An introduction to shape and topology optimization

Éric Bonnetier* and Charles Dapogny†

* Institut Fourier, Université Grenoble-Alpes, Grenoble, France
† CNRS & Laboratoire Jean Kuntzmann, Université Grenoble-Alpes, Grenoble, France

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Homogenization and non existence of optimal design (I)

- Let us consider the following shape optimization problem, in the two-phase conductivity setting:

$$\min_{\Omega \subset D} J(\Omega), \text{ where } J(\Omega) := \int_D j(u_\Omega)\,\mathrm{d}x, \qquad \text{(SO)}$$

and $j : \mathbb{R} \to \mathbb{R}$ is a given, smooth function such that:

$$|j(u)| \leq C(1 + |u|^2),$$
$$|j'(u)| \leq C(1 + |u|) \text{ and } |j''(u)| \leq C.$$

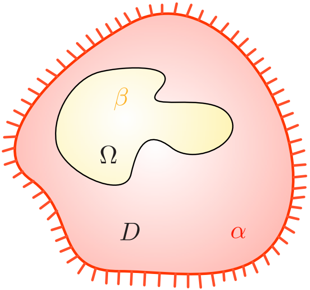
- The temperature $u_\Omega \in H^1_0(D)$ is the solution to:

$$\begin{cases} -\mathrm{div}(\gamma_\Omega \nabla u_\Omega) = f & \text{in } D, \\ u_\Omega = 0 & \text{on } \partial D, \end{cases}$$

where the conductivity γ_Ω is of the form:

$$\gamma_\Omega(x) = \alpha + \chi_\Omega(x)(\beta - \alpha), \quad x \in D.$$

- According to the ersatz material trick, this approximates the one-phase and void problem as $\alpha \to 0$.

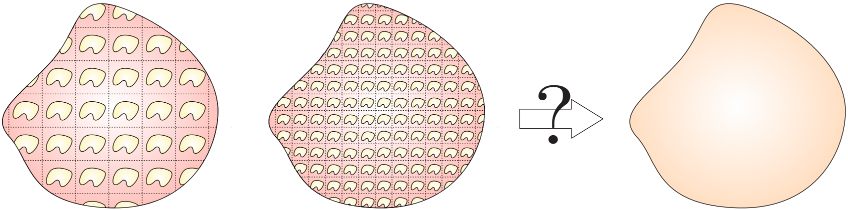


Homogenization and non existence of optimal design (II)

- The shape optimization problem (SO) does not have a solution in general.
- The main reason is the homogenization phenomenon: there exist minimizing sequences of shapes Ω^n, i.e.

$$J(\Omega_n) \xrightarrow{n \to \infty} \inf_{\Omega \subset D} J(\Omega)$$

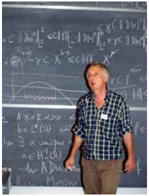
but Ω_n develops smaller and smaller features as $n \to \infty$, and has no limit as a "true" shape.



- One remedy is relaxation: enlarge the set of admissible designs so that it contain the "limiting" or "effective behaviors" of such minimizing sequences.

A wee bit of history

- The first investigations about effective modulus theory in averaged media dates back to Poisson (1781-1840).
- The term "homogenization" was coined by I. Babuska.
- The variational theory of homogenization was developed by F. Murat and L. Tartar.


- An introductory reference to this theory is Chapter 7 in [Allc]; see [Allh] for a more exhaustive and technical presentation.

Part IV

Mathematical homogenization

The direct method in the calculus of variations (I)

Let $(V, ||\cdot||)$ be a Banach space. We consider the minimization problem:

$$\min_{v \in A} J(v), \text{ where } \begin{cases} J : V \to \mathbb{R} \text{ is an objective function;} \\ A \subset V \text{ is a set of admissible points.} \end{cases} \qquad (M)$$

Theorem 1.

Assume that:

1. *V is a reflexive Banach space;*
2. *The function J "tends to infinity at infinity":*
$$\forall M > 0, \; \exists C > 0 \text{ s.t. } ||v|| > C \Rightarrow J(v) > M.$$
3. *The function J is sequentially lower semi-continuous for the weak convergence:*
$$\text{If } v_n \xrightarrow{n\to\infty} v \text{ weakly, then } \; J(v) \leq \liminf_{n\to\infty} J(v_n).$$
4. *The set A is closed for the weak topology of V, i.e. for any sequence $v_n \in A$,*
$$v_n \xrightarrow{n\to\infty} v \text{ weakly in } V \Rightarrow v \in A.$$

Then the problem (M) has a minimum point.

The direct method in the calculus of variations (II)

Proof:

- Let v_n be a minimizing sequence for $J(v)$:

$$v_n \in A, \text{ and } J(v_n) \xrightarrow{n\to\infty} \inf_{v\in A} J(v).$$

Such a sequence exists by the very definition of the infimum.

- The sequence v_n is bounded. Indeed, if it were not the case, there would exist a subsequence v_{n_k} such that:

$$||v_{n_k}|| \xrightarrow{k\to\infty} \infty.$$

Since $J(v)$ "tends to infinity at infinity", this would imply that:

$$J(v_{n_k}) \xrightarrow{k\to\infty} \infty,$$

in contradiction with the fact that v_n is a minimizing sequence for $J(v)$.

- Hence, since V is reflexive, there exists a subsequence v_{n_k} and $v \in V$ such that:

$$v_{n_k} \xrightarrow{k\to\infty} v \text{ weakly in } V.$$

The direct method in the calculus of variations (III)

- Since the set A is weakly closed, the limit v belongs to A.
- From the sequential lower semi-continuity of $J(v)$, it follows:

$$J(v) \leq \liminf_{k \to \infty} J(v_{n_k}) = \inf_{v \in A} J(v),$$

and so the element v satisfies

$$J(v) = \inf_{v \in A} J(v).$$

□

The direct method in the calculus of variations (IV)

In practice, it often happens that:

- The function $J(v)$ fails to be lower semi-continuous;
- The admissible set A fails to be closed for the weak convergence in V.

This reflects the fact that minimizing sequences v_n for $J(v)$ "go to nowhere", i.e. A lacks compactness.

A natural remedy is relaxation, which consists in enlarging the set A as:

$$A^* = \{v \in V \text{ is the weak * limit of some sequence } v_n \in A\},$$

and correspondingly extending the definition of $J(v)$ to $v \in A^*$:

$$J^*(v) = \inf\left\{\liminf J(v_n),\ \ v_n \xrightarrow{n\to\infty} v \text{ weakly}\right\},$$

with the hope to retrieve compactness.

The direct method in the calculus of variations (V)

In typical shape optimization problems of the form

$$\min_{\Omega \in \mathcal{U}_{\text{ad}}} J(\Omega),$$

the set of shapes is not closed in any possible sense, i.e.

Minimizing sequences of shapes converge (in an adapted sense) to "something of a different nature."

This raises the following questions.

- What is the set of relaxed designs?
- How can we characterize this set?
- How can we relate the relaxed problem to the original one?
- How can we take advantage of this relaxation procedure in the context of topology optimization?

Part IV

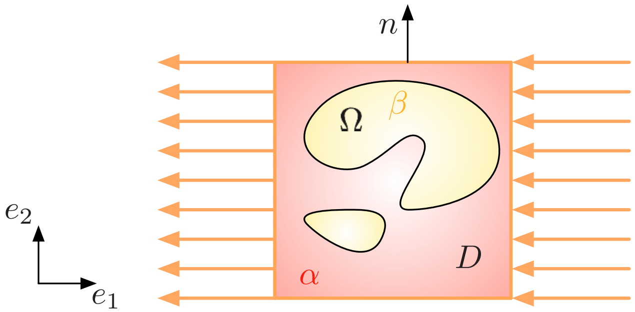
Mathematical homogenization

A non existence example (I)

- Let D be the unit square in $\mathbb{R}^2$, and $n : \partial D \to \mathbb{R}^d$ be the unit normal vector to ∂D.
- For a given subset $\Omega \subset D$, we consider the unique solution $u_\Omega \in H^1(D)/\mathbb{R}$ (that is, up to constants) to:

$$\left\{ \begin{array}{ll} -\text{div}(\gamma_\Omega \nabla u_\Omega) = 0 & \text{in } D, \\ \gamma_\Omega \dfrac{\partial u_\Omega}{\partial n} = n_1 & \text{on } \partial D, \end{array} \right. \quad \text{where } \gamma_\Omega(x) = \left\{ \begin{array}{ll} \beta & \text{in } \Omega, \\ \alpha & \text{in } D \setminus \Omega, \end{array} \right.$$

$0 < \alpha \leq \beta$ and $n_1 = n \cdot e_1$.



