An introduction to shape and topology optimization

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Part V

Topology optimization

3. Homogenization

Assume, we want to find the optimal distribution of a given volume of a given material with Hooke's law A, in a given set $\Omega \subset \mathbb{R}^3$ so as to minimize the compliance under given loading conditions

Find $\chi \in L^\infty(\Omega, \{0,1\})$ which minimizes

$$J(\chi) \quad = \quad \int_\Omega Ae(u_\chi) : e(u_\chi) + \lambda \int_\Omega \chi(x)$$

where u_χ is the solution to

$$\begin{cases} \mathrm{div}(Ae(u_\chi)) & = & 0 & \text{in } \Omega \cap \{\chi = 1\} \\ Ae(u_\chi)n & = & g & \text{on } \Gamma_N \subset \partial\Omega \\ u_\chi & = & 0 & \text{on } \Gamma_D \subset \partial\Omega \end{cases}$$

Homogenization (2)

To make the problem less singular, one could replace voids in Ω by a very soft material, with Hooke's law ηA, where $\eta << 1$ is a fixed parameter

The material coefficients then take the form

$$A_\chi(x) \quad = \quad \chi(x)\, A + (1 - \chi(x))\, \eta A$$

and the previous PDE is set in the whole of Ω

The optimization problem then becomes : Find $\chi \in L^\infty(\Omega, \{0, 1\})$ such that χ minimizes

$$J(\chi) \quad = \quad \int_\Omega A_\chi(x) e(u_\chi) : e(u_\chi) + \lambda \int_\Omega \chi(x)$$

where u_χ is the solution to

$$\left\{ \begin{array}{rcl} \mathrm{div}(A_\chi(x) e(u_\chi)) & = & 0 \quad \text{in } \Omega \\ A_\chi e(u_\chi) n & = & g \quad \text{on } \Gamma_N \subset \partial\Omega \\ u_\chi & = & 0 \quad \text{on } \Gamma_D \subset \partial\Omega \end{array} \right. \tag{1}$$

Homogenization (3)

The shape optimization problem then becomes one of finding an optimal distribution of a mixture of 2 phases, with Lamé coefficients A and ηA

One strategy may consist in filling in the whole of Ω with material A and then replacing this material by the weak material ηA at places where the former is least necessary, to match the volume constraint while optimizing the overall rigidity

One could remove material A in big chunks or by drilling many tiny holes

Removing many tiny holes often proves more advantageous. It allows to reduce weight while maintaning some structural rigidity

When the holes become infinitesimally small, the structure effectively behaves like a composite material

Homogenization (4)

In the direct method of the calculus of variation, existence of minimizers was shown by studying the behavior of minimizing sequences

In the context of a mixture of 2 phases, studying minimizing sequences raises the following questions :

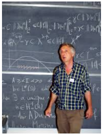
1. Admissible designs χ_n are characteristic functions, thus any minimizing sequence is uniformly bounded in L^∞ : if it converges, its limit θ_* is likely to be a density
2. The associated displacements u_n are bounded in $H^1(\Omega)$. By weak compactness a subsequence converges to a limit u_*. Does u_* satisfy a PDE similar to (**??**) ? What would be the associated (effective) Hooke's law A^* ? What is the relation between θ_* and A^* ?
3. Is there a relation between $\lim_n J(\chi_n)$ and u_*, A^* ?

