An introduction to shape and topology optimization

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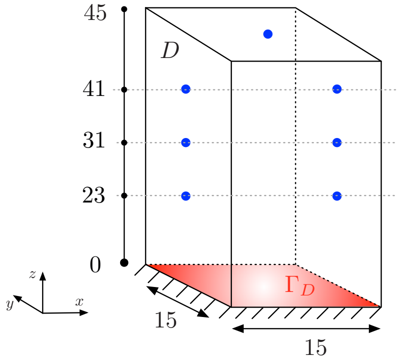
Part VII

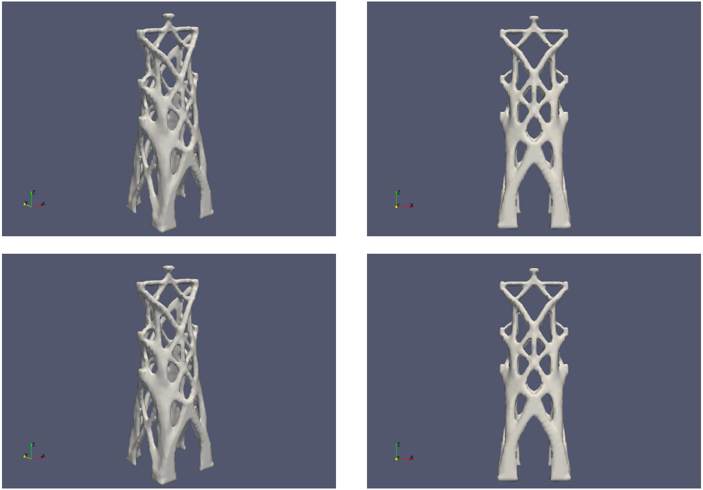
Perspectives and future challenges

Optimization of the shape of a pylon (I)

- A pylon Ω is fixed on a region Γ_D of its boundary;
- Several load scenarii may occur, depending on the conditions of the ambient medium (wind, etc.);
- The physical behavior of Ω is described by the linearized elasticity equations;
- The mean compliance of Ω under the various possible scenarii is minimized, under a volume constraint.



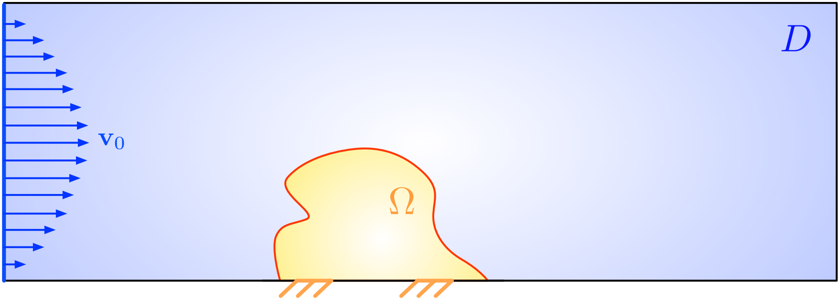
Optimization of the shape of a pylon (II)



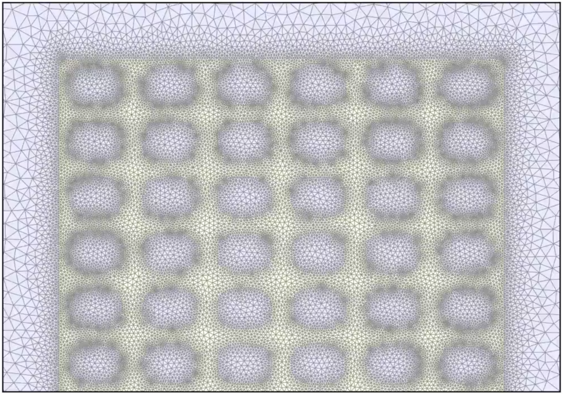
(Top / bottom) Two optimized pylons associated to various sets of constraints.

An example in fluid mechanics (I)

- A fluid is flowing through a pipe D with a given incoming velocity profile.
- An obstacle Ω occupies the pipe, which contains a non optimizable region.
- The fluid is governed by the Navier-Stokes equations, and the behavior of the obstacle is described by the linearized elasticity system.
- The work of the fluid on Ω is minimized under a volume constraint.