A non existence example (II)

We consider the shape optimization problem:

$$\min_{\Omega \in \mathcal{U}_{\mathrm{ad}}} J(\Omega) \text{ s.t. } \mathrm{Vol}(\Omega) = \eta |D|, \quad (|D| = 1), \tag{SO}$$

where

- The set $\mathcal{U}_{\mathrm{ad}}$ of admissible shapes is made of all measurable subsets $\Omega \subset D$;
- $\eta \in (0,1)$ is an imposed volume fraction for Ω;
- The objective function $J(\Omega)$ is the compliance of Ω:

$$J(\Omega) = \int_{\partial D} n_1 u_\Omega \, \mathrm{d}s = \int_D \gamma_\Omega \nabla u_\Omega \cdot \nabla u_\Omega \, \mathrm{d}x.$$

Theorem 2.

The shape optimization problem (SO) does not have a global minimum point.

A non existence example (III)

Let us start with a preliminary result about an alternative, "dual" expression for $J(\Omega)$.

Lemma 3 (Minimization of the complementary energy).

The function $J(\Omega)$ *rewrites:*

$$J(\Omega) = \min_{\sigma \in \Sigma} \int_D \gamma_\Omega^{-1} \sigma \cdot \sigma \,\mathrm{d}x,$$

where the set Σ *is defined by:*

$$\Sigma := \left\{ \sigma \in L^2(D)^2, \; \mathrm{div}\sigma = 0 \text{ in } D \text{ and } \sigma \cdot n = n_1 \text{ on } \partial D \right\}.$$

Remarks:

- See [KoMi] for an elementary and interesting discussion about this trick.
- The big picture behind this result is the duality theory for the Legendre transform; see [BauCom].

A non existence example (IV)

Proof of the lemma:

- From the Lax-Milgram theorem, it holds:

$$J(\Omega) = -2 \min_{u \in H^1(D)} \left(\frac{1}{2} \int_D \gamma_\Omega \nabla u \cdot \nabla u \, \mathrm{d}x - \int_{\partial D} n_1 u \, \mathrm{d}s \right),$$

  where u_Ω is the unique solution (up to constants) to the minimization problem.

- Elementary fact: For any vector $\xi \in \mathbb{R}^2$, and any symmetric, positive definite 2×2 matrix $A \in \mathbb{R}^{2\times 2}$,

$$\frac{1}{2} A\xi \cdot \xi = \max_{\sigma \in \mathbb{R}^2} \left(\xi \cdot \sigma - \frac{1}{2} A^{-1} \sigma \cdot \sigma \right),$$

  where the maximum is uniquely attained at $\sigma = A^{-1}\xi$.

- Then $J(\Omega)$ rewrites

$$J(\Omega) = \max_{u \in H^1(D)} \min_{\sigma \in L^2(D)^2} \mathcal{L}(u, \sigma),$$

  where we have defined

$$\begin{aligned} \mathcal{L}(u,\sigma) &= -2 \int_D \sigma \cdot \nabla u \, \mathrm{d}x + \int_D \gamma_\Omega^{-1} \sigma \cdot \sigma \, \mathrm{d}x + 2 \int_{\partial D} n_1 u \, \mathrm{d}s \\ &= 2 \int_D (\mathrm{div}\sigma) u \, \mathrm{d}x - 2 \int_{\partial D} u \sigma \cdot n \, \mathrm{d}s + \int_D \gamma_\Omega^{-1} \sigma \cdot \sigma \, \mathrm{d}x + 2 \int_{\partial D} n_1 u \, \mathrm{d}s. \end{aligned}$$

A non existence example (V)

- We know from what precedes that $(u, \sigma) = (u_\Omega, \gamma_\Omega \nabla u_\Omega)$ realizes the max-min: it is therefore a saddle point for $\mathcal{L}(u, \sigma)$, and so the min and max can be interchanged:

$$J(\Omega) = \min_{\sigma \in L^2(D)^2} \max_{u \in H^1(D)} \mathcal{L}(u, \sigma).$$

- Now, for a given $\sigma \in L^2(D)^2$, we verify that:

$$\max_{u \in H^1(D)} \mathcal{L}(u, \sigma) = \begin{cases} \int_D \gamma_\Omega^{-1} \sigma \cdot \sigma \, \mathrm{d}x & \text{if } \sigma \in \Sigma, \\ +\infty & \text{otherwise.} \end{cases}$$

where:

$$\Sigma := \left\{ \sigma \in L^2(D)^2, \ \mathrm{div}\sigma = 0 \text{ in } D \text{ and } \sigma \cdot n = n_1 \text{ on } \partial D \right\}.$$

A non existence example (VI)

- Indeed, if $\sigma \in \Sigma$, one has immediately:

$$\mathcal{L}(u,\sigma) = \int_D \gamma_\Omega^{-1} \sigma \cdot \sigma \,\mathrm{d}x.$$

On the other hand, if $\sigma \notin \Sigma$,

  - Either $\mathrm{div}\sigma$ does not vanish identically on D, and so

$$\text{there exists } \varphi \in \mathcal{C}_c^\infty(D) \text{ such that } \int_D (\mathrm{div}\sigma)\varphi \,\mathrm{d}x > 0,$$

  which implies:

$$\max_{u \in H^1(D)} \mathcal{L}(u,\sigma) \geq \mathcal{L}(\lambda\varphi,\sigma) \xrightarrow{\lambda\to+\infty} +\infty.$$

  - Or $\sigma \cdot n$ does not coincide with n_1 on ∂D, and by the same token:

$$\max_{\sigma} \mathcal{L}(u,\sigma) = +\infty.$$

- As a result of this discussion, $J(\Omega)$ has the desired expression:

$$J(\Omega) = \min_{\sigma\in\Sigma} \int_D \gamma_\Omega^{-1}\sigma \cdot \sigma \,\mathrm{d}x.$$

A non existence example (VII)

Sketch of proof of the non existence result: We proceed in three steps.

Step 1: We derive a lower bound for $J(\Omega)$.

- When $\Omega \subset D$ has measure $\mathrm{Vol}(\Omega) = \eta|D|$, the mean value of the conductivity γ_Ω over D is:

$$\gamma_0 := \frac{1}{|D|}\int_D \gamma_\Omega \,\mathrm{d}x = (1-\eta)\alpha + \eta\beta,$$

- The mean value of any $\sigma \in \Sigma$ is exactly e_1, as a result of the following integration by parts, for $i = 1, 2$:

$$\left(\frac{1}{|D|}\int_D \sigma \,\mathrm{d}x - e_1\right)\cdot e_i = \frac{1}{|D|}\int_D (\sigma - e_1)\cdot \nabla x_i \,\mathrm{d}x =$$

$$\frac{1}{|D|}\int_{\partial D} (\sigma - e_1)\cdot n\, x_i \,\mathrm{d}s - \frac{1}{|D|}\int_D (\mathrm{div}\sigma) x_i \,\mathrm{d}x = 0.$$

We now rely on the next lemma, which follows from a straightforward calculation.

Lemma 4.

The function

$$\mathcal{I} : (0,\infty) \times \mathbb{R}^2 \to \mathbb{R},\ \ \mathcal{I}(a,\sigma) = a^{-1}|\sigma|^2$$

has the following exact Taylor expansion about any point (a_0, σ_0)*:*

$$\mathcal{I}(a,\sigma) = \mathcal{I}(a_0,\sigma_0) + \mathcal{I}'(a_0,\sigma_0)(a-a_0,\sigma-\sigma_0) + \mathcal{I}\left(a, \sigma - \frac{a}{a_0}\sigma_0\right),$$

where $$\mathcal{I}'(a_0,\sigma_0)(a-a_0,\sigma-\sigma_0) = -\frac{|\sigma_0|^2}{a_0^2}(a-a_0) + \frac{2}{a_0}\sigma_0 \cdot (\sigma-\sigma_0).$$

It follows that for any measurable subset $\Omega \subset D$ with $\mathrm{Vol}(\Omega) = \eta|D|$,

$$\begin{aligned} J(\Omega) &= \min_{\sigma\in\Sigma} \int_D \mathcal{I}(\gamma_\Omega, \sigma)\,\mathrm{d}x \\ &= \int_D \mathcal{I}(\gamma_0, e_1)\,\mathrm{d}x + \min_{\sigma\in\Sigma}\int_D \mathcal{I}\left(\gamma_\Omega, \sigma - \frac{\gamma_\Omega}{\gamma_0}e_1\right)\mathrm{d}x \\ &\geq \int_D \mathcal{I}(\gamma_0, e_1)\,\mathrm{d}x \\ &= |D|\left((1-\eta)\alpha + \eta\beta\right)^{-1}, \end{aligned}$$

which is the desired lower bound for $J(\Omega)$.

A non existence example (IX)

Second step: The lower bound is not attained.

