

An introduction to shape and topology optimization

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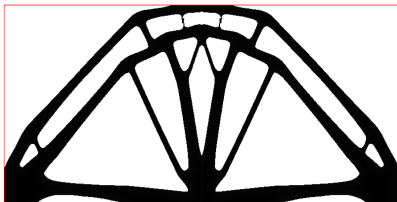
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Foreword

- **Shape optimization** is about the minimization of an **objective** function $J(\Omega)$, depending on a **shape** Ω of \mathbb{R}^2 or \mathbb{R}^3 , under certain **constraints**.
- Such problems have come up early in the history of sciences, and they are ubiquitous in nature.
- Nowadays, they arouse a tremendous enthusiasm in engineering.
- They are at the interface between mathematics, physics, mechanical engineering and computer science.
- Shape optimization is a burning field of research!



Contents

- The present course is composed of
 - 12 **lectures**, covering the main theoretical aspects;
 - A set of **appendices**, at the end of the slides, where basic notions are recalled, and topics related to those of the course are broached.
 - A set of **codes**, dedicated to the numerical implementation of basic shape and topology optimization algorithms, written in FreeFem++.
- All the material for the course (slides of the lectures and commented, demonstration programs) is available on the webpage of the course:

<https://ljk.imag.fr/membres/Charles.Dapogny/coursoptim.html>

- For any comment, suggestion or question, do not hesitate to contact either of the instructors:

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Part I

Introduction, history and generalities about shape optimization

- 1 Some selected milestones in shape optimization
 - Dido's problem and the isoperimetric inequality
 - Shape optimization in architecture
 - Towards "modern" shape and topology optimization
- 2 Generalities about shape optimization problems and examples

Dido's problem (I)

- **Dido's problem** is reported in the myth of the **foundation of Carthage** by Phœnician princess Dido, in 814 B.C. (cf. Virgil's *Aeneid*, ≈ 100 B.C.).
- Dido fled from Tyr (actual Lebanon) after her husband got murdered by her brother Pygmalion.
- Accompanied by her fellows, she reached the Tunisian shore, where she required a land from local king Jarbas...
- *... They came to this spot, where to-day you can behold the mighty Battlements and the rising citadel of New Carthage,
And purchased a site, which was named 'Bull's Hide' after the bargain
By which they should get as much land as they could enclose with a bull's hide...*

[Virgil, Aeneid]

Dido's problem (II)

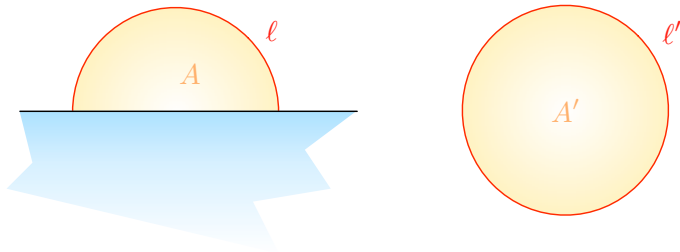


W. Turner: *"Dido Building Carthage" or "The Rise of the Carthaginian Empire"* (1815).

Dido's problem (III)

Using modern terminology:

How to surround the largest possible area A with a given contour length ℓ ?



(Left) The solution to Dido's problem in the case where the surrounded domain is limited by the sea; (right) an "unconstrained" version of Dido's problem.

The isoperimetric inequality (I)

- Without knowing it, Queen Dido had just discovered the **isoperimetric inequality**:

Let $\Omega \subset \mathbb{R}^2$ be a domain with “smooth enough” boundary $\partial\Omega$. Let A be the area covered by Ω , and ℓ be the length of $\partial\Omega$. Then,

$$4\pi A \leq \ell^2,$$

where equality holds if and only if Ω is a disk.

- Equivalently,

- Among all domains $\Omega \subset \mathbb{R}^2$ with **prescribed perimeter**, that with **maximum area** is the disk.
- Among all domains $\Omega \subset \mathbb{R}^2$ with **prescribed area**, that with **minimum perimeter** is the disk.

- Multiple variants of this problem exist.

Example: One may impose that the boundary of Ω should contain a non-optimizable region (a segment).

The isoperimetric inequality (II)

- However intuitive, the first proof of this fact was obtained in 1838 by J. Steiner, ... but the proof was false! Actually, J. Steiner proved that, **assuming** that an optimal shape exist... it should then be a disk.
- However, many shape optimization problems **do not have a solution**, for deep mathematical and physical reasons.
- Only in 1860 did K. Weierstrass complete the proof of the isoperimetric inequality in two dimensions.
- The isoperimetric inequality holds in more general contexts, for instance in three space dimensions (H. Schwarz, 1884):

Among all domains $\Omega \subset \mathbb{R}^3$ with **prescribed volume**, that with **minimum surface** is the ball.

Another occurrence of the isoperimetric inequality

Medieval cities often have a circular shape so as to minimize the perimeter of the necessary fortifications around a given population (i.e. their area).



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Map of Paris during the Dark Ages.