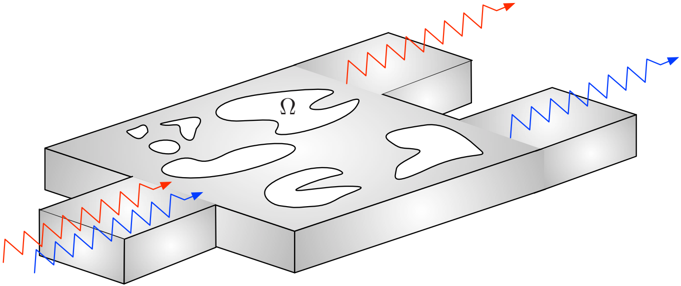
An example in fluid mechanics (II)



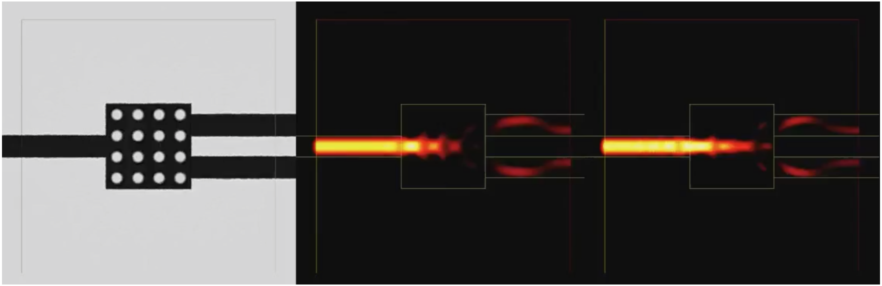
Optimized reinforcement of a pillar subjected to the pressure of a fluid *(Thanks: Florian Feppon).*

An example in electromagnetism (I)

- An electromagnetic wave is conveyed by a waveguide.
- Its eletric and magnetic fields fufilill the Maxwell equations.
- A demultiplexer is a device aimed to steer the incoming wave to different output ports depending on its wavelength.
- The aim is to optimize the distribution Ω of silica in the demultiplexer so as to impose this behavior?



An example in electromagnetism (II)



Optimized shape of a demultiplexer (Thanks: Nicolas Lebbe).

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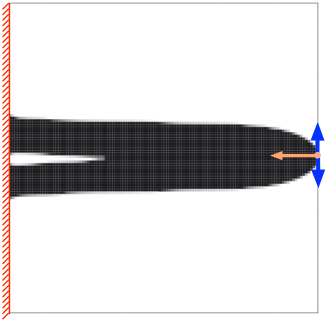
Shape optimization and robustness

- The physical situations in which shape optimization problems arise are characterized by data :
  - The magnitude of the applied loads on an elastic structure;
  - The location where they are applied;
  - The wavelength of a wave conveyed by a guide;
  - The viscosity of a fluid passing through a channel.
- Often, these data are estimated, or measured, in a quite imprecise way.
- It is therefore natural to require that a shape be robust with respect to small perturbations on these data.

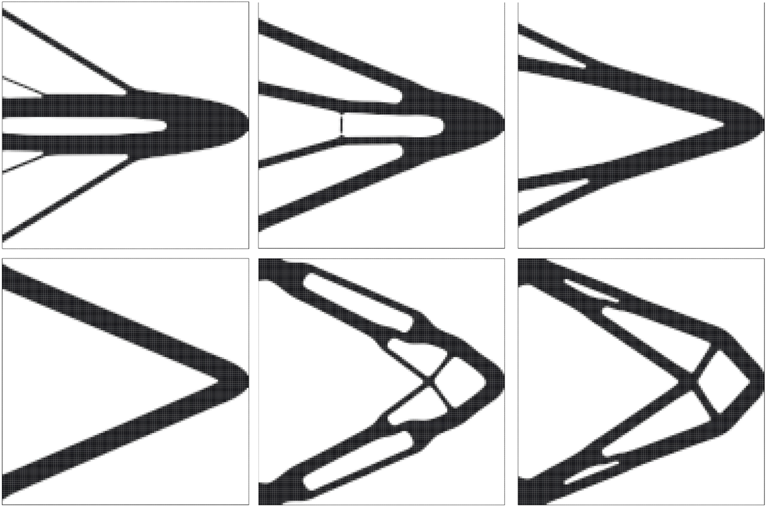
A revealing example [deGAlJou] (I)

- A cantilever beam is submitted to a horizontal load.
- 'Small' vertical perturbations are expected on this load.
- The shape of the beam is optimized under a volume constraint.
- In this particular situation, uncertainties can be taken into account in a rigorous way.



Optimized shape of the beam without taking into account uncertainties on the load.

A revealing example [deGAlJou] (II)



Optimized shape of the cantilever beam under larger and larger uncertainties on the vertical component of the load.

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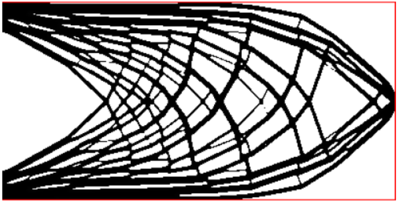
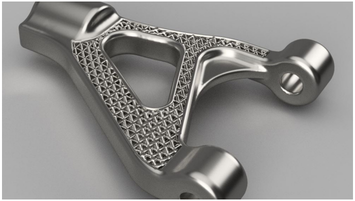
Perspectives and future challenges

Manufacturing the optimized shapes

- Optimized shapes are often too complex to be manufactured by traditional fabrication processes (milling, casting, etc.); for instance:
  - They show thin bars, which are likely to break in the course of the assembly;
  - They contain thick regions, where the cooling of the molten material is difficult.
- One burning challenge is to model the constraints imposed by the fabrication process on the optimized shape Ω
- The modern additive manufacturing (or 3d printing) techniques are expected to allow much more freedom in terms of the constructible designs.