- Let us look more carefully at the derivation of the lower bound in Step 1:

$$\begin{aligned} J(\Omega) &= \min_{\sigma\in\Sigma} \int_D \mathcal{I}(\gamma_\Omega, \sigma)\,\mathrm{d}x \\ &= \int_D \mathcal{I}(\gamma_0, e_1)\,\mathrm{d}x + \min_{\sigma\in\Sigma} \int_D \mathcal{I}\left(\gamma_\Omega, \sigma - \frac{\gamma_\Omega}{\gamma_0} e_1\right)\,\mathrm{d}x \\ &\geq \int_D \mathcal{I}(\gamma_0, e_1)\,\mathrm{d}x \\ &= |D|\left((1-\eta)\alpha + \eta\beta\right)^{-1}, \end{aligned}$$

- Equality is solely lost in the third line; hence, equality holds if and only if:

$$\mathcal{I}\left(\gamma_\Omega, \sigma - \frac{\gamma_\Omega}{\gamma_0} e_1\right) = \gamma_\Omega^{-1} \left|\sigma(x) - \frac{\gamma_\Omega}{\gamma_0} e_1\right|^2 = 0 \text{ for a.e. } x \in D.$$

- In turn, this only happens when $\sigma = \frac{\gamma_\Omega}{\gamma_0} e_1$ a.e. in D, which is impossible since then σ would not satisfy $\sigma \cdot n = n_1$ on ∂D, as is required from elements of Σ.

We have thus proved the strict lower bound:

$$\forall \Omega \subset D \text{ measurable with } \mathrm{Vol}(\Omega) = \eta|D|, \quad J(\Omega) > |D|\left((1-\eta)\alpha + \eta\beta\right)^{-1}.$$

A non existence example (X)

Third step: Construction of a minimizing sequence.

- We construct a sequence of measurable shapes $\Omega^n \subset D$ such that:

$$\mathrm{Vol}(\Omega^n) = \eta|D|, \text{ and } J(\Omega^n) \to |D|((1-\eta)\alpha + \eta\beta) \text{ as } n \to \infty.$$

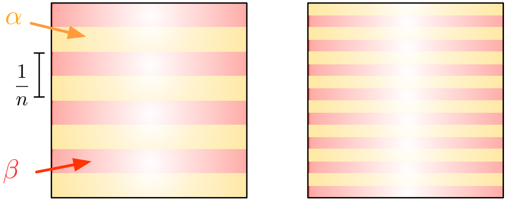
- Let indeed $\chi : (0,1) \to \mathbb{R}$ be the function defined by:

$$\chi(t) = \begin{cases} 1 & \text{if } t \leq \eta, \\ 0 & \text{otherwise,} \end{cases}$$

and let Ω^n be the domain;

$$\Omega^n = \{(x_1, x_2) \in D, \ \chi(nx_2) = 1\};$$

roughly speaking, Ω^n is made of n regularly spaced horizontal strips with width $\frac{\eta}{n}$.



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A non existence example (XI)

- The theory of homogenization will allow to prove that

$$u_{\Omega^n} \xrightarrow{n\to\infty} u^* \text{ weakly in } H^1(D),$$

where $u^* \in H^1(D)/\mathbb{R}$ is the solution to the homogenized problem:

$$\begin{cases} -\mathrm{div}(A^* \nabla u^*) &= 0 & \text{in } D, \\ A^* \nabla u^* \cdot n &= n_1 & \text{on } \partial D, \end{cases}$$

and the homogenized matrix A^* reads:

$$A^* = \begin{pmatrix} (1-\eta)\alpha + \eta\beta & 0 \\ 0 & ((1-\eta)\alpha^{-1} + \eta\beta^{-1})^{-1} \end{pmatrix}.$$

- The function u^* can be calculated in closed form; one verifies indeed that:

$$A^* \nabla u^* = e_1, \text{ and } u^*(x) = ((1-\eta)\alpha + \eta\beta)^{-1} x_1.$$

Finally, taking limits in the definition of $J(\Omega)$ yields immediately:

$$J(\Omega^n) = \int_{\partial D} n_1 u_{\Omega^n} \, \mathrm{d}s \longrightarrow \int_{\partial D} n_1 u^* \, \mathrm{d}s = ((1-\eta)\alpha + \eta\beta)^{-1},$$

which is the desired value.

Non existence in shape and topology optimization

This example reflects a quite general situation in shape and topology optimization:

- When Ω_n is a minimizing sequence for $J(\Omega)$, the corresponding sequence $\chi_{\Omega_n} \in L^\infty(D, \{0,1\})$ of characteristic functions is bounded (by 1).
- Hence, up to a subsequence, χ_{Ω_n} converges weakly * to a density $\theta \in L^\infty(D, [0,1])$.
- The associated sequence u_{Ω_n} is bounded in $H^1(D)$, and so (up to a subsequence) it converges weakly to some element $u^* \in H^1(D)$.
- How can we characterize u^* (via a PDE)?
- What does the energy

$$\int_D \gamma_{\Omega_n} \nabla u_{\Omega_n} \cdot \nabla u_{\Omega_n} \, \mathrm{d}x$$

converge to?

Part IV

Mathematical homogenization

The two-phase conductivity setting

Let us recall our model problem:

- We consider the following shape optimization problem, in the two-phase conductivity setting:

$$\min_{\Omega \subset D} J(\Omega), \text{ where } J(\Omega) := \int_D j(u_\Omega)\, \mathrm{d}x, \qquad \text{(SO)}$$

  and $j : \mathbb{R} \to \mathbb{R}$ is a given, smooth function.

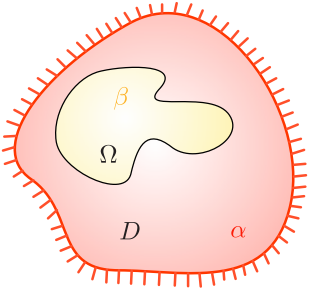
- The temperature $u_\Omega \in H^1_0(D)$ is the solution to:

$$\begin{cases} -\mathrm{div}(\gamma_\Omega \nabla u_\Omega) & = & f & \text{in } D, \\ u_\Omega & = & 0 & \text{on } \partial D, \end{cases} \qquad \text{(C)}$$

  where the conductivity γ_Ω is of the form:

$$\gamma_\Omega(x) = \alpha + \chi_\Omega(x)(\beta - \alpha), \quad x \in D.$$

- How to give a meaning to (C) and (SO) at the "limit" when Ω develops infinitely many, infinitely small patterns?



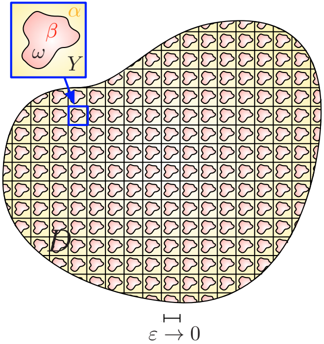
Periodic homogenization

- We mainly deal with the periodic setting.
- Let $Y = (0,1)^d$ be the unit periodicity cell.
- The conductivity inside Y associated to the pattern $\omega \subset Y$ is:

$$A(y) = \begin{cases} \beta & \text{if } y \in \omega, \\ \alpha & \text{if } y \in D \setminus \overline{\omega}. \end{cases}$$

- This pattern induces a conductivity distribution $A\left(\frac{x}{\varepsilon}\right)$ for $x \in D$, by rescaling $A(y)$ at size ε and periodization.
- The state $u_\varepsilon \in H^1_0(D)$ solves:

$$\begin{cases} -\text{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f & \text{in } D, \\ u_\varepsilon = 0 & \text{on } \partial D. \end{cases}$$



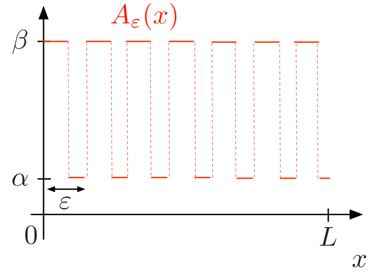
What does u_ε look like, as $\varepsilon \to 0$?

In 1d, let $D = (0, L)$; we solve the problem:

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(A\left(\dfrac{x}{\varepsilon}\right)\dfrac{\mathrm{d}u_\varepsilon}{\mathrm{d}x}\right) = f \quad \text{in } D, \\ u_\varepsilon(0) = u_\varepsilon(1) = 0, \end{cases}$$

where, for $y \in Y = (0, 1)$,

$$A(y) = \begin{cases} \alpha & \text{if } \; y < \theta, \\ \beta & \text{otherwise.} \end{cases}$$



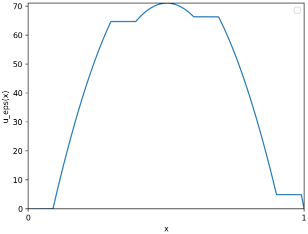
We shall see in a more general context that:

- The sequence u_ε converges to the solution u^* to the partial differential equation

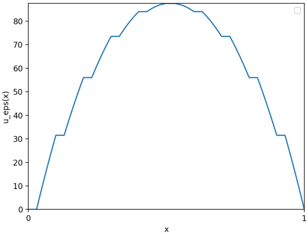
$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(A^*\dfrac{\mathrm{d}u_\varepsilon}{\mathrm{d}x}\right) = f \quad \text{in } D, \\ u_\varepsilon(0) = u_\varepsilon(1) = 0, \end{cases} \quad \text{where } A^* = \left(\frac{\theta}{\alpha} + \frac{(1-\theta)}{\beta}\right)^{-1}.$$

- The convergence is weak in $H^1_0(0, L)$: u_ε oscillates around u^*:
  - The function u_ε converges to u^* in $L^2(0, L)$;
  - The derivative $\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x}$ converges only weakly in $L^2(0, L)$ to $\frac{\mathrm{d}u^*}{\mathrm{d}x}$.