Part I

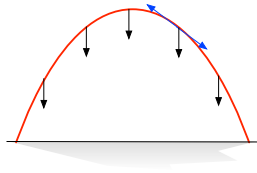
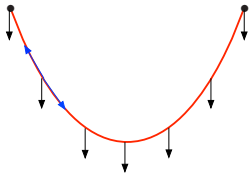
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The quest of architects for optimal design (I)

- Structural optimization has long been a central concern in architectural design.
- One crucial step towards modern design: the **Hooke's theorem** (1675)

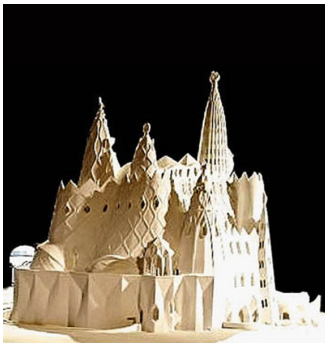
"As hangs the flexible chain, so but inverted will stand the rigid arch."



(Left) A chain hanging in equilibrium under the action of gravity and tension forces; (right) an arch standing in equilibrium under gravity and compression forces.

The quest of architects for optimal design (II)

- A. Gaudi sketched the plans of the church of the Colònia Güell (1889-1914) by relying on a **funicular model** so as to determine a stable assembly of columns and vaults.

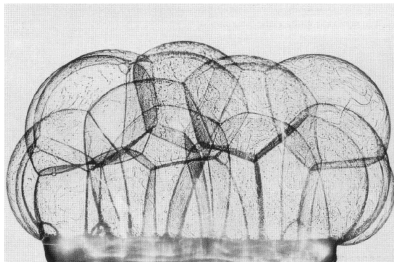


(Left) Gaudi's experimental device, (right) model of the Colònia Güell (Photo credits: <http://www.gaudidesigner.com>).

The quest of architects for optimal design (III)

Since then, optimal design concepts have attracted the attention of world-renowned architects: Heinz Isler, Gustave Eiffel, Frei Otto, etc.

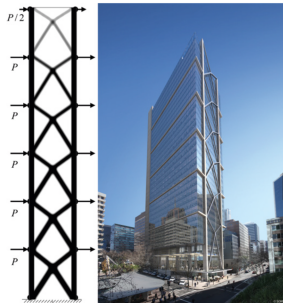
- They allow to model complex geometric criteria, related to the **aesthetics**, the **constructibility**, and the **mechanical performance** of structures.
- Optimized shapes with respect to mechanical considerations have often “elegant” outlines: their **organic** nature is very appreciated by architects.



(Left) A soap-film structure, coined by Frei Otto, (right) interior view of the Manheim Garden festival.

The quest of architects for optimal design (IV)

- Nowadays, modern **structural optimization** techniques are currently employed for the design of large-scale buildings.



(Left) Entrance of the Qatar National Convention Center, in Doha [Sasaki et al]. (Right) Sketch of a 288m high skyscraper in Australia by Skidmore, Owings & Merrill.

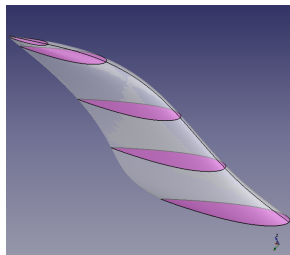
Part I

Introduction, history and generalities about shape optimization

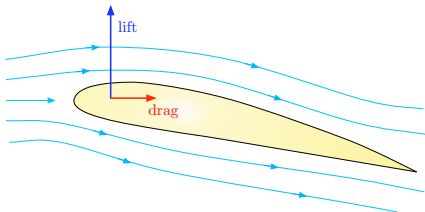
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Towards “modern” shape and topology optimization (I)

- More advanced shape optimization methods have emerged since the 1960's, mainly due to
 - The development of efficient numerical tools for simulating complex physical phenomena (notably the [finite element method](#));
 - The increase in computational power.
- One of the first fields involved is [aeronautics](#), where engineers were motivated to optimize [airfoils](#) so as to
 - Minimize the [drag](#) of aircrafts;
 - Increase their [lift](#).



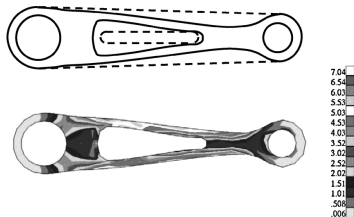
Sketch of the wing of an aircraft



An airfoil subjected to the reaction of air

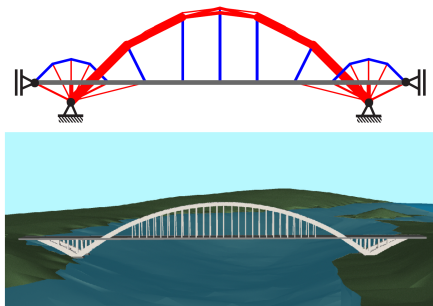
Towards “modern” shape and topology optimization (II)

Concurrently, such computer-aided methods have aroused a great enthusiasm in civil and mechanical engineering.



Optimization of a torque arm (from

[KiWan])



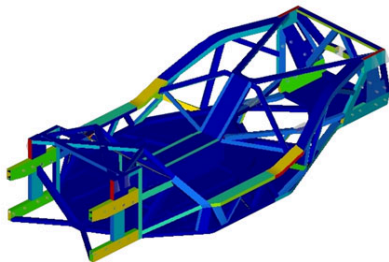
Optimization of an arch bridge (from [ZhaMa])

Towards “modern” shape and topology optimization (III)

- Since then, much headway has been made in the mathematical and algorithmic practice of shape and topology optimization.
- Nowadays, shape and topology optimization techniques are consistently used in industry in a wide variety of situations.
- Several industrial softwares are available: OptiStruct, Ansys, Tosca, etc.