Homogenization (5)

Homogenization is a mathematical theory of composite materials : it helps answer the above questions

Historically, the first works on effective modulus theory may date back to Poisson (1781-1840)

The term homogenization is due to I. Babuška, and the variational theory was essentially developped by F. Murat and L. Tartar


Here, we are only concerned with periodic homogenization of 2nd order elliptic PDE's

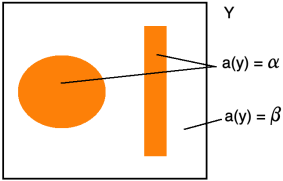
3.1. A formal expansion

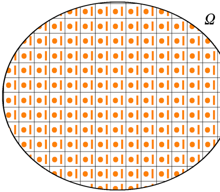
Let Ω be a smooth bounded open set in $\mathbb{R}^d$ and let $Y = (0,1)^d \subset \mathbb{R}^d$

Let $a(y)$ be a Y-periodic function in $\mathbb{R}^d$ such that

$$0 < \alpha_* \leq a(y) \leq \alpha^*, \quad a.e.\ y \in Y$$

and set $a_\varepsilon(x) = a(x/\varepsilon)$ for $x \in \Omega$ and $\varepsilon = 1/n > 0$





A model example in electrostatics (2)

Given $f \in L^2(\Omega)$, we consider the conduction equation

$$\left\{ \begin{array}{lcll} -\mathrm{div}(a_\varepsilon(x)\nabla u_\varepsilon(x)) & = & f & \text{in } \Omega \\ u_\varepsilon(x) & = & 0, & \text{on } \partial\Omega \end{array} \right. \tag{2}$$

which has a unique solution $u_\varepsilon \in H^1_0(\Omega)$

What does u_ε look like when $\varepsilon \to 0$?

A formal expansion

Because of the periodic character of the coefficient a_ε, it is tempting to look for u_ε in the form

$$u_\varepsilon(x) \quad = \quad u_0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \cdots \qquad (3)$$

where the functions $u_j(x, y)$ are Y-periodic functions of the fast variable $y = x/\varepsilon$

Injecting the ansatz (**??**) in the PDE, using that

$$\frac{\partial}{\partial x_j} u_\varepsilon(x) \quad = \quad \sum_p \varepsilon^p \Big(\frac{\partial u_p}{\partial x_j}(x, x/\varepsilon) + \frac{1}{\varepsilon} \frac{\partial u_p}{\partial y_j}(x, x/\varepsilon) \Big)$$

and regrouping terms in powers of ε, one obtains (denoting $y = x/e$) :

A model example in electrostatics (3)

$$
\begin{aligned}
&\mathrm{div}(a_\varepsilon(x)\nabla u_\varepsilon(x)) \\
&= \quad (\mathrm{div}_x + \frac{1}{\varepsilon}\mathrm{div}_y)\Big(a(y)(\nabla_x + \frac{1}{\varepsilon}\nabla_y)\Big)u_\varepsilon \\
&= \quad \frac{1}{\varepsilon^2}\mathrm{div}_y(a(y)\nabla_y u_0) \\
&\quad + \frac{1}{\varepsilon}\Big(\mathrm{div}_y(a(y)\nabla_y u_1) + \mathrm{div}_y(a(y)\nabla_x u_0) + \mathrm{div}_x(a(y)\nabla_y u_0)\Big) \\
&\quad + \varepsilon^0\Big(\mathrm{div}_y(a(y)\nabla_y u_2) + \mathrm{div}_y(a(y)\nabla_x u_1) + \mathrm{div}_x(a(y)\nabla_y u_1) + \mathrm{div}_x(a(y)\nabla_x u_0)\Big) \\
&\quad + \varepsilon \ldots \\
&= \quad -\varepsilon^0 f
\end{aligned}
$$

A model example in electrostatics (4)

Identifying the powers of ε yields

- Terms in ε^{-2} :

$$\mathrm{div}_y(a(y)\nabla_y u_0(x,y)) = 0 \tag{4}$$

which we view as an equation for the Y-periodic function $u_0(x,\cdot)$, considering x as a parameter

Let $H^1_\#(Y)$ denote the closure of the space of Y-periodic $\mathcal{C}^\infty$ functions for the H^1 norm, and let $g \in L^2(Y)$