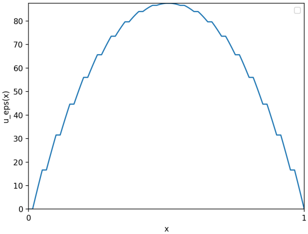
Appetizer: the 1d case (II)



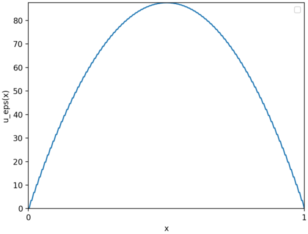
$\varepsilon = 0.3$



$\varepsilon = 0.1$



$\varepsilon = 0.05$



$\varepsilon = 0.01$

Plot of the function $u_\varepsilon(x)$ for $f = 1$, $\theta = 0.3$, $\alpha = 1$, $\beta = 0.001$ and various values of the period ε.

Part IV

Mathematical homogenization

Formal two-scale expansions: rationale (I)

- For given values $0 < \alpha < \beta$, let us denote

$$\mathcal{M}_{\alpha,\beta} := \left\{ M \in \mathbb{R}_s^{d\times d},\ \text{s.t. } \forall \xi \in \mathbb{R}^d,\ \ \alpha|\xi|^2 \leq M\xi\cdot\xi \leq \beta|\xi|^2 \right\},$$

where $\mathbb{R}_s^{d\times d}$ stands for the set of symmetric $d \times d$ matrices.

- We aim to guess the limiting behavior of the solution $u_\varepsilon \in H_0^1(D)$ to

$$\begin{cases} -\mathrm{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f & \text{in } \ D, \\ u_\varepsilon = 0 & \text{on } \ \partial D, \end{cases} \quad \text{where } A(y) \in L^\infty(Y, \mathcal{M}_{\alpha,\beta}).$$

- To this end, we rely on the formal, heuristic two-scale expansion method.

Formal two-scale expansions: rationale (II)

- We postulate an expansion of the form:

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots,$$

  where each term $u_i(x, y)$

  - depends in a smooth way on the macroscopic variable $x \in D$,
  - is a periodic function of the microscopic (or "fast") variable $y \in Y$.

- In order to identify each term, we insert this particular structure into the equation

$$-\mathrm{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f,$$

  and we identify terms with equal powers in ε.

The Sobolev space $H^1_\#(Y)$ of Y-periodic functions (I)

The functions space $H^1_\#(Y)$ of Y-periodic H^1 functions is defined by:

$$H^1_\#(Y) := \left\{ u \in H^1_{\text{loc}}(\mathbb{R}^d), \;\; u(x+e_i) = u(x), \;\; i = 1, \dots, d \text{ and a.e. } x \in \mathbb{R}^d \right\},$$

and it is equipped with the norm $|| \cdot ||_{H^1(Y)}$.

Proposition 5.

- *The subset $\mathcal{C}^\infty_\#(Y)$ of smooth Y-periodic functions is dense in $H^1_\#(Y)$.*
- *For any function $u \in H^1_\#(Y)$, it holds:* $\displaystyle\int_Y \frac{\partial u}{\partial x_i} \,\mathrm{d}x = 0.$

<u>*Hint of proof:*</u>

- The first point is proved by a classical approximation and truncation argument.
- The second point follows from the first one by density: for $u \in \mathcal{C}^\infty_\#(Y)$, Green's formula implies:

$$\int_Y \frac{\partial u}{\partial x_i} \,\mathrm{d}x = \int_{\partial Y} u n_i \,\mathrm{d}s = 0.$$

$\square$

The Sobolev space $H^1_\#(Y)$ of Y-periodic functions (II)

We shall use repeatedly the following lemma:

Lemma 6.

Let $g \in L^2(Y)$ with $\int_Y g \,\mathrm{d}y = 0$; then the equation

$$\begin{cases} -\mathrm{div}(A(y)\nabla u) = g & \text{in } Y, \\ y \mapsto u(y) & \text{is } Y\text{-periodic}, \end{cases}$$

has a unique solution in $H^1_\#(Y)/\mathbb{R}$ (i.e. up to constants).

Proof: A variational formulation for this problem is:

$$\text{Search for } u \in H^1_\#(Y)/\mathbb{R} \text{ s.t. } \int_Y A(y)\nabla u \cdot \nabla v \,\mathrm{d}y = \int_Y gv \,\mathrm{d}y.$$

- The mapping

$$(u, v) \mapsto \int_Y A(y)\nabla u \cdot \nabla v \,\mathrm{d}y$$

  is a continuous, coercive bilinear form on $H^1_\#(Y)/\mathbb{R}$.
- $v \mapsto \int_Y gv \,\mathrm{d}y$ is a continuous linear form on $H^1_\#(Y)/\mathbb{R}$ because $\int_Y g \,\mathrm{d}y = 0$.

The result then follows from the Lax-Milgram theorem.

Formal two-scale expansions (I)

For any of the terms $u_i\left(x, \frac{x}{\varepsilon}\right)$, the chain rule yields:

$$\nabla\left(u_i\left(x, \frac{x}{\varepsilon}\right)\right) = \left(\nabla_x u_i(x,y) + \frac{1}{\varepsilon}\nabla_y u_i(x,y)\right)\Big|_{(x,y)=\left(x,\frac{x}{\varepsilon}\right)},$$

and so

$$-\operatorname{div}\left[A\left(\frac{x}{\varepsilon}\right)\nabla\left(u_i\left(x,\frac{x}{\varepsilon}\right)\right)\right] = -\frac{1}{\varepsilon^2}\underbrace{\left(\operatorname{div}_y\left(A(y)\nabla_y u_i(x,y)\right)\right)\Big|_{(x,y)=\left(x,\frac{x}{\varepsilon}\right)}}_{:=\operatorname{div}_y(A\nabla_y u_i)\left(x,\frac{x}{\varepsilon}\right)}$$

$$-\frac{1}{\varepsilon}\underbrace{\left(\operatorname{div}_x(A(y)\nabla_y u_i(x,y))\right)\Big|_{(x,y)=\left(x,\frac{x}{\varepsilon}\right)}}_{:=\operatorname{div}_x(A\nabla_y u_i)\left(x,\frac{x}{\varepsilon}\right)} - \frac{1}{\varepsilon}\underbrace{\left(\operatorname{div}_y(A(y)\nabla_x u_i(x,y))\right)\Big|_{(x,y)=\left(x,\frac{x}{\varepsilon}\right)}}_{:=\operatorname{div}_y(A\nabla_x u_i)\left(x,\frac{x}{\varepsilon}\right)}$$

$$-\underbrace{\left(\operatorname{div}_x(A(y)\nabla_x u_i(x,y))\right)\Big|_{(x,y)=\left(x,\frac{x}{\varepsilon}\right)}}_{:=\operatorname{div}_x(A\nabla_x u_i)\left(x,\frac{x}{\varepsilon}\right)}.$$

Formal two-scale expansions (II)

Inserting the expansion

$$u_\varepsilon(x) = \sum_{i=0}^{\infty} u_i \left(x, \frac{x}{\varepsilon}\right)$$

into the conductivity equation

$$-\mathrm{div}(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon) = f \text{ in } D,$$

results in:

$$\begin{aligned}
f(x) \quad = \quad & -\frac{1}{\varepsilon^2}\mathrm{div}_y\left(A\nabla_y u_0\right)\left(x, \frac{x}{\varepsilon}\right) \\
& -\frac{1}{\varepsilon}\Big(\mathrm{div}_x(A\nabla_y u_0) + \mathrm{div}_y(A(\nabla_x u_0 + \nabla_y u_1))\Big)\left(x, \frac{x}{\varepsilon}\right) \\
& -\sum_{i=0}^{\infty}\varepsilon^i\Bigg(\mathrm{div}_x(A(\nabla_x u_i + \nabla_y u_{i+1})) \\
& \qquad\qquad + \mathrm{div}_y(A(\nabla_x u_{i+1} + \nabla_y u_{i+2}))\Bigg)\left(x, \frac{x}{\varepsilon}\right).
\end{aligned}$$

Formal two-scale expansions (III)

Identification of the terms of order ε^{-2}.