Optimization of a hip prosthesis (Photo credits: [AI])



Optimization of an automotive chassis

(from [CaBa])



Disclaimer

- This course is **very** introductory, and by no means exhaustive, as well for theoretical as for numerical purposes.
- See the (non exhaustive) References section to go further.

Part I

Introduction, history and generalities about shape optimization

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- 2 Generalities about shape optimization problems and examples
 - What is a shape optimization problem?
 - Examples of model problems

What is a shape optimization problem? (I)

- A typical shape optimization problem arises under the form:

$$\min_{\Omega \in \mathcal{U}_{\text{ad}}} J(\Omega), \text{ s.t. } C(\Omega) \leq 0,$$

where

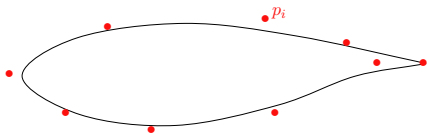
- Ω is the **shape**, or the **design variable**;
 - $J(\Omega)$ is an **objective function** to be minimized;
 - $C(\Omega)$ is a **constraint function**;
 - \mathcal{U}_{ad} is a set of **admissible shapes**;
-
- In this course, the considered problems are motivated by **mechanical** or **physical applications**; $J(\Omega)$ and $C(\Omega)$ often depend on Ω via a **state** u_Ω , solution to a **PDE** posed on Ω (e.g. the linear elasticity system, or the Stokes equations).

What is a shape optimization problem? (II)

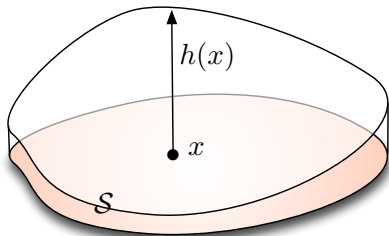
- A shape optimization process is the combination of:
 - A **physical model**, often based on PDE (e.g. the linear elasticity equations, the Stokes system, etc...) describing the mechanical behavior of shapes,
 - A **mathematical representation** of shapes and their variations (e.g. as sets of parameters, density functions, etc...),
 - A **numerical description** of shapes (e.g. by a mesh, a spline model, etc...)
- These choices are strongly inter-dependent and they are often guided by the particular application.
- Roughly speaking, shape and topology optimization problems fall into three main categories: **parametric**, **geometric** and **topology** optimization.
- This classification is quite arbitrary; it mainly reflects a point of view about what is important in the problem. The associated mathematical and numerical methods share a lot of common features.

I. Parametric optimization

The considered shapes are described by means of a set of physical **parameters** $\{p_i\}_{i=1,\dots,N}$, typically thicknesses, curvature radii, etc...



Description of a wing by NURBS; the parameters of the representation are the control points p_i .



A plate with fixed cross-section S is parametrized by its thickness function $h : S \rightarrow \mathbb{R}$.

Various settings for shape optimization (II)

- The parameters describing shapes are the only **optimization variables**, and the shape optimization problem rewrites:

$$\min_{\{p_i\} \in \mathcal{P}_{\text{ad}}} J(p_1, \dots, p_N),$$

where \mathcal{P}_{ad} is a set of **admissible parameters**.

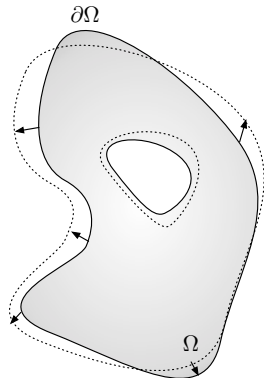
- Parametric shape optimization is eased by the fact that it is straightforward to account for **variations** of a shape $\{p_i\}_{i=1, \dots, N}$:

$$\{p_i\}_{i=1, \dots, N} \rightarrow \{p_i + \delta p_i\}_{i=1, \dots, N}.$$

- However, the variety of possible designs is severely restricted, and the use of such methods relies on an a priori knowledge about the sought optimal design.

II. Geometric shape optimization

- The **topology** of shapes (i.e. their number of holes in 2d) is fixed.
- The whole **boundary** $\partial\Omega$ of shapes Ω is the optimization variable.
- Geometric optimization allows for more freedom than parametric optimization, since no a priori knowledge of the relevant regions of shapes to act on is required.

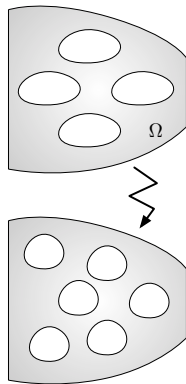


Optimization of Ω via “free” perturbations of the boundary $\partial\Omega$.

III. Topology optimization

- In many applications, the suitable **topology** of shapes is unknown, and it should also be subject to optimization.
- In this context, it is often preferred not to represent the boundaries of shapes, but to employ different descriptions which allow for a more natural account of **topological changes**.

Example Describing shapes Ω as **density functions** $h : D \rightarrow [0, 1]$.



Optimizing a shape by acting on its topology.

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A simplified, academic example (I)

A cavity $D \subset \mathbb{R}^d$ is filled with a material with **thermal conductivity** $h : D \rightarrow \mathbb{R}$.

