J(\chi_p) \quad \rightarrow \quad |\Omega| \Big(\frac{|\sigma_0|^2}{a_{\theta_*}} + \lambda(1-\theta_*) \Big) \; = \; F(\theta_*) \; = \; I_\lambda$$

Non existence (12)

The main reason for this non existence of optimal solution is the homogenization effect: the values of $J(\Omega)$ are improved by sequences of shapes showing smaller and smaller features.



A sequence of shapes showing smaller and smaller features, making $J(\Omega)$ better and better.

Functional analysis (1)

The previous example, where the objective functional involes the compliance shows that

- a sequence of admissible designs $(\chi_n) \subset L^\infty(\Omega, \{0,1\})$ is naturally uniformly bounded

- a subsequence naturally converges to some density $\theta \in L^\infty(\Omega, [0,1])$ in the weak-* topology

- the associated fields u_n are naturally bounded in $H^1(\Omega)$ and a subsequence converges to some $u_* \in H^1(\Omega)$ for the weak topology

- so the question is : what do the energies $\displaystyle\int_\Omega A(\chi_n)\nabla u_n \cdot \nabla u_n$ converge to ?

Functional analysis (2)

Def : Let E be a Banach space with norm $||\cdot||_E$, and E' its dual

- The sequence $(f_n)_n \subset E$ converges strongly to $f \in E$ if

$$||f_n - f||_E \quad \to \quad 0 \quad \text{as } n \to \infty$$

- The sequence $(f_n)_n \subset E$ converges weakly to $f \in E$ if

$$\forall\, \varphi \in E', \quad < f_n, \varphi >_{E,E'} \quad \to \quad < f_n, \varphi >_{E,E'} \quad \text{as } n \to \infty$$

  We write $f_n \rightharpoonup f$

- The sequence $(\varphi_n)_n \subset E'$ converges weakly-* to $\varphi \in E'$ if

$$\forall\, f \in E, \quad < f, \varphi_n >_{E,E'} \quad \to \quad < f, \varphi >_{E,E'} \quad \text{as } n \to \infty$$

  We write $\varphi_n \rightharpoonup \varphi$ as well

Functional analysis (3)

Weak topologies express some form of convergence 'in average'

We are mostly interested in the cases when $E = L^p(\Omega)$ or $E = W^{1,p}(\Omega), 1 \leq p \leq \infty$

Functional analysis (4)

- For $1 < p < \infty$, the dual space of $L^p(\Omega)$ is $(L^p(\Omega))' = L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$f_n \rightharpoonup f \quad \text{weakly in } L^p \quad \Leftrightarrow \quad \int_\Omega f_n \varphi \to \int_\Omega f \varphi \qquad \forall\, \varphi \in L^q(\Omega)$$

- When $p = 1$, $L^1(\Omega)' = L^\infty(\Omega)$

$$f_n \rightharpoonup f \quad \text{weakly in } L^1 \quad \Leftrightarrow \quad \int_\Omega f_n \varphi \to \int_\Omega f \varphi \qquad \forall\, \varphi \in {}^\infty(\Omega)$$

- When $p = \infty$, $(L^\infty(\Omega))'$ is strictly larger than $L^1(\Omega)$ and can be identified as the space of Radon measures

  So weak-* convergence matters in this case

$$f_n \rightharpoonup f \quad \text{weakly-* in } L^\infty \quad \Leftrightarrow \quad \int_\Omega f_n \varphi \to \int_\Omega f \varphi \qquad \forall\, \varphi \in {}^1(\Omega)$$

Thm :

1. If $u_n \to u$ strongly in $L^p(\Omega), 1 \leq p \leq \infty$ there exists $h \in L^p(\Omega)$ and a subsequence such that

$$u_n \to u(x) \ a.e. x \in \Omega, \qquad |u_n(x)| \leq h(x) \ a.e. x \in \Omega$$

2. If $(u_n)_n$ is bounded in $L^p(\Omega)$ and $u_n(x) \to u(x)$ a.e. $x \in \Omega$, then $u_n \to u$ strongly in $L^r(\Omega)$ for any $1 \leq r < p$

3. If $u_n \to u$ strongly in $L^p(\Omega)$, then

$$u_n \quad \rightharpoonup \quad u \quad \text{weakly in } L^p(\Omega)$$

Functional analysis (6)