- We obtain:
$$-\mathrm{div}_y(A(y)\nabla_y u_0(x,y)) = 0, \quad x \in D, \;\; y \in Y.$$

- We have seen that for any $g \in L^2(Y)$ with $\int_Y g(y)\,\mathrm{d}y = 0$, the equation
$$\begin{cases} -\mathrm{div}_y(A(y)\nabla_y v(y)) = g(y) & \text{in } Y, \\ y \mapsto v(y) & \text{is } Y-\text{periodic}, \end{cases}$$
has a unique solution in $H^1_\#(Y)$, up to constants.

- For fixed $x \in D$, 0 is one solution $y \mapsto u_0(x,y)$ to
$$-\mathrm{div}_y(A(y)\nabla_y u_0(x,y)) = 0, \quad y \in Y.$$

- Hence, $u_0(x,y)$ is a function of x only, that we rewrite $u_0(x)$.

Formal two-scale expansions (IV)

Identification of the terms of order ε^{-1}.

- We obtain:

$$-\mathrm{div}_y(A(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y))) = 0, \quad x \in D, \ \ y \in Y.$$

- Let us introduce the cell functions $\chi_i \in H^1_\#(Y)/\mathbb{R}$, $i = 1, \dots, d$, solutions to:

$$\begin{cases} -\mathrm{div}_y(A(y)(\nabla_y \chi_i(y) + e_i)) = 0 & \text{in } Y, \\ y \mapsto \chi_i(y) & \text{is } Y\text{-periodic}, \end{cases}$$

- Since $\nabla u_0(x) = \sum_{i=1}^{d} \frac{\partial u_0}{\partial x_i} e_i$, we obtain by linearity, owing to the well-posedness Lemma 6:

$$u_1(x,y) = \sum_{i=1}^{d} \frac{\partial u_0}{\partial x_i}(x)\chi_i(y) + r(x),$$

where $r(x)$ is a function of x only.

Identification of the terms of order ε^0.

- We obtain:

$$- \operatorname{div}_x(A(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y))) \\ - \operatorname{div}_y(\underbrace{A(y)(\nabla_x u_1(x,y) + \nabla_y u_2(x,y))}_{Y-\text{periodic}}) = f(x).$$

- Integrating over $y \in Y$ yields:

$$- \int_Y \operatorname{div}_x(A(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y)))\,\mathrm{d}y = f(x), \quad x \in D.$$

- Now using the expression for $u_1(x,y)$, we obtain:

$$-\operatorname{div}_x \left(\sum_{j=1}^{d} \left(\int_Y (A(y)(e_j + \nabla_y \chi_j(y))\,\mathrm{d}y \right) \frac{\partial u_0}{\partial x_j}(x) \right) = f(x).$$

Formal two-scale expansions (VI)

Identification of the terms of order ε^0 (continued).

- This rewrites:

$$-\mathrm{div}(A^*\nabla u_0) = f, \text{ where } A^*_{ij} = \int_Y A(y)(e_j + \nabla_y \chi_j(y)) \cdot e_i \, \mathrm{d}y, \\ = \int_Y A(y)(e_j + \nabla_y \chi_j(y)) \cdot (e_i + \nabla_y \chi_i(y)) \, \mathrm{d}y$$

  where we have used the variational formulation for χ_i.

- By linearity, it holds:

$$\forall \xi \in \mathbb{R}^d, \quad A^*\xi \cdot \xi = \int_Y A(y)(\xi + \nabla_y \chi_\xi(y)) \cdot (\xi + \nabla_y \chi_\xi(y)) \, \mathrm{d}y,$$

  where

$$\begin{cases} -\mathrm{div}_y(A(y)(\nabla_y \chi_\xi(y) + \xi)) = 0 & \text{in } Y, \\ y \mapsto \chi_\xi(y) & \text{is } Y\text{-periodic,} \end{cases}$$

- One may prove that the matrix A^* is symmetric, positive definite.

- Using the same type of expansion on ∂D yields $u_0(x) = 0$ on ∂D.

Formal two-scale expansions: summary

The function $u_\varepsilon(x)$ has the expansion:

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots$$

- The function $u_0(x)$ is the solution to the equation

$$\begin{cases} -\mathrm{div}(A^*\nabla u_0) = f & \text{in } D, \\ u_0 = 0 & \text{on } \partial D. \end{cases}$$

  The symmetric, positive definite homogenized tensor A^* is defined by:

$$A^*_{ij} = \int_Y A(y)\Big(e_i + \nabla_y \chi_i(y)\Big) \cdot \Big(e_j + \nabla_y \chi_j(y)\Big)\,\mathrm{d}y,$$

  where the cell functions $\chi_i \in H^1_\#(Y)/R$ are the solutions to

$$\begin{cases} -\mathrm{div}\Big(A(y)(e_i + \nabla\chi_i)\Big) = 0 & \text{in } D, \\ y \mapsto \chi_i(y) & \text{is } Y-\text{periodic}. \end{cases}$$

- The first-order term $u_1(x,y)$ is "smooth" in x, and Y-periodic in y:

$$u_1(x,y) = \sum_{i=1}^d \frac{\partial u_0}{\partial x_i}(x)\chi_i(y) + r(x).$$

Part IV

Mathematical homogenization

H-Convergence

For given values $0 < \alpha < \beta$ of the conductivity, let us recall the definition

$$\mathcal{M}_{\alpha,\beta} := \left\{ M \in \mathbb{R}_s^{d\times d},\ \text{s.t. } \forall \xi \in \mathbb{R}^d,\ \ \alpha|\xi|^2 \leq M\xi\cdot\xi \leq \beta|\xi|^2 \right\},$$

where $\mathbb{R}_s^{d\times d}$ stands for the set of symmetric $d \times d$ matrices.

Definition 1.

A sequence $A^\varepsilon(x)$ in $L^\infty(D, \mathcal{M}_{\alpha,\beta})$ H-converges to $A^*(x) \in L^\infty(D, \mathcal{M}_{\alpha,\beta})$ if for any $f \in H^{-1}(D)$, the sequence $u_\varepsilon \in H_0^1(D)$ of solutions to:

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x)\nabla u_\varepsilon(x)) = f(x) & \text{in } D, \\ u_\varepsilon = 0 & \text{on } \partial D, \end{cases}$$

satisfies

$$u_\varepsilon \to u^* \text{ weakly in } H_0^1(D), \text{ and } A^\varepsilon \nabla u_\varepsilon \to A^* \nabla u^* \text{ weakly in } L^2(D)^d,$$

where u^* is the unique solution to the homogenized equation

$$\begin{cases} -\mathrm{div}(A^*(x)\nabla u^*(x)) = f(x) & \text{in } D, \\ u^* = 0 & \text{on } \partial D. \end{cases}$$

The convergence result (I)

- This course deals mainly with periodic homogenization.
- In this context, the following result about the weak convergence of periodized functions is crucial.

Lemma 7.

Let $f \in L^2_\#(Y)$, and let f_ε be the sequence in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ defined by:

$$f_\varepsilon(x) := f\left(\frac{x}{\varepsilon}\right), \text{ a.e. } x \in \mathbb{R}^d.$$

Then for any bounded set $D \subset \mathbb{R}^d$, $f_\varepsilon(x)$ converges weakly in $L^2(D)$ to the constant function with value equal to the average $\int_Y f(y)\,\mathrm{d}y$:

$$\forall g \in L^2(D), \quad \int_D f_\varepsilon(x) g(x)\,\mathrm{d}x \xrightarrow{\varepsilon \to 0} \left(\int_Y f(y)\,\mathrm{d}y\right) \int_D g(x)\,\mathrm{d}x.$$

The convergence result (II)

Theorem 8.

Let $A(y) \in L^\infty_\#(Y, \mathcal{M}_{\alpha,\beta})$ be a periodic matrix field on the unit cell, and define

$$A^\varepsilon(x) := A\left(\frac{x}{\varepsilon}\right), \quad a.e. \quad x \in D.$$

The sequence $A^\varepsilon(x)$ H-converges to the constant matrix $A^* \in \mathcal{M}_{\alpha,\beta}$ with entries

$$A^*_{ij} = \int_Y A(y)(e_i + \nabla\chi_i(y)) \cdot (e_j + \nabla\chi_j(y))\,\mathrm{d}y, \quad i,j = 1,\dots,d,$$

where the $\chi_i \in H^1_\#(Y)/\mathbb{R}$ are the cell functions, solution to:

$$\begin{cases} -\mathrm{div}(A(y)(e_i + \nabla\chi_i(y))) = 0 & in \quad Y, \\ y \mapsto \chi_i(y) & is\ Y - periodic. \end{cases}$$

Sketch of proof: Let $u_\varepsilon \in H^1_0(D)$ be the unique solution to the conductivity equation:

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x)\nabla u_\varepsilon(x)) = f(x) & \text{in } D, \\ u_\varepsilon = 0 & \text{on } \partial D, \end{cases}$$

and let $\sigma_\varepsilon := A^\varepsilon \nabla u_\varepsilon \in L^2(D)^d$ be the associated flux.