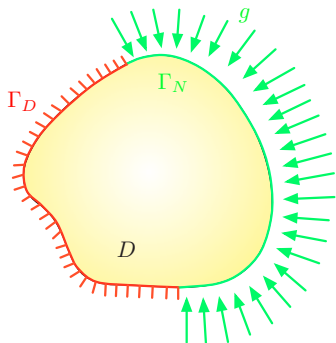
- A region $\Gamma_D \subset \partial D$ is kept at temperature 0.
- A **heat flux** g is applied on $\Gamma_N := \partial D \setminus \overline{\Gamma_D}$.
- A **heat source** or **sink** $f : D \rightarrow \mathbb{R}$ is acting inside D .

The **temperature** $u_h : D \rightarrow \mathbb{R}$ within the cavity is the solution to the **conductivity equation**:

$$\begin{cases} -\operatorname{div}(h\nabla u_h) & = f & \text{in } D, \\ u_h & = 0 & \text{on } \Gamma_D, \\ h \frac{\partial u_h}{\partial n} & = g & \text{on } \Gamma_N. \end{cases}$$

Parametric optimization problem: the **design variable** is the conductivity distribution $h \in \mathcal{U}_{\text{ad}}$, where

$$\mathcal{U}_{\text{ad}} = \{h \in L^\infty(D), \alpha \leq h(x) \leq \beta, x \in D\}.$$



The considered cavity

A simplified, academic example (II)

Examples of objective functions:

- The **compliance** $C(h)$ of the cavity D :

$$C(h) = \int_D h |\nabla u_h|^2 dx = \int_{\Omega} f u_h dx + \int_{\Gamma_N} g u_h ds,$$

measuring the heat power inside D , or the work of the heat sources on D .

- A **least-square error** between u_h and a target temperature u_0 :

$$D(h) = \left(\int_D k(x) |u_h - u_0|^\alpha dx \right)^{\frac{1}{\alpha}},$$

where α is a fixed parameter, and $k(x)$ is a weight factor.

- The opposite of the **first eigenvalue** of the cavity:

$$-\lambda_1(h), \text{ where } \lambda_1(h) = \min_{\substack{u \in H^1(D) \\ u=0 \text{ on } \Gamma_D}} \frac{\int_D h |\nabla u|^2 dx}{\int_{\Omega} u^2 dx},$$

which characterizes the decay rate of the heat inside D in the **transient** version of the conductivity equation.

A simplified, academic example (III)

This problem has a **geometric optimization** variant, where the conductivity inside D takes

- A high value β inside a region $\Omega \subset D$;
- A low value α inside $D \setminus \bar{\Omega}$;

that is:

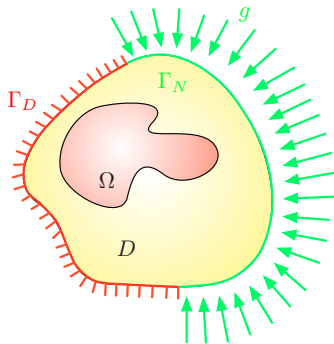
$$h_\Omega = \alpha + \chi_\Omega(\beta - \alpha),$$

where χ_Ω is the characteristic function of Ω .

The **temperature** $u_\Omega : D \rightarrow \mathbb{R}$ is the solution to the **conductivity equation**:

$$\begin{cases} -\operatorname{div}(h_\Omega \nabla u_\Omega) & = f & \text{in } D, \\ u_\Omega & = 0 & \text{on } \Gamma_D, \\ h_\Omega \frac{\partial u_\Omega}{\partial n} & = g & \text{on } \Gamma_N. \end{cases}$$

Geometric optimization problem: the design variable is the geometry Ω of the good conducting phase.



The two-phase conductivity setting

Shape optimization in structural mechanics (I)

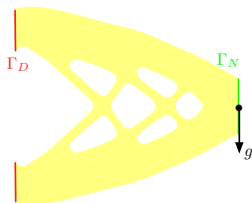
We consider a **structure** $\Omega \subset \mathbb{R}^d$, that is, a bounded domain which is

- **Fixed** on a part $\Gamma_D \subset \partial\Omega$ of its boundary,
- Submitted to **surface loads** g , applied on $\Gamma_N \subset \partial\Omega$,
 $\Gamma_D \cap \Gamma_N = \emptyset$.

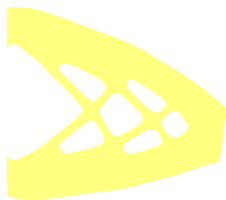
The **displacement** vector field $u_\Omega : \Omega \rightarrow \mathbb{R}^d$ is governed by the **linear elasticity system**:

$$\begin{cases} -\operatorname{div}(Ae(u_\Omega)) & = 0 & \text{in } \Omega, \\ u_\Omega & = 0 & \text{on } \Gamma_D, \\ Ae(u_\Omega)n & = g & \text{on } \Gamma_N, \\ Ae(u_\Omega)n & = 0 & \text{on } \Gamma := \partial\Omega \setminus (\Gamma_D \cup \Gamma_N), \end{cases}$$

where $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the strain tensor field, and A is the Hooke's law of the material.



A "Cantilever" beam



The deformed cantilever

Examples of objective functions:

- The work of the external loads g or **compliance** $C(\Omega)$ of domain Ω :

$$C(\Omega) = \int_{\Omega} A e(u_{\Omega}) : e(u_{\Omega}) \, dx = \int_{\Gamma_N} g \cdot u_{\Omega} \, ds$$

- A **least-square discrepancy** between the displacement u_{Ω} and a target displacement u_0 (useful when designing **micro-mechanisms**):

$$D(\Omega) = \left(\int_{\Omega} k(x) |u_{\Omega} - u_0|^{\alpha} \, dx \right)^{\frac{1}{\alpha}},$$

where α is a fixed parameter, and $k(x)$ is a weight factor.