Under our hypotheses on the coefficient a, we have

Lemma 1 : The variational problem : find $v \in H^1_\#(Y)$ such that

$$-\mathrm{div}(a(y)\nabla_y v(y)) = g, \quad \text{in } Y$$

has a unique solution in $H^1_\#(Y)/\mathbb{R}$ provided $\int_Y g(y)\,dy = 0$

The Lemma thus shows that the first term $u_0(x, y) \sim u_0(x)$ is independent of y

- Terms in ε^{-1} :

$$\begin{aligned}
&\Big(\mathrm{div}_y(a(y)\nabla_y u_1) + \mathrm{div}_y(a(y)\nabla_x u_0) + \mathrm{div}_x(a(y)\nabla_y u_0)\Big) \\
= \quad &\Big(\mathrm{div}_y(a(y)\nabla_y u_1) + \mathrm{div}_y(a(y)\nabla_x u_0)\Big) \;=\; 0
\end{aligned}$$

which we rewrite as and equation for the y-periodic function $u_1(x, \cdot)$

$$-\,\mathrm{div}_y(a(y)\nabla_y u_1) \;=\; \sum_j \frac{\partial u_0}{\partial x_j}(x)\,\mathrm{div}(a(y)e_j) \quad \text{in } Y \tag{5}$$

The periodic character of a shows that one can apply Lemma 1, which yields a solution $u_1 \in H^1_\#(Y)$ to this equation (unique up to a constant w.r.t. y, which however may depend on x)

Note that u_1 depends linearly on the data of equation (**??**) and thus can be written

$$u_1(x, y) \;=\; \sum_{j=1}^{d} \frac{\partial u_0}{\partial x_j}(x)\chi_j(y) \;+\; U_1(x)$$

where the functions $\chi_j, 1 \le j \le d$, are solutions to the cell problems

$$\begin{cases} \mathrm{div}\Big(a(y)\nabla(\chi_j(y) + y_j)\Big) = 0 \quad \text{in } Y \\ \chi_j \in H^1_\#(y) \end{cases} \tag{6}$$

and are called correctors

(or the vector-valued function $\chi = (\chi_j)_{1 \le j \le n}$)

A model example in electrostatics (7)

- Terms in ε^0 : we rewrite them in the form

$$-\mathrm{div}_y(a(y)\nabla_y u_2) \quad = \quad \mathrm{div}_y(a(y)\nabla_x u_1) + \mathrm{div}_x(a(y)\nabla_y u_1) + \mathrm{div}_x(a(y)\nabla_x u_0) + f$$

Invoking Lemma 1 again, this problem is well-posed in $H^1_\#(Y)/\mathbb{R}$ if the RHS has zero average w.r.t. y, i.e.

$$\int_Y \mathrm{div}_y(a(y)\nabla_x u_1) + \mathrm{div}_x(a(y)\nabla_y u_1) + \mathrm{div}_x(a(y)\nabla_x u_0) + f \quad = \quad 0$$

Using the fact that $a(y)\nabla_x u_1$ is Y-periodic, one sees that

$$\int_Y \mathrm{div}_y(a(y)\nabla_x u_1) \quad = \quad 0$$

so that in view of the expression of u_1, the compatibility condition reduces to

$$-\mathrm{div}_x\Big(\int_Y a(y)\Big[I + \nabla\chi(y)\Big]dy\ \nabla_x u_0\Big) \quad = \quad \Big(\int_Y dy\Big) f(x) \ = \ f(x)$$

Thus, u_0 is the solution to a PDE of the form

$$\left\{ \begin{array}{rcll} -\mathrm{div}(A^* \nabla u_0) & = & f & \text{in } \Omega \\ u_0 & = & 0 & \text{on } \partial\Omega \end{array} \right. \tag{7}$$

where the effective conductivity is the constant matrix

$$A^*_{ij} = \int_Y a(y) \left[\delta_{ij} + \frac{\partial \chi_i}{\partial y_j} \right] dy$$

A model example in electrostatics (9)

Remarks :