Tartar's energy method (I)

Step 1: we derive a priori estimates for u_ε and σ_ε.

- The variational formulation for u_ε reads:

$$\forall v \in H^1_0(D), \quad \int_D A^\varepsilon(x)\nabla u_\varepsilon \cdot \nabla v \,\mathrm{d}x = \int_D fv \,\mathrm{d}x. \qquad \text{(VF)}$$

- Taking $v = u_\varepsilon$ in the above identity, we obtain:

$$||\nabla u_\varepsilon||^2_{L^2(D)^d} \leq ||f||_{H^{-1}(D)}||u_\varepsilon||_{H^1(D)}.$$

  Thanks to the Poincaré inequality, this yields:

$$||u_\varepsilon||_{H^1(D)} \leq C, \text{ and so } ||\sigma_\varepsilon||_{L^2(D)^d} \leq C.$$

- Hence, there exists a subsequence (still labelled by ε) and $u^* \in H^1(D)$, $\sigma^* \in L^2(D)^d$ such that:

$$u_\varepsilon \to u^* \text{ weakly in } H^1(D), \text{ and strongly in } L^2(D) \text{ by the Rellich theorem},$$

$$\sigma_\varepsilon \to \sigma^* \text{ weakly in } L^2(D)^d.$$

- In particular, taking limits in (VF) yields:

$$\forall v \in H^1_0(D), \quad \int_D \sigma^* \cdot \nabla v \,\mathrm{d}x = \int_D fv \,\mathrm{d}x.$$

Tartar's energy method (II)

Our aim is then to prove that σ^* and u^* are related as:

$$\sigma^*(x) = A^* \nabla u^*(x),$$

where A^* is the homogenized matrix.

Step 2: we construct judicious test functions for the variational formulation of u_ε.

- For $i = 1, \ldots, d$, let us recall the cell functions $\chi_i \in H^1_\#(Y)/\mathbb{R}$:

$$\left\{ \begin{array}{ll} -\mathrm{div}(A(y)(e_i + \nabla \chi_i)) = 0 & \text{in } Y, \\ y \mapsto \chi_i(y) & \text{is } Y-\text{periodic}, \end{array} \right. \quad \text{(CF)}$$

  and let us define:

$$w^i(y) = \chi_i(y) + y_i, \quad y \in Y.$$

- We thence construct functions on D by ε-periodization of the w^i:

$$w^i_\varepsilon(x) := \varepsilon w^i\left(\frac{x}{\varepsilon}\right), \quad x \in D.$$

- As a consequence of the definition (CF), it holds that

$$-\mathrm{div}(A^\varepsilon(x) \nabla w^i_\varepsilon) = 0 \text{ in } H^{-1}(D).$$

Tartar's energy method (III)

- It is easily seen that:

$$w_\varepsilon^i(x) = \varepsilon\chi_i\left(\frac{x}{\varepsilon}\right) + x_i \xrightarrow{\varepsilon\to 0} x_i \text{ strongly in } L^2(D).$$

- Likewise, using the Lemma about weak convergence of periodized functions,

$$\nabla w_\varepsilon^i(x) = e_i + (\nabla\chi_i)\left(\frac{x}{\varepsilon}\right) \xrightarrow{\varepsilon\to 0} e_i + \underbrace{\int_Y \nabla\chi_i(y)\,\mathrm{d}y}_{\substack{=0 \text{ by Green's formula} \\ \text{since } \chi_i(y) \text{ is } Y-\text{periodic}}} \quad \text{weakly in } L^2(D)^d,$$

and so,

$$\nabla w_\varepsilon^i(x) \xrightarrow{\varepsilon\to 0} e_i \text{ weakly in } L^2(D)^d.$$

Tartar's energy method (IV)

Step 3: "Compensated compactness".

- **Main idea**: use "clever" test functions in the variational formulation (VF) of u_ε in order to compensate the oscillations of the conductivity A_ε.
- The functions w_ε^i are used to "modulate" test functions in the variational formulation (VF).
- Let $\phi \in \mathcal{C}_c^\infty(D)$ be an arbitrary function. We insert $v(x) = \phi(x) w_\varepsilon^i(x)$ in (VF).
- This yields:

$$\begin{aligned}
\int_D f \phi w_\varepsilon^i \,\mathrm{d}x &= \int_D A^\varepsilon(x) \nabla u_\varepsilon \cdot \nabla(\phi w_\varepsilon^i) \,\mathrm{d}x, \\
&= \int_D \phi A^\varepsilon(x) \nabla u_\varepsilon \cdot \nabla w_\varepsilon^i \,\mathrm{d}x + \int_D w_\varepsilon^i A^\varepsilon(x) \nabla u_\varepsilon \cdot \nabla \phi \,\mathrm{d}x, \\
&= \underbrace{\int_D A^\varepsilon(x) \nabla w_\varepsilon^i \cdot \nabla(\phi u_\varepsilon) \,\mathrm{d}x}_{=0 \text{ since } \mathrm{div}(A^\varepsilon \nabla w_\varepsilon^i)=0} - \int_D u_\varepsilon A^\varepsilon(x) \nabla \phi \cdot \nabla w_\varepsilon^i \,\mathrm{d}x \\
&\quad + \int_D w_\varepsilon^i A^\varepsilon(x) \nabla u_\varepsilon \cdot \nabla \phi \,\mathrm{d}x.
\end{aligned}$$

Tartar's energy method (V)

- We are left with the following identity, in which we expect to take limits:

$$\int_D f\phi w_\varepsilon^i \,\mathrm{d}x = \underbrace{-\int_D u_\varepsilon A^\varepsilon(x)\nabla\phi\cdot\nabla w_\varepsilon^i\,\mathrm{d}x}_{=:I_\varepsilon^1} + \underbrace{\int_D w_\varepsilon^i A^\varepsilon(x)\nabla u_\varepsilon\cdot\nabla\phi\,\mathrm{d}x}_{=:I_\varepsilon^2}.$$

- The integral I_ε^1 rewrites:

$$I_\varepsilon^1 = -\int_D (A\nabla w^i)\left(\frac{x}{\varepsilon}\right)\cdot(u_\varepsilon\nabla\phi)\,\mathrm{d}x$$

  Since

$$\begin{cases} (A\nabla w^i)\left(\frac{x}{\varepsilon}\right) \xrightarrow{\varepsilon\to 0} A(y)\nabla w^i(y) & \text{weakly in } L^2(D)^d, \\ u_\varepsilon\nabla\phi \xrightarrow{\varepsilon\to 0} u^*\nabla\phi & \text{strongly in } L^2(D), \end{cases}$$

  this yields:

$$I_\varepsilon^1 \xrightarrow{\varepsilon\to 0} -\left(\int_D A(y)\nabla w^i(y)\,\mathrm{d}y\right)\cdot\int_D u^*\nabla\phi\,\mathrm{d}x.$$

- By the same token we prove the convergence:

$$I_\varepsilon^2 \xrightarrow{\varepsilon\to 0} \int_D x_i\sigma^*\cdot\nabla\phi\,\mathrm{d}x.$$

Tartar's energy method (VI)

- All things considered, we have the identity:

$$\int_D f\phi x_i \,\mathrm{d}x = -\left(\int_D A(y)\nabla w^i(y)\,\mathrm{d}y\right)\cdot\int_D u^*\nabla\phi\,\mathrm{d}x + \int_D x_i\sigma^*\cdot\nabla\phi\,\mathrm{d}x.$$

- On a different note, the variational formulation for σ^* implies that:

$$\begin{aligned}\int_D x_i\sigma^*\cdot\nabla\phi\,\mathrm{d}x &= \int_D \sigma^*\cdot\nabla(x_i\phi)\,\mathrm{d}x - \int_D(\sigma^*\cdot e_i)\phi\,\mathrm{d}x\\ &= \int_D f\phi x_i\,\mathrm{d}x - \int_D \sigma_i^*\phi\,\mathrm{d}x.\end{aligned}$$

- As a result, it follows from integration by parts that:

$$\int_D \sigma_i^*\phi\,\mathrm{d}x = \int_D\left(\left(\int_D A(y)\nabla w^i(y)\,\mathrm{d}y\right)\cdot\nabla u^*\right)\phi\,\mathrm{d}x.$$

Tartar's energy method (VII)

- Since this identity holds for any test function $\phi \in \mathcal{C}_c^\infty(D)$, this entails:

$$\sigma_i^*(x) = \left(\int_D A(y) \nabla w^i(y) \, \mathrm{d}y \right) \cdot \nabla u^*(x) \quad \text{a.e. } x \in D.$$

- After inspection, this is the expected result:

$$\sigma^*(x) = A^* \nabla u(x).$$

$\square$

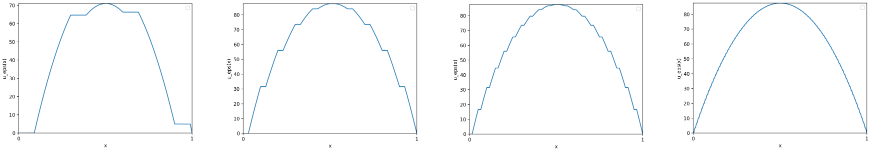
Remark Strictly speaking, we have proved that for a subsequence of the ε, it holds:

$$\begin{cases} u_\varepsilon(x) \to u^*(x) & \text{weakly in } H_0^1(D), \\ \text{and } (A^\varepsilon \nabla u_\varepsilon)(x) \to A^* \nabla u^*(x) & \text{weakly in } L^2(D). \end{cases}$$

Actually, a classical argument based on the uniqueness of the limit reveals that the above convergence holds for the whole sequence ε.