Shape optimization in structure mechanics (III)

Examples of constraints:

- A constraint on the **volume** $\text{Vol}(\Omega)$, or the **perimeter** $\text{Per}(\Omega)$ of shapes.

$$\text{Vol}(\Omega) = \int_{\Omega} dx, \quad \text{Per}(\Omega) = \int_{\partial\Omega} ds.$$

- A constraint on the **total stress** developed in shapes:

$$S(\Omega) = \int_{\Omega} \|\sigma(u_{\Omega})\|^2 dx,$$

where $\sigma(u) = Ae(u)$ is the **stress tensor**.

- **Geometric constraints**, e.g.
 - Constraints on the **minimal** and **maximum** thickness of shapes;
 - Constraints on their **curvature radii**;

such constraints are often imposed by the **manufacturing process**.

Shape optimization in fluid mechanics (I)

An incompressible fluid with kinematic viscosity ν occupies a domain $\Omega \subset \mathbb{R}^d$.

- The flow u_{in} through the **input boundary** Γ_{in} is known.
- A pressure profile p_{out} is imposed on the **exit boundary** Γ_{out} .
- No slip boundary conditions are considered on the **free boundary** $\partial\Omega \setminus (\Gamma_{\text{in}} \cup \Gamma_{\text{out}})$.

The velocity $u_\Omega : \Omega \rightarrow \mathbb{R}^d$ and pressure $p_\Omega : \Omega \rightarrow \mathbb{R}$ of the fluid satisfy the **Stokes equations**:

$$\begin{cases} -2\nu \operatorname{div}(D(u)) + \nabla p = f & \text{in } \Omega \\ \operatorname{div}(u) = 0 & \text{in } \Omega \\ u = u_{\text{in}} & \text{on } \Gamma_{\text{in}} \\ u = 0 & \text{on } \Gamma \\ \sigma(u)n = -p_{\text{out}} & \text{on } \Gamma_{\text{out}} \end{cases},$$

where $D(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the **rate of strain tensor**.

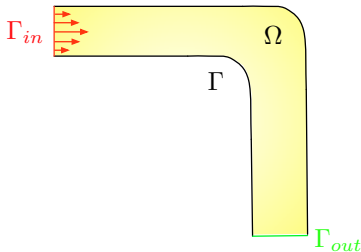
Shape optimization in fluid mechanics (II)

Model problem I: *Optimization of the shape of a pipe.*

- The shape Ω is a pipe, connecting the (fixed) input area Γ_{in} and output area Γ_{out} .
- One is interested in minimizing the **total work of the viscous forces** inside Ω :

$$J(\Omega) = 2\nu \int_{\Omega} D(u_{\Omega}) : D(u_{\Omega}) \, dx.$$

- A constraint on the volume $\text{Vol}(\Omega)$ of the pipe is enforced.

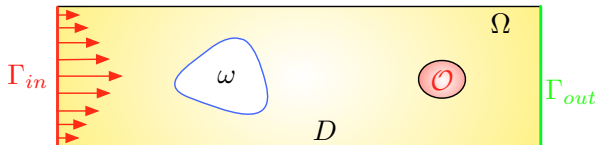


Shape optimization in fluid mechanics (III)

Model problem II: *Reconstruction of the shape of an obstacle.*

- An obstacle of unknown shape ω is immersed in a fixed domain D filled by the considered fluid.
- Given a measure u_{meas} of the velocity u_{Ω} of the fluid inside a **small observation area** \mathcal{O} , one aims at reconstructing the shape of ω .
- The optimized domain is $\Omega := D \setminus \bar{\omega}$, and only the part $\partial\omega$ of $\partial\Omega$ is optimized. One then minimizes the **least-square criterion**:

$$J(\Omega) = \int_{\mathcal{O}} |u_{\Omega} - u_{\text{meas}}|^2 dx.$$



And yet more examples

- *Optimization of the shape of an airfoil:* reducing the **drag** acting on airplanes (even by a few percents) has been a tremendous challenge in the aerodynamic industry for decades.
- *Optimization of the microstructure of composite materials:* in linear elasticity, one is interested in the design of **negative Poisson ratio materials**, etc...
- *Optimization of the shape of **wave guides*** (e.g. optical fibers), in order to minimize the power loss of conducted electromagnetic waves.
- etc...

Why are shape optimization problems difficult?

- *From the modeling viewpoint:* difficulty to describe the physical problem at stake by a model which is relevant (thus complicated enough), yet tractable (i.e. simple enough).
- *From the theoretical viewpoint:* often, optimal shapes do not exist, and shape optimization problems enjoy at best **local optima**.
- *From both theoretical and numerical viewpoints:* the optimization variable is the domain! Hence the need for of a means to **differentiate functions depending on the domain**, and before that, to **parametrize shapes and their variations**.
- *On the numerical side:* difficulty to represent shapes and their evolutions.
- *On the numerical side:* shape optimization problems may be **very sensitive** and can be completely dominated by discretization errors.