- The effective conductivity is generally anisotropic, albeit in the case of this example the conductivities $a_\varepsilon(x)$, with fast variations at the microscopic scale, are isotropic

- A^* is symmetric and positive definite

- A^* is given by the following variational principle : for any $\xi \in \mathbb{R}^d$

$$A^*\xi \cdot \xi \quad = \quad \inf \left(\int_Y a(y)(\xi + \nabla w(y)) \cdot (\xi + \nabla w(y))\, dy, \quad w \in H^1_\#(Y) \right)$$

- Assume that $a(y) = \alpha\chi(y) + \beta(1 - \chi(y))$ describes the mixture of 2 phases :

  What are all the A^* that can be achieved by mixing the phases α and β with a given volume fraction of α ?

  = the problem of G-closure

3.2. A convergence result for periodic homogenization

Thm : (Tartar's energy proof)

Assume that the conductivity $a \in L^\infty(\Omega)$ is uniformly elliptic in Ω

$$0 < \alpha_* \leq a(y) \leq \alpha^*, \quad \text{a.e. in } \Omega$$

Let $u_* \in H_0^1(\Omega)$ denote the solution to the homogenized problem

$$a_*(u_*, v) \quad := \quad \int_\Omega A^* \nabla u_* \cdot \nabla v \; = \int_\Omega fv \qquad \forall\, v \in H_0^1(\Omega)$$

Then the solutions $u_\varepsilon \in H_0^1(\Omega)$ to

$$a_\varepsilon(u_\varepsilon, v) \quad := \quad \int_\Omega a_\varepsilon \nabla u_\varepsilon \cdot \nabla v \; = \int_\Omega fv \qquad \forall\, v \in H_0^1(\Omega)$$

converge weakly in H^1 to u_*.

In addition, the energies converge $\displaystyle \int_\Omega a_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \to \int_\Omega A^* \nabla u_* \cdot \nabla u_*$

A convergence result (2)

Remark : Note that in general, the functions u_ε do not converge strongly to u_* in H^1 (in particular their gradients only converge weakly in L^2)

Proof :

- Step 1 : A priori estimates

Recall that $u_\varepsilon \in H_0^1(\Omega)$ solves

$$\forall\, v \in H_0^1(\Omega) \quad \int_\Omega a_\varepsilon \nabla u_\varepsilon \cdot \nabla v \quad = \quad \int_\Omega fv \tag{8}$$

Choosing $v = u_\varepsilon$ shows that

$$\int_\Omega a_\varepsilon |\nabla u_\varepsilon|^2 \quad \leq \quad ||f||_{H^{-1}} ||u_\varepsilon||_{H^1}$$

It follows from the ellipticity and the Poincaré inequality that for some $M > 0$,

$$||u_\varepsilon||_{H^1} \leq M \qquad ||a_\varepsilon \nabla u_\varepsilon||_{L^2} \leq M$$

A convergence result (3)

We thus can extract a subsequence (not re-named) such that

$$\begin{cases} u_\varepsilon & \rightharpoonup & u_* & \text{weakly in } H^1(\Omega) \\ \sigma_\varepsilon := a_\varepsilon \nabla u_\varepsilon & \rightharpoonup & \sigma_* & \text{weakly in } L^2(\Omega) \end{cases}$$

Passing to the limit in (**??**) we se that σ_* satisfies the following equation

$$\forall\, v \in H_0^1(\Omega) \quad \int_\Omega \sigma_* \cdot \nabla v \;=\; \int_\Omega f v \tag{9}$$

• Step 2 : Fix $1 \leq j \leq n$ and consider the j-th corrector

$$\begin{cases} -\mathrm{div}_y\Big(a(y)\nabla_y(\chi_j + y_j)\Big) = 0 \quad \text{in } Y \\ \chi_j \in H^1_\#(Y)/\mathbb{R} \end{cases}$$

Set $w(y) = \chi_j(y) + y_j$ and $w_\varepsilon(x) = \varepsilon w(x/\varepsilon)$