Correctors (I)

- We have only proved the weak $H_0^1(D)$ convergence of u_ε to the solution u^* to the homogenized equation.
- In particular, the gradient ∇u_ε *does not* converge to ∇u^* strongly in $L^2(D)$ (and not pointwise).
- This is due to the fact that u_ε converges to u^* by "oscillating around u^*".



$\varepsilon = 0.3$ $\varepsilon = 0.1$ $\varepsilon = 0.05$ $\varepsilon = 0.01$

Behavior of the solution u_ε in a one-dimensional problem.

- In order to improve this convergence result (i.e. get strong H^1 convergence), we have to introduce correctors to capture this oscillating behavior.

Correctors (II)

The following result captures the oscillatory nature of the gradient ∇u_ε.

Theorem 9.

Let u_ε, $u^* \in H_0^1(D)$ be as before, and suppose that u^* is more regular: $u^* \in H^2(D)$. Then the following corrector result holds:

$$\left\| u_\varepsilon(x) - u^*(x) - \sum_{i=1}^{d} \frac{\partial u^*}{\partial x_i}(x)\chi_i\left(\frac{x}{\varepsilon}\right) \right\|_{H^1(D)} \xrightarrow{\varepsilon \to 0} 0.$$

Remarks

- See for instance [All2s] for a proof.
- The assumption that $u^* \in H^2(D)$ is not very restrictive in practice. For instance, by elliptic regularity, it holds as soon as Ω is regular and $f \in L^2(D)$.

Part IV

Mathematical homogenization

Extension to the non periodic case

- We have hitherto considered the periodic homogenization setting, but the theory may consider much more general situations.
- We have the following general sequential compactness result about H-convergence.

Theorem 10.

For any sequence $A^\varepsilon(x) \in L^\infty(D, \mathcal{M}_{\alpha,\beta})$, there exists a subsequence of indices (still denoted by ε) and a homogenized matrix field $A^*(x) \in L^\infty(D, \mathcal{M}_{\alpha,\beta})$ such that

$$A^\varepsilon(x) \text{ H-converges to } A^*(x).$$

Remarks

- Contrary to the periodic setting, we cannot expect that the *whole* sequence $A^\varepsilon(x)$ H-converges.
- See [Allh], Th. 1.2.4.2 for a proof of this result, based on the compensated compactness technique.

Extension to the non periodic case

- H-convergence was defined as the weak convergence of the solution (and the flux of) u_ε to the conductivity equation with homogeneous Dirichlet B.C..
- H-convergence actually does not depend on the boundary conditions featured by the considered problem.

Theorem 11.

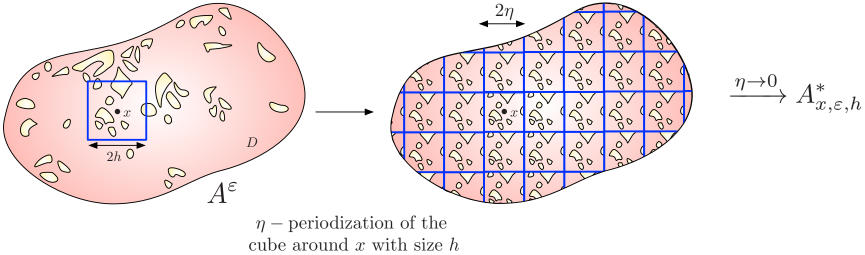
Let $A^\varepsilon(x)$ be any sequence of matrices in $L^\infty(D, \mathcal{M}_{\alpha,\beta})$ which H-converges to $A^*(x) \in L^\infty(D, \mathcal{M}_{\alpha,\beta})$. Let z_ε be any sequence in $H^1(D)$ such that:

$$\begin{cases} -\operatorname{div}\left(A^\varepsilon \nabla z_\varepsilon\right) = f_\varepsilon \xrightarrow{\varepsilon \to 0} f & \text{strongly in } H^{-1}_{\mathrm{loc}}(D), \\ z_\varepsilon \xrightarrow{\varepsilon \to 0} z & \text{weakly in } H^1_{\mathrm{loc}}(D); \end{cases}$$

then it holds:

$$A^\varepsilon \nabla z_\varepsilon \xrightarrow{\varepsilon \to 0} A^* \nabla z \text{ weakly in } L^2_{\mathrm{loc}}(D)^d.$$

The H-convergence of a general matrix sequence $A^{\varepsilon}(x) \in L^{\infty}(D, \mathcal{M}_{\alpha,\beta})$ can be understood locally as the limit predicted by periodic homogenization.



For fixed ε and h, let $A_{x,\varepsilon,h}$ be the tensor obtained by homogenization of the periodic pattern given by $y \mapsto A^{\varepsilon}(x + hy)$, i.e. induced by the values of A^{ε} on a small cube with size h around x, periodized.

The following theorem is proved in [Allh], Th. 1.3.4.6.

Theorem 12.

Let A_ε be a sequence of matrices in $L^\infty(D, \mathcal{M}_{\alpha,\beta})$ which H-converges to some limit $A^*(x)$. For any $x \in D$ and $h > 0$ small enough, let us define

$$(A^*_{x,\varepsilon,h})_{ij} = \int_Y A^\varepsilon(x + hy)(e_i + \nabla w^i_{x,\varepsilon,h}) \cdot (e_j + \nabla w^j_{x,\varepsilon,h})\,\mathrm{d}y,$$

where $\nabla w^i_{x,\varepsilon,h}$ is the solution to the cell problem:

$$\begin{cases} -\mathrm{div}(A(x + hy)(e_i + \nabla w^i_{x,\varepsilon,h})) = 0 & \text{in } Y, \\ y \mapsto w^i_{x,\varepsilon,h} & \text{is } Y-\text{periodic.} \end{cases}$$

Then there exists a subsequence $h \to 0$ such that:

$$\lim_{h\to 0} \lim_{\varepsilon \to 0} A^*_{x,\varepsilon,h} = A^*(x) \text{ for a.e. } x \in D.$$

Other homogenization techniques: two-scale convergence

- The notion of two-scale convergence is very well-suited to the study of periodic homogenization problems.

- It gives a rigorous meaning to the convergence of an oscillating function $u_\varepsilon(x)$ to a function of both macroscopic and microscopic variables $x \in D$ and $y \in Y$.

- For instance, in the periodic homogenization context, it holds:

$$u_\varepsilon(x) \to u_0(x) \text{ weakly in } H_0^1(D)$$

and

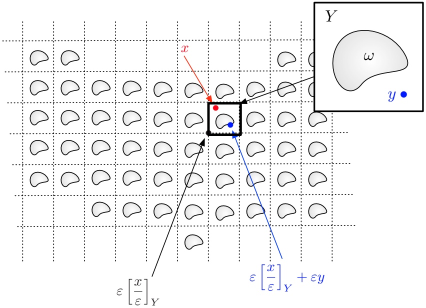
$$\text{“ } \nabla u_\varepsilon(x) \xrightarrow{2-\text{scale}} \nabla u_0(x) + \nabla_y u_1(x, y).\text{”}$$

- See [Ngue] and [All2s] for further explanations.

Other homogenization techniques: periodic unfolding

The periodic unfolding method [CioDaGr] features an extension procedure of a function $v : D \to \mathbb{R}$ to both macroscopic and microscopic scales:

$$v(x) \rightsquigarrow v(x,y) = v\left(\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon y\right).$$



It allows to compare $u_\varepsilon : D \to \mathbb{R}$ with oscillatory expansions of the form

$$u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \dots$$

Extension to the context of linear elasticity (I)

Consider the counterpart of the previous situation in the realm of linear elasticity.