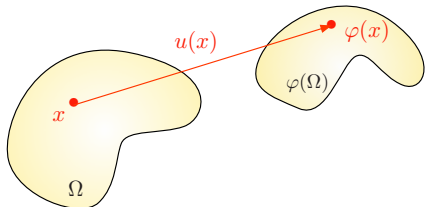
Appendix: physical models

A primer in linear elasticity

Elasticity is the ability of a structure Ω to resist an input **stress**, and to return exactly to its original state when the stress is relieved (\neq **plasticity** or **fracture**).

The motion of an elastic shape Ω is described by:

- The **deformation** $\varphi : \Omega \rightarrow \varphi(\Omega)$;
- The **displacement** $u(x) = \varphi(x) - x$.



Lagrangian point of view: the considered quantity is the motion $u(x)$ of the constituent particles x of the structure at rest Ω , which serves as reference.

The strain tensor (I)

The **Cauchy-Green strain tensor** $C(x)$ measures how φ distorts lengths.

A curve $(0, 1) \ni t \mapsto \gamma(t) \in \Omega$ with **length**

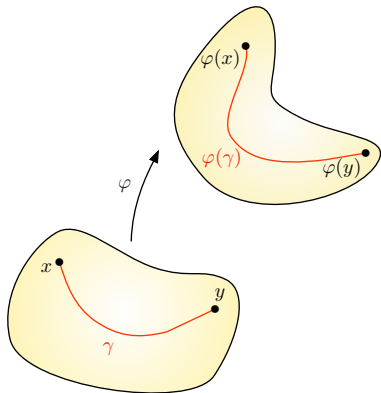
$$l(\gamma) := \int_0^1 |\gamma'(t)| dt;$$

is transformed into $t \mapsto \varphi(\gamma(t))$, with length:

$$l(\varphi(\gamma)) = \int_0^1 \sqrt{(C(\gamma(t))\gamma'(t)) \cdot \gamma'(t)} dt,$$

where

$$\begin{aligned} C(x) &= (\nabla\varphi(x))^T (\nabla\varphi(x)) \\ &= (I + \nabla u(x))^T (I + \nabla u(x)). \end{aligned}$$



The strain tensor (II)

Geometric linearity Assuming that the displacement u is “small”, one approximates:

$$C(x) \approx I + \nabla u(x) + \nabla u^T(x).$$

- The **linearized strain tensor**

$$e(u)(x) := \frac{1}{2}(\nabla u(x) + \nabla u^T(x))$$

then satisfies:

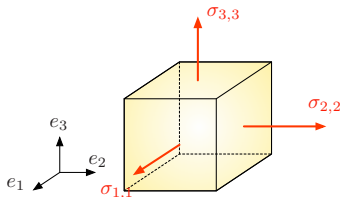
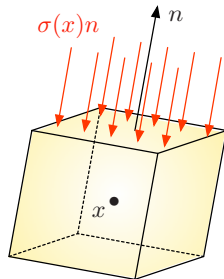
$$\ell(\varphi(\gamma)) \approx \ell(\gamma) + \int_0^1 \left(e(u)(\gamma(t)) \frac{\gamma'(t)}{|\gamma'(t)|} \right) \cdot \frac{\gamma'(t)}{|\gamma'(t)|} |\gamma'(t)| dt.$$

- In the usual Cartesian coordinates, $e(u)$ is defined by:

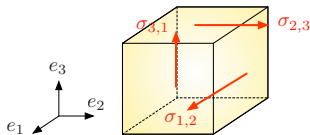
$$e(u)_{i,j} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, \dots, d.$$

The stress tensor

- The **stress tensor** σ encodes the **internal efforts** within the body.
- For $x \in \Omega$, $n \in \mathbb{R}^d$ with $|n| = 1$, $\sigma(x)n \in \mathbb{R}^d$ is the force applied by the surrounding medium on the face oriented by n of a small cube around x .
- **Cauchy's theorem**: as a consequence of balance of momentum, $\sigma(x)$ is a $d \times d$ **symmetric** matrix.



The diagonal entries of σ account for **traction** and **compression** forces.



The off-diagonal entries of σ account for **shear** effects.

The equilibrium equations

- The **equilibrium** equations relate internal efforts with external stresses.
- If **body forces** $f : \Omega \rightarrow \mathbb{R}^d$ (e.g. gravity) occur, it holds, for any subset $\mathcal{V} \subset \Omega$,

$$\underbrace{\int_{\partial\mathcal{V}} \sigma \cdot n \, ds}_{\text{Efforts applied on } \partial\mathcal{V}} + \underbrace{\int_{\mathcal{V}} f \, dx}_{\text{Body efforts within } \mathcal{V}} = 0,$$

and so, using **Green's formula**:

$$\int_{\mathcal{V}} (\operatorname{div}(\sigma) + f) \, dx = 0.$$

Since the latter relation holds for any subset $\mathcal{V} \subset \Omega$, it follows:

$$-\operatorname{div}(\sigma) = f \text{ in } \Omega.$$

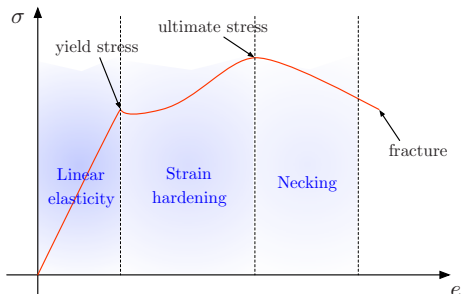
- Likewise, if **external forces** (e.g. traction loads) $g : \partial\Omega \rightarrow \mathbb{R}^d$ are applied, it holds:

$$\sigma n = g \text{ on } \partial\Omega.$$

The constitutive relation (I)