4. If $u_n \rightharpoonup u$ weakly in $L^p(\Omega), 1 \leq p < \infty$, then u_n is bounded and
$$||u||_{L^p} \quad \leq \quad \liminf_{n\to\infty} ||u_n||_{L^p}$$

5. If $u_n \rightharpoonup u$ weakly in $L^p(\Omega), 1 \leq p < \infty$, and $v_n \to v$ strongly in $(L^p)'(\Omega)$ then
$$\int_\Omega u_n v_n \quad \to \quad \int_\Omega uv$$

However if $u_n \rightharpoonup u$ weakly, one does not have $f(u_n) \rightharpoonup f(u)$ when f is a nonlinear expression

Functional analysis (7)

If $\dim(E) = \infty$, the weak topology contains less open (and closed) sets than the strong topology

However, it contains more compact sets

Thm : (Banach-Alaoglu)

The unit ball $B_{E'} = \{\varphi \in E', \quad \text{s.t.} \quad ||\varphi||_{E'} \leq 1\}$ is compact for the weak-* topology

Consequences for the L^p spaces

- When $1 < p < \infty$, any bounded sequence in $L^p(\Omega)$ contains a weakly convergent subsequence
- When $p = \infty$, any bounded sequence in $L^\infty(\Omega)$ contains a subsequence that converges weakly-*

Functional analysis (8)

Closed sets for the weak topology are also closed for the strong topology

The converse is false in general, except for convex sets

Thm : Let $C \subset E$ be a convex set. Then C is closed for the weak topology if and only if C is closed for the strong topology

Thm : Let $J : E \to]-\infty, +\infty]$ be a convex function which is continuous (respectively lsc) for the strong topology

Then it is continous (rep. lsc) for the weak topology

In particular (in the lsc case)

$$f_n \rightharpoonup f \quad \Rightarrow \quad J(f) \leq \liminf_n J(f_n)$$

Functional analysis (9)

Prop : An important exemple for shape optimization

Let Ω be a bounded open set in $\mathbb{R}^d$ and let $Y = [0,1]^d$ denote the unit cube in $\mathbb{R}^d$

Let $\chi \in L^\infty(Y)$ and extend it as a $Y-$periodic function to the whole $\mathbb{R}^d$

Define for $n \geq 1$ $\quad \chi_n(x) = \chi(nx), \quad x \in \Omega$

Then $\quad \chi_n \rightharpoonup \theta \quad$ weakly-* in $L^\infty \Omega$, where θ is the constant function

$$\theta \quad = \quad \int_Y \chi(y)\, dy$$

Proof : in the 1-d case

Let $\Omega =]a, b[$ be a bounded interval in $\mathbb{R}$, $Y = [0, 1]$ and $\chi(x) \in L^\infty([0, 1])$ extended by periodicity in $\mathbb{R}$

We have to show that for any $\varphi \in L^1(\Omega)$

$$\int_a^b \chi(nx)\varphi(x)\,dx \quad \to \quad \theta \int_a^b \varphi(y)\,dy$$

By density, it suffices to show this for functions φ of the form $\quad \varphi(x) = 1_{]\alpha,\beta[}(x)$

Let $n \geq 1$ and write $\quad \alpha = [n\alpha]/n + r_\alpha, \quad \beta = [n\beta]/n + r_\beta \quad , 0 \leq r_\alpha, r_\beta < 1/n$

Functional analysis (11)

Then we can write for n large enough

$$\int_a^b \chi(nx)\, 1_{]\alpha,\beta[}(x)\, dx \;=\; \int_{[n\alpha]/n+r_\alpha}^{[n\beta]/n+r_\beta} \chi(nx)\, dx$$

$$= \int_{[n\alpha]/n+r_\alpha}^{([n\alpha]+1)/n} \chi(nx)\, dx + \sum_{j=[n\alpha]+1}^{[n\beta]} \int_{j/n}^{(j+1)/n} \chi(nx)\, dx + \int_{[n\beta]/n}^{[n\beta]/n+r_\beta} \chi(nx)\, dx$$

$$= O(\frac{||\chi||_{L^\infty}}{n}) \;+\; \sum_{j=[n\alpha]+1}^{[n\beta]} \frac{1}{n} \int_0^1 \chi(y)\, dy$$

$$\to \Big(\int_0^1 \chi(y)\, dy\Big)(\beta-\alpha) \;=\; \theta \int_a^b 1_{]\alpha,\beta[}(x)\, dx$$

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