- The Hooke's law inside Y induced by the pattern $\omega \subset Y$ is:

$$A(y) = \begin{cases} B & \text{if } \; y \in \omega, \\ A & \text{if } y \in D \setminus \overline{\omega}. \end{cases}$$

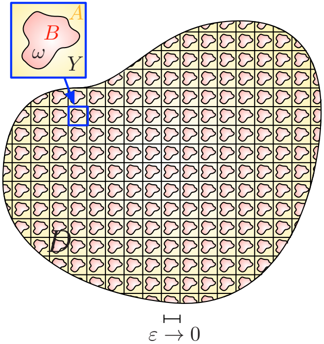
- This pattern induces a Hooke's law $A\left(\frac{x}{\varepsilon}\right)$ for $x \in D$, by rescaling $A(y)$ at size ε and periodization.

- The displacement $u_\varepsilon \in H_0^1(D)^d$ of D solves:

$$\begin{cases} -\mathrm{div}\left(A\left(\frac{x}{\varepsilon}\right) e(u_\varepsilon)\right) = f & \text{in } \; D, \\ u_\varepsilon = 0 & \text{on } \; \partial D, \end{cases}$$

where the strain tensor $e(u)$ is:

$$e(u) = \frac{1}{2}(\nabla u + \nabla u^T).$$



What does u_ε look like, as $\varepsilon \to 0$?

Extension to the context of linear elasticity (II)

- The formal two-scale asymptotic expansion argument reveals that the effective behavior $u_0 \in H^1_0(D)^d$ of u_ε satisfies:

$$\left\{ \begin{array}{cl} -\text{div}\,(A^* e(u_0)) = f & \text{in} \quad D, \\ u_0 = 0 & \text{on} \quad \partial D, \end{array} \right.$$

  where the homogenized tensor A^* is defined by:

$$\forall \xi \in \mathbb{R}^{d\times d}_s, \quad A^*\xi : \xi = \int_Y A(y)(\xi + \chi_\xi(y)) : (\xi + \chi_\xi(y))\, \text{d}y,$$

  involving the cell function $\chi_\xi \in H^1_\#(Y)$, solution to:

$$\left\{ \begin{array}{cl} -\text{div}(A(y)(\xi + e(\chi_\xi(y)))) = 0 & \text{in} \;\; Y, \\ y \mapsto \chi_\xi(y) & \text{is} \;\; Y-\text{periodic}. \end{array} \right.$$

- The general properties of H-convergence remain true in this context, namely:
  - The compactness of $\mathcal{M}_{\alpha,\beta}$ for H-convergence;
  - The corrector result to obtain strong $H^1(D)$ convergence of u_ε;
  - The irrelevance of boundary conditions in the definition of H-convergence;
  - The genericity of periodic configurations to describe the nature of H-convergent sequences.

Towards shape optimization (I)

Let us recall the shape optimization problem of interest.

- The problem reads

$$\min_{\Omega \subset D} J(\Omega), \text{ where } J(\Omega) := \int_D j(u_\Omega)\,\mathrm{d}x, \qquad \text{(SO)}$$

  and $j : \mathbb{R} \to \mathbb{R}$ is a given, smooth function.

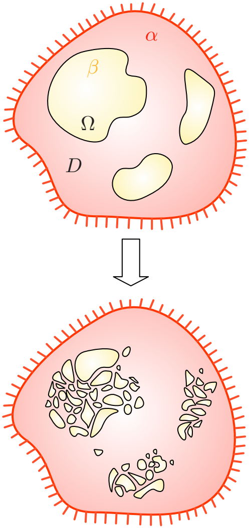
- The temperature $u_\Omega \in H^1_0(D)$ is the solution to the two-phase conductivity equation:

$$\begin{cases} -\mathrm{div}(\gamma_\Omega \nabla u_\Omega) = f & \text{in } D, \\ u_\Omega = 0 & \text{on } \partial D, \end{cases}$$

  where γ_Ω reads:

$$\gamma_\Omega(x) = \alpha + \chi_\Omega(x)(\beta - \alpha), \quad x \in D.$$

- We aim to allow "limits of classical designs" in (SO), showing increasingly many small features.



Towards shape optimization (II)

We have seen that a periodic conductivity distribution $A^\varepsilon(x)$, defined by:

$$A^\varepsilon(x) := A_\omega\left(\frac{x}{\varepsilon}\right), \quad x \in D,$$

where $A_\omega(y)$ is made from a pattern $\omega \subset Y$:

$$A_\omega(y) := \begin{cases} \beta & \text{if } y \in \omega, \\ \alpha & \text{if } y \in Y \setminus \omega \end{cases} \qquad y \in Y,$$

has an effective behavior described by the homogenized tensor A^*_ω.

Definition 2.

For $\theta \in [0,1]$, G_θ is the set of all conductivity matrices obtained by homogenization of the phases α and β in proportions $(1-\theta)$ and θ:

$$G_\theta = \overline{\{A^*_\omega, \;\; \omega \subset Y\}} \subset \mathcal{M}_{\alpha,\beta}.$$

Towards shape optimization (III)

This gives us a hint of how to relax the shape optimization problem (SO):

$$\min_{(\theta,A)\in\mathcal{CD}} J(\theta,A), \text{ where } J(\theta,A) = \int_D j(u_{\theta,A})\,\mathrm{d}x,$$

where the set $\mathcal{CD}$ of composite designs is defined by:

$$\mathcal{CD} := \left\{(\theta, A^*) \in L^\infty(D,[0,1]) \times L^\infty(D,\mathcal{M}_{\alpha,\beta}), \;\; A^*(x) \in G_{\theta(x)} \text{ a.e. } \; x \in D\right\}.$$

This program raises (at least!) three questions:

- How to characterize more explicitly (e.g. to parametrize) the set G_θ?
- How to justify this procedure?
- How to use this in a numerical method?

Part IV

Mathematical homogenization

On first step towards characterizing G_θ is the following lemma.

Proposition 13 (Voigt-Reuss bounds).

Let $A(y) \in L^\infty(Y, \mathcal{M}_{\alpha,\beta})$ be a matrix field; then the matrix A^* obtained by periodic homogenization satisfies the following bounds:

$$\forall \xi \in \mathbb{R}^d, \quad , \underline{A}\xi \cdot \xi \leq A^*\xi \cdot \xi \leq \overline{A}\xi \cdot \xi,$$

where $\underline{A}$ and $\overline{A}$ are respectively the harmonic and arithmetic means of $A(y)$, namely:

$$\underline{A} := \left(\int_Y A^{-1}(y)\, \mathrm{d}y \right)^{-1} \text{ and } \overline{A} := \int_Y A(y)\, \mathrm{d}y.$$

Proof of the upper bound: We recall the variational principle satisfied by A^*:

$$\forall \xi \in \mathbb{R}^d, \quad A^*\xi \cdot \xi = \min_{w \in H^1_\#(Y)/\mathbb{R}} \int_Y A(y)\Big(\xi + \nabla w(y)\Big) \cdot \Big(\xi + \nabla w(y)\Big)\, \mathrm{d}y.$$

Choosing in particular $w = 0$ in the latter minimization yields, for $\xi \in \mathbb{R}^d$:

$$A^*\xi \cdot \xi \leq \left(\int_Y A(y)\, \mathrm{d}y \right) \xi \cdot \xi,$$

which is the desired upper bound.

Proof of the lower bound: We use the same duality trick as in the proof of Lemma 3.

- For all $\xi \in \mathbb{R}^d$, and positive definite $d \times d$ matrix $A \in \mathbb{R}^{d\times d}$,

$$\frac{1}{2}A\xi \cdot \xi = \max_{\sigma \in \mathbb{R}^d} \left(\xi \cdot \sigma - \frac{1}{2}A^{-1}\sigma \cdot \sigma\right),$$

  where the maximum is uniquely attained at $\sigma = A^{-1}\xi$.

- It follows that, for a.e. $y \in Y$,

$$A(y)\Big(\xi + \nabla w(y)\Big) \cdot \Big(\xi + \nabla w(y)\Big) \geq \max_{\sigma \in \mathbb{R}^d} \Big(2\Big(\xi + \nabla w(y)\Big) \cdot \sigma - A^{-1}\sigma \cdot \sigma\Big).$$

The G-closure problem (III)

- Hence, we obtain,

$$\begin{aligned} A^*\xi\cdot\xi &\geq \min_{w\in H^1_\#(Y)/\mathbb{R}} \int_Y \max_{\sigma(y)\in\mathbb{R}^d} \Big(2\big(\xi+\nabla w(y)\big)\cdot\sigma(y) - A^{-1}\sigma(y)\cdot\sigma(y)\Big)\,\mathrm{d}y \\ &\geq \min_{w\in H^1_\#(Y)/\mathbb{R}} \max_{\sigma\in\mathbb{R}^d} \int_Y \Big(2\big(\xi+\underbrace{\nabla w(y)}_{\substack{\int