- The system of equations is completed with a **constitutive relation** between the stress tensor σ and the strain tensor $e(u)$, which describes, equivalently,
 - The deformation of a piece of material caused by a given stress;
 - The internal stress induced by an imposed deformation.
- **Material linearity:** σ depends **linearly** on $e(u)$, via the **Hooke's law**:

$$\sigma = Ae(u).$$



The constitutive relation (II)

The **Hooke's law** A of an **isotropic** material reads:

$$\forall e \in \mathcal{S}_d(\mathbb{R}), Ae = 2\mu e + \lambda \text{tr}(e)I.$$

where the **Lamé parameters** λ, μ are related to the more physical quantities E and ν :

$$\mu = \frac{E}{2(1+\nu)}, \text{ and } \lambda = \frac{E\nu}{(1+\nu)(1+\nu(1-d))}.$$

- The **Young's modulus**

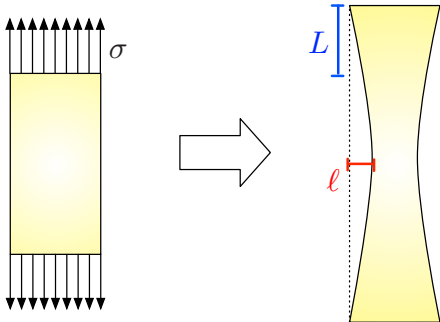
$$E = \sigma/L$$

measures the resistance to deformation under traction;

- The **Poisson's ratio**

$$\nu = -\ell/L$$

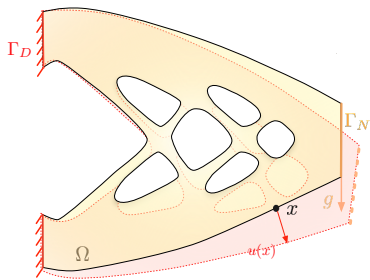
accounts for the relative transverse displacement for a given longitudinal deformation.



Linear elasticity in a nutshell

In most concrete situations,

- The elastic shape Ω is **attached** on a region Γ_D of its boundary;
- **Body forces** $f : \Omega \rightarrow \mathbb{R}^d$ are at play;
- **Surface loads** $g : \Gamma_N \rightarrow \mathbb{R}^d$ are applied on a region $\Gamma_N \subset \partial\Omega$;
- The remaining region $\Gamma := \partial\Omega \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N})$ of $\partial\Omega$ is traction-free.



The **displacement** $u : \Omega \rightarrow \mathbb{R}^d$ of the shape in this context is the unique solution (in $H^1(\Omega)^d$) to:

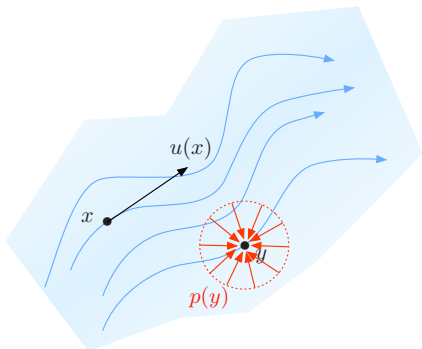
$$\begin{cases} -\operatorname{div}(Ae(u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ Ae(u)n = g & \text{on } \Gamma_N, \\ Ae(u)n = 0 & \text{on } \Gamma. \end{cases}$$

A glimpse of incompressible fluid mechanics

- A **fluid** medium is characterized by its inability to resist a permanent **shear stress**.
- **Eulerian** description: one looks at the properties inside the fluid domain Ω at all positions x , independently of the attached particle (the latter may change).
- We focus on the **steady-state Stokes equations**; see [ChoMar] for more advanced models: time-dependent, Navier-Stokes equations, turbulence phenomena, etc.

The state of the fluid is described in terms of

- The **velocity** $u : \Omega \rightarrow \mathbb{R}^d$;
- The **pressure** $p : \Omega \rightarrow \mathbb{R}$ inside the fluid.



Rate of strain and vorticity

Two important quantities related to the velocity of the fluid are:

- The **rate of strain tensor**:

$$D(u) = \frac{1}{2}(\nabla u + \nabla u^T).$$

- The **vorticity** ω is a scalar field if $d = 2$, and a vector field if $d = 3$:

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \text{ if } d = 2;$$

$$\omega = \nabla \times u = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \text{ if } d = 3.$$

Physical interpretation: A Taylor expansion around $x \in \Omega$ yields (for $d = 3$):

$$u(x+h) = u(x) + D(u)(x)h + \frac{1}{2}\omega(x) \times h + \mathcal{O}(|h|^2);$$

- The transformation $h \mapsto D(u)(x)h$ induces a **stretching of lengths** encoded by the eigenvalues of the symmetric matrix $D(u)$;
- The mapping $h \mapsto \omega(x) \times h$ is a **rotation** with axis $\frac{\omega(x)}{|\omega(x)|}$ and velocity $|\omega(x)|$.