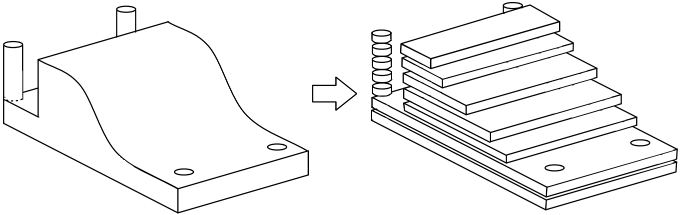




(Left) One 'optimized' shape; (right) one shape assembled by additive manufacturing.

Additive manufacturing in a nutshell (I)

- All additive manufacturing techniques begin with a slicing procedure: the shape is converted into a series of horizontal layers.
- These 2d layers are constructed one atop the other, according to the selected technology.



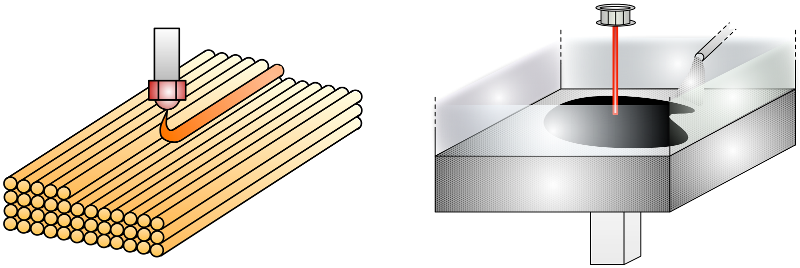
The slicing procedure, initiating any additive manufacturing process.

- In principle, these techniques allow to construct 'arbitrarily complex' shapes.

Additive manufacturing in a nutshell (II)

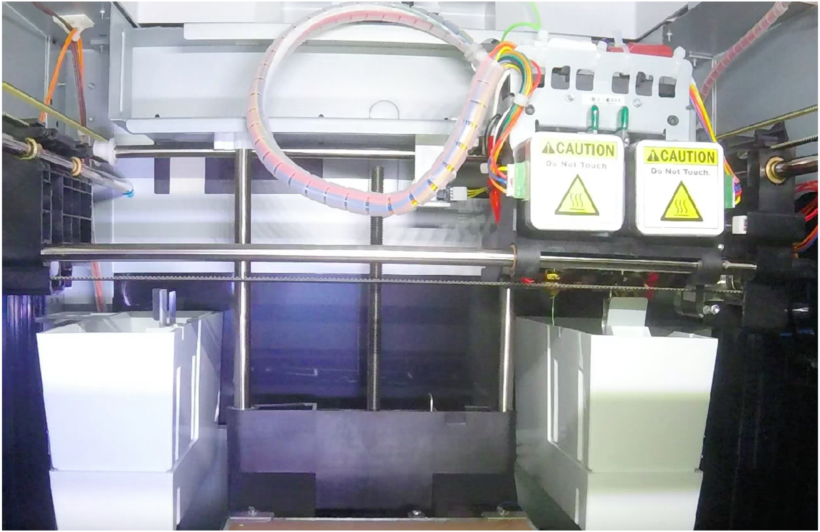
Two of the most popular additive manufacturing technologies are:

- Material extrusion methods (e.g. FDM): they proceed by depositing a molten filament of polymer (e.g. ABS) under the form of rasters.
- Powder bed fusion methods (e.g. EBM, SLS) are mainly used to process metal. The construction of each 2d layer starts by spreading metallic powder inside the build chamber, which is then selectively molten owing to a laser.



Sketch of the (left) FDM technology, and (right) EBM process.

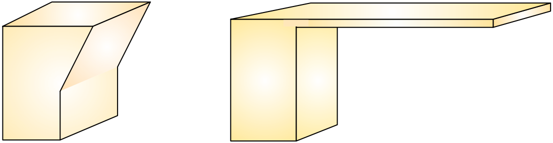
Additive manufacturing in a nutshell (III)



One machine tool for the FDM process in action.

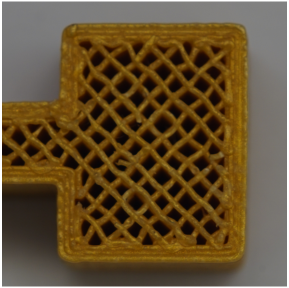
New challenges in connection with additive manufacturing!