The w_ε satisfies the following equation in $\mathcal{D}'(\Omega)$

$$\mathrm{div}_x\Big(a_\varepsilon(x)\nabla_x w_\varepsilon(x)\Big) = 0 \qquad (10)$$

A convergence result (5)

In addition, note that

$$
\begin{aligned}
w_\varepsilon(x) &= \varepsilon\chi_j(x/\varepsilon) + x_j \to x_j \qquad \text{strongly in } L^2 \\
\frac{\partial w_\varepsilon}{\partial x_i} &= \delta_{ij} + \left(\frac{\partial \chi_j}{\partial y_i}\right)\left(\frac{x}{\varepsilon}\right) \\
&\rightharpoonup \delta_{ij} + \mathcal{M}\left(\frac{\partial \chi_j}{\partial y_i}\right) = \delta_{ij}
\end{aligned}
$$

where $\mathcal{M}(\psi) = \int_Y \psi(y)\,dy$ and the last convergence is in L^2 weak

Note that this last convergence results from the periodicity of $\left(\frac{\partial \chi_j}{\partial y_i}\right)\left(\frac{x}{\varepsilon}\right)$

In short : $w_\varepsilon \rightharpoonup x_j$ weakly in H^1

We would like to use w_ε as a test function, however is does not satisfy the BC's

A convergence result (6)

• Step 3 : Let $\phi \in \mathcal{C}_c^\infty(\Omega)$. Choosing $v = \phi w_\varepsilon$ in the variational formulation (**??**) gives

$$\begin{aligned}
\int_\Omega f\left(\phi w_\varepsilon\right) &= \int_\Omega a_\varepsilon \nabla u_\varepsilon \cdot \nabla(\phi w_\varepsilon) \\
&= \int_\Omega a_\varepsilon \nabla u_\varepsilon \cdot \left[\phi \nabla w_\varepsilon + w_\varepsilon \nabla \phi\right] \\
&= \int_\Omega a_\varepsilon \nabla w_\varepsilon \cdot \nabla\left(\phi u_\varepsilon\right) - a_\varepsilon \nabla w_\varepsilon \cdot \left(u_\varepsilon \nabla \phi\right) + \sigma_\varepsilon w_\varepsilon \nabla \phi \\
&= \int_\Omega -\left(a \nabla w\right)\left(\frac{x}{\varepsilon}\right) \cdot u_\varepsilon \nabla \phi + \sigma_\varepsilon w_\varepsilon \nabla \phi
\end{aligned}$$

where we have used the fact that w_ε satisfies (**??**)

A convergence result (7)

$$\int_\Omega f\left(\phi w_\varepsilon\right) \quad = \quad \int_\Omega -\left(a\nabla w\right)\left(\frac{x}{\varepsilon}\right)\cdot u_\varepsilon \nabla\phi + \sigma_\varepsilon w_\varepsilon \nabla\phi$$

Recall that $\begin{cases} u_\varepsilon, w_\varepsilon & \rightharpoonup & u_*, x_j & \text{weakly in } H^1 & \text{and thus strongly in } L^2 \\ \sigma_\varepsilon & \rightharpoonup & \sigma_* & \text{weakly in } L^2 \end{cases}$

Noting that we can take limits as $\varepsilon \to 0$ in products where one of the terms converges strongly and the other weakly, we obtain

$$\int_\Omega f\left(\phi x_j\right) \quad = \quad \int_\Omega -\mathcal{M}\left(a\nabla w\right)\cdot u_* \nabla\phi + \sigma_* \cdot x_j \nabla\phi$$

A convergence result (8)

so that recalling the equation satisfied by σ_* yields

$$\int_\Omega \sigma_* \cdot \left(\phi \nabla x_j + x_j \nabla \phi\right)$$

$$= \int_\Omega \sigma_* \cdot \nabla\left(\phi x_j\right) = \int_\Omega f\left(\phi x_j\right)$$

$$= \int_\Omega -\mathcal{M}\left(a \nabla w\right) \cdot u_* \nabla \phi + \sigma_* \cdot x_j \nabla \phi$$

which we simplify after integration by parts to get

$$\int_\Omega \left(\sigma_* \cdot \nabla x_j\right) \phi = \int_\Omega \left(\mathcal{M}\left(a \nabla w\right) \cdot \nabla u_*\right) \phi$$

A convergence result (9)

As ϕ was arbitrary, we see that

$$\sigma_* e_j = \Big[\int_Y a(y)\nabla(y_j + \chi_j)\, dy\Big]\nabla u_*$$

$$\sigma_* = \Big[\int_Y a(y)\Big(I + \nabla\chi\Big)\, dy\Big]\nabla u_* = A^*\nabla u_*$$

We conclude that $u_\varepsilon \rightharpoonup u_*$ weakly in H^1, where u_* is the solution in $H_0^1(\Omega)$ to

$$\forall\ v \in H_0^1(\Omega) \quad \int_\Omega \sigma_* \cdot \nabla v = \int_\Omega A^*\nabla u_* \cdot \nabla v = \int_\Omega fv$$

A few remarks

- Essentially the same approach can be carried out for any 2nd order elliptic PDE or system of strongly elliptic PDE's

  In particular one can homogenize the Helmholtz equations, the Maxwell equations, the system of elasticity.

- In the latter case, the tensor of homogenized coefficients is given in terms of a cell problem in the form : for any $\xi \in \mathbb{M}^3_s$

$$A^*\xi : \xi \quad = \quad \inf \left\{ \int_\Omega A(y)(\xi + e(w)) : (\xi + e(w))\, dy, \quad w \in H^1_\#(Y, \mathbf{R}^3) \right\}$$

where $A(y)$ is the microscopic tensor of Lamé coefficients

A few remarks (3)

- Homogenization can be generalized to non periodic settings : quasi-periodicity, G and H- convergence (De Giorgi, Murat-Tartar) and to other notions of variational convergence (Γ-convergence)

Def : A sequence of fields A_ε H-converges to a field A_* if for any $f \in V'$, the solutions $u_\varepsilon \in V$ to

$$\forall\, v \in V \quad \int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla v \;=\; < f, v >$$

converge weakly in V to the solution $u_* \in V$ of

$$\forall\, v \in V \quad \int_\Omega A_* \nabla u_* \cdot \nabla v \;=\; < f, v >$$

A celebrated theorem of Tartar's shows that any uniformly elliptic and uniformly bounded sequence of fields A_ε has a H-converging subsequence

A few remarks (3)

- Homogenization has a local character : a result of Tartar (-Kohn-Dal Maso) states that if A_* is a field that can be obtained as the H-limit of mixtures of 2 phases, then for $a.e.x \in \Omega$ the tensor $A_*(x)$ can be constructed by periodic homogenization

- Extensions exist to degenerate cases : perforated media, porous media-Darcy law, assemblages of thin structures, high contrast coefficients, random coefficients...

- There exist a rich and vast body of work on homogenization : homogenization via Floquet-Bloch expansions, 2-scale convergence, homogenization of eigenvalue problems, of rough boundaries, homogenization in the case of dilute phases,...

A few remarks (4)

- The above proof is due to Tartar, who had the idea to use oscillating test functions in the variational formulation for the u_ε's to *compensate* for the oscillating nature of the latter

  This has lead to the theory of compensated compactness and to the notion of 2-scale convergence

Functional analysis (1)

The previous example, where the objective functional involes the compliance shows that

- a sequence of admissible designs $(\chi_n) \subset L^\infty(\Omega, \{0,1\})$ is naturally uniformly bounded

- a subsequence naturally converges to some density $\theta \in L^\infty(\Omega, [0,1])$ in the weak-* topology

- the associated fields u_n are naturally bounded in $H^1(\Omega)$ and a subsequence converges to some $u_* \in H^1(\Omega)$ for the weak topology

- so the question is : what do the energies $\displaystyle\int_\Omega A(\chi_n)\nabla u_n \cdot \nabla u_n$ converge to ?

Functional analysis (2)

Def : Let E be a Banach space with norm $||\cdot||_E$, and E' its dual

- The sequence $(f_n)_n \subset E$ converges strongly to $f \in E$ if

$$||f_n - f||_E \quad \to \quad 0 \quad \text{as } n \to \infty$$

- The sequence $(f_n)_n \subset E$ converges weakly to $f \in E$ if

$$\forall\, \varphi \in E', \quad < f_n, \varphi >_{E,E'} \quad \to \quad < f_n, \varphi >_{E,E'} \quad \text{as } n \to \infty$$

  We write $f_n \rightharpoonup f$

- The sequence $(\varphi_n)_n \subset E'$ converges weakly-* to $\varphi \in E'$ if

$$\forall\, f \in E, \quad < f, \varphi_n >_{E,E'} \quad \to \quad < f, \varphi >_{E,E'} \quad \text{as } n \to \infty$$

  We write $\varphi_n \rightharpoonup \varphi$ as well

Functional analysis (3)

Weak topologies express some form of convergence 'in average'

We are mostly interested in the cases when $E = L^p(\Omega)$ or $E = W^{1,p}(\Omega), 1 \leq p \leq \infty$

Functional analysis (4)

- For $1 < p < \infty$, the dual space of $L^p(\Omega)$ is $(L^p(\Omega))' = L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$f_n \rightharpoonup f \quad \text{weakly in } L^p \quad \Leftrightarrow \quad \int_\Omega f_n\varphi \to \int_\Omega f\varphi \qquad \forall\, \varphi \in L^q(\Omega)$$

- When $p = 1$, $L^1(\Omega)' = L^\infty(\Omega)$

$$f_n \rightharpoonup f \quad \text{weakly in } L^1 \quad \Leftrightarrow \quad \int_\Omega f_n\varphi \to \int_\Omega f\varphi \qquad \forall\, \varphi \in {}^\infty(\Omega)$$

- When $p = \infty$, $(L^\infty(\Omega))'$ is strictly larger than $L^1(\Omega)$ and can be identified as the space of Radon measures

  So weak-* convergence matters in this case

$$f_n \rightharpoonup f \quad \text{weakly-* in } L^\infty \quad \Leftrightarrow \quad \int_\Omega f_n\varphi \to \int_\Omega f\varphi \qquad \forall\, \varphi \in {}^1(\Omega)$$

Functional analysis (5)

Thm :

1. If $u_n \to u$ strongly in $L^p(\Omega), 1 \leq p \leq \infty$ there exists $h \in L^p(\Omega)$ and a subsequence such that

$$u_n \to u(x) \; a.e. x \in \Omega, \qquad |u_n(x)| \leq h(x) \; a.e. x \in \Omega$$

2. If $(u_n)_n$ is bounded in $L^p(\Omega)$ and $u_n(x) \to u(x)$ a.e. $x \in \Omega$, then $u_n \to u$ strongly in $L^r(\Omega)$ for any $1 \leq r < p$

3. If $u_n \to u$ strongly in $L^p(\Omega)$, then

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega)$$

Functional analysis (6)

4. If $u_n \rightharpoonup u$ weakly in $L^p(\Omega), 1 \leq p < \infty$, then u_n is bounded and
$$||u||_{L^p} \quad \leq \quad \liminf_{n\to\infty} ||u_n||_{L^p}$$

5. If $u_n \rightharpoonup u$ weakly in $L^p(\Omega), 1 \leq p < \infty$, and $v_n \to v$ strongly in $(L^p)'(\Omega)$ then
$$\int_\Omega u_n v_n \quad \to \quad \int_\Omega uv$$

However if $u_n \rightharpoonup u$ weakly, one does not have $f(u_n) \rightharpoonup f(u)$ when f is a nonlinear expression

Functional analysis (7)

If $\dim(E) = \infty$, the weak topology contains less open (and closed) sets than the strong topology

However, it contains more compact sets

Thm : (Banach-Alaoglu)

The unit ball $B_{E'} = \{\varphi \in E', \quad \text{s.t.} \quad ||\varphi||_{E'} \leq 1\}$ is compact for the weak-* topology

Consequences for the L^p spaces

- When $1 < p < \infty$, any bounded sequence in $L^p(\Omega)$ contains a weakly convergent subsequence
- When $p = \infty$, any bounded sequence in $L^\infty(\Omega)$ contains a subsequence that converges weakly-*

Functional analysis (8)

Closed sets for the weak topology are also closed for the strong topology

The converse is false in general, except for convex sets

Thm : Let $C \subset E$ be a convex set. Then C is closed for the weak topology if and only if C is closed for the strong topology

Thm : Let $J : E \to]-\infty, +\infty]$ be a convex function which is continuous (respectively lsc) for the strong topology

Then it is continous (rep. lsc) for the weak topology

In particular (in the lsc case)

$$f_n \rightharpoonup f \quad \Rightarrow \quad J(f) \leq \liminf_n J(f_n)$$

Prop : An important exemple for shape optimization

Let Ω be a bounded open set in $\mathbb{R}^d$ and let $Y = [0,1]^d$ denote the unit cube in $\mathbb{R}^d$

Let $\chi \in L^\infty(Y)$ and extend it as a $Y-$periodic function to the whole $\mathbb{R}^d$

Define for $n \geq 1$ $\quad \chi_n(x) = \chi(nx), \quad x \in \Omega$

Then $\quad \chi_n \rightharpoonup \theta \quad$ weakly-* in $L^\infty\Omega$, where θ is the constant function

$$\theta \quad = \quad \int_Y \chi(y)\, dy$$

Proof : in the 1-d case

Let $\Omega =]a, b[$ be a bounded interval in $\mathbb{R}$, $Y = [0, 1]$ and $\chi(x) \in L^\infty([0, 1])$ extended by periodicity in $\mathbb{R}$

We have to show that for any $\varphi \in L^1(\Omega)$

$$\int_a^b \chi(nx)\varphi(x)\,dx \quad \to \quad \theta \int_a^b \varphi(y)\,dy$$

By density, it suffices to show this for functions φ of the form $\quad \varphi(x) = 1_{]\alpha,\beta[}(x)$

Let $n \geq 1$ and write $\quad \alpha = [n\alpha]/n + r_\alpha, \quad \beta = [n\beta]/n + r_\beta \quad, 0 \leq r_\alpha, r_\beta < 1/n$

Then we can write for n large enough

$$\int_a^b \chi(nx)\, 1_{]\alpha,\beta[}(x)\, dx \;=\; \int_{[n\alpha]/n+r_\alpha}^{[n\beta]/n+r_\beta} \chi(nx)\, dx$$

$$= \int_{[n\alpha]/n+r_\alpha}^{([n\alpha]+1)/n} \chi(nx)\, dx + \sum_{j=[n\alpha]+1}^{[n\beta]} \int_{j/n}^{(j+1)/n} \chi(nx)\, dx + \int_{[n\beta]/n}^{[n\beta]/n+r_\beta} \chi(nx)\, dx$$

$$= O(\frac{||\chi||_{L^\infty}}{n}) \;+\; \sum_{j=[n\alpha]+1}^{[n\beta]} \frac{1}{n} \int_0^1 \chi(y)\, dy$$

$$\to \Big(\int_0^1 \chi(y)\, dy\Big)(\beta-\alpha) \;=\; \theta \int_a^b 1_{]\alpha,\beta[}(x)\, dx$$

Bibliography

Ref. for today's lecture I

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