The equilibrium equations (I)

- As in the case of elasticity, the **balance of momentum** at equilibrium implies that:

$$-\operatorname{div}(\sigma) = f \text{ in } \Omega,$$

where:

- $\sigma : \Omega \rightarrow \mathbb{R}^{d \times d}$ is the **stress tensor**;
 - $f : \Omega \rightarrow \mathbb{R}^d$ represents volumic forces (e.g. gravity).
- The fluid is assumed to be **incompressible**: at equilibrium, the mass contained inside each subset $\mathcal{V} \subset \Omega$ is **conserved**:

$$\int_{\mathcal{V}} u \cdot n \, ds = 0,$$

and so, by virtue of **Green's formula**:

$$\operatorname{div}(u) = 0 \text{ in } \Omega.$$

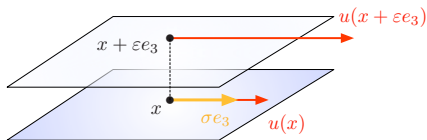
Newton's law

The stress tensor σ is related to the characteristics u , p of the fluid via **Newton's law**:

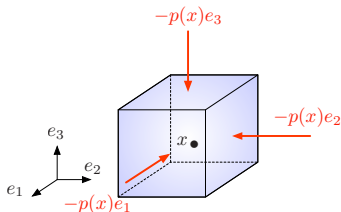
$$\sigma = 2\nu D(u) - pI.$$

Here,

- The **viscous forces** $2\nu D(u)$ are frictional, slowing effects, proportional to the variations of the velocity within the fluid, via the **viscosity** coefficient ν .
- The **pressure** p induces a normal stress on every region of Ω .



The difference between the velocity at x and $x + \varepsilon e_3$ causes a friction force at x , proportional to the viscosity of the fluid.



Pressure forces act in a normal fashion.

In most practical situations,

- The fluid is subjected to **internal forces** $f : \Omega \rightarrow \mathbb{R}^d$;
- The fluid enters the domain Ω via the region $\Gamma_{\text{in}} \subset \partial\Omega$, with a known velocity profile $u_{\text{in}} : \Gamma_{\text{in}} \rightarrow \mathbb{R}^d$;
- The fluid leaves Ω through the region $\Gamma_{\text{out}} \subset \partial\Omega$, with no applied stress;
- The fluid satisfies **no slip boundary conditions** on the remaining region $\Gamma := \partial\Omega \setminus (\overline{\Gamma_{\text{in}}} \cup \overline{\Gamma_{\text{out}}})$, i.e. the fluid “sticks” to the wall.

The **Stokes equations** read in this context:

$$\left\{ \begin{array}{ll} -2\nu \operatorname{div}(D(u)) + \nabla p = f & \text{in } \Omega, \\ \operatorname{div}(u) = 0 & \text{in } \Omega, \\ \sigma(u)n = 0 & \text{on } \Gamma_{\text{out}}, \\ u = u_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ u = 0 & \text{on } \Gamma. \end{array} \right.$$

Appendix: technical facts

Notation for differential calculus (I)

Let Ω be an open subset of \mathbb{R}^d ;

- The **gradient** of a (differentiable) real-valued function $u : \Omega \rightarrow \mathbb{R}$ is the vector field:

$$\forall x \in \Omega, \nabla u(x) = \begin{pmatrix} \frac{\partial u}{\partial x_1}(x) \\ \vdots \\ \frac{\partial u}{\partial x_d}(x) \end{pmatrix}.$$

- The **derivative** of a vector-valued function $v : \Omega \rightarrow \mathbb{R}^d$ at $x \in \Omega$ is the matrix (tensor) field:

$$\forall x \in \Omega, \nabla v(x) = \begin{pmatrix} \frac{\partial v_1}{\partial x_1}(x) & \cdots & \frac{\partial v_1}{\partial x_d}(x) \\ \vdots & \vdots & \vdots \\ \frac{\partial v_d}{\partial x_1}(x) & \cdots & \frac{\partial v_d}{\partial x_d}(x) \end{pmatrix}$$

Notation for differential calculus (II)

- The **divergence** of a vector field $v : \Omega \rightarrow \mathbb{R}^d$ is the function:

$$\begin{aligned}\forall x \in \Omega, \operatorname{div}(v)(x) &= \operatorname{tr}(\nabla v(x)) \\ &= \frac{\partial v_1}{\partial x_1}(x) + \dots + \frac{\partial v_d}{\partial x_d}(x)\end{aligned}$$

- The **divergence** of a tensor field $\sigma : \Omega \rightarrow \mathbb{R}^{d \times d}$ is the vector field whose entries are the divergences of the rows of σ :

$$\forall x \in \Omega, \operatorname{div}(\sigma)(x) = \begin{pmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \dots + \frac{\partial \sigma_{1d}}{\partial x_d} \\ \vdots \\ \frac{\partial \sigma_{d1}}{\partial x_1} + \dots + \frac{\partial \sigma_{dd}}{\partial x_d} \end{pmatrix}.$$

The Green's formula

The **Green's formula** is a generalization of **integration by parts** to the case of multiple space dimensions.

Theorem 1 (Green's formula).

Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain. Then, for any function $u \in W^{1,1}(\Omega)$,

$$\int_{\Omega} \frac{\partial u}{\partial x_i} dx = \int_{\partial\Omega} u n_i ds, \quad i = 1, \dots, d,$$

where $n = (n_1, \dots, n_d)$ is the unit normal vector to $\partial\Omega$, pointing outward Ω .

The Green's formula has a number of useful avatars, such as:


Corollary 2.

Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain. Then, for any $u \in H^2(\Omega)$, $v \in H^1(\Omega)$:

$$\int_{\Omega} \Delta u v dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds - \int_{\Omega} \nabla u \cdot \nabla v dx.$$






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