- It is difficult to erect large overhanging regions with these techniques.



(Left) One 'small' and (right) one 'large' overhang.

- Materials assembled by additive manufacturing do not have the exact same physical properties as those predicted by theory.



One 2d layer of a structure produced by FDM.

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Conclusions and take-home messages

- Shape and topology optimization methods arise in a wide range of physical contexts: heat conduction, structure mechanics, fluid mechanics, electromagnetism, etc. They are also used in imaging, or shape reconstruction.
- Several vantages are available: parametric, geometric or topology optimization, depending on the nature of the sought design, and on the available means (data, CPU, etc.).
- Accordingly, various notions of shape sensitivity exist: 'classical' Fréchet derivatives, shape derivatives, topological derivatives,...
- ... paving the way to multiple numerical frameworks: parametric optimization algorithms, algorithms featuring an evolving mesh, fixed mesh level set methods, etc.
- Recent, burning challenges have arisen in this thriving field of mathematics, physics and computer science!

Thank you for your attention!

Technical appendix

Change of variable formulas (I)

The next theorem is an extension of the usual change of variables formula (involving a $\mathcal{C}^1$ diffeomorphism) to the case of a Lipschitz diffeomorphism; see [EGar], Chap. 3.

Theorem 1 (Lipschitz change of variables in volume integrals).

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz bounded domain, and $\varphi : \Omega \to \mathbb{R}^d$ be a Lipschitz diffeomorphism of $\mathbb{R}^d$. Then, for any function $f \in L^1(\varphi(\Omega))$, $f \circ \varphi$ is in $L^1(\Omega)$ and:

$$\int_{\varphi(\Omega)} f \, dx = \int_\Omega |\det(\nabla\varphi)| f \circ \varphi \, dx.$$

Remark: The Jacobian determinant $|\det(\nabla\varphi)|$ exists a.e. in Ω, as a consequence of the Rademacher theorem:

A Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ is almost everywhere differentiable.

Change of variable formulas (II)

The following theorem is a version of the change of variables formula adapted to surface integrals; see [HenPi], Prop. 5.4.3.

Theorem 2 (Change of variables in surface integrals).

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $\mathcal{C}^1$ with boundary Γ and unit normal vector n pointing outward Ω. Let $\varphi : \Omega \to \mathbb{R}^d$ be a $\mathcal{C}^1$ diffeomorphism of $\mathbb{R}^d$. Then, for any function $g \in L^1(\varphi(\Gamma))$, $g \circ \varphi$ belongs to $L^1(\Gamma)$ and:

$$\int_{\varphi(\Gamma)} g \, ds = \int_\Gamma |\mathrm{Com}(\nabla \varphi) n| g \circ \varphi \, ds,$$

where $\mathrm{Com}(M)$ is the cofactor matrix of a $d \times d$ matrix.

Remark: The integrand

$$|\mathrm{Com}(\nabla \varphi) n| = |\det(\nabla \varphi)| |\nabla \varphi^{-T} n|$$

is sometimes called the tangential Jacobian of the diffeomorphism φ.

The implicit function theorem

Let us recall the implicit function theorem; see [La], Chap. *I*, Th. 5.9.

Theorem 3 (Implicit function theorem).

Let Θ, E, F be Banach spaces, $\mathcal{V} \subset \Theta$, $U \subset E$ be open sets. and $\mathcal{F} : \mathcal{V} \times U \to G$ be a function of class $\mathcal{C}^p$ for $p \geq 1$. Let $(\theta_0, u_0) \in \mathcal{V} \times U$ be such that $\mathcal{F}(\theta_0, u_0) = 0$ and assume that:

$$d_u\mathcal{F}(\theta_0, u_0) : F \to G \text{ is a linear isomorphism.}$$

Then there exist an open subset $\mathcal{V}' \subset \mathcal{V}$ of θ_0 in Θ and a mapping $g : \mathcal{V}' \to U$ of class $\mathcal{C}^p$ satisfying the properties:

1. $g(\theta_0) = u_0$,
2. *For all $\theta \in \mathcal{V}'$, the equation $\mathcal{F}(\theta, u) = 0$ has a unique solution $u \in E$, given by $u = g(\theta)$.*

The Lax-Milgram theorem

Theorem 4 (Lax-Milgram theorem).

Let V be a Hilbert space, and let $(u,v) \mapsto a(u,v)$ and $v \mapsto \ell(v)$ be a bilinear and a linear form on V, respectively, such that:

- *a is continuous;*
- *a is elliptic (or coercive): there exists $\alpha > 0$ such that:*

$$\forall u \in V, a(u,u) \geq \alpha ||u||^2;$$

- *ℓ is continuous.*

Then the variational problem

$$\text{Search for } u \in V \text{ s.t. } \forall v \in V, \; a(u,v) = \ell(v)$$

has a unique solution in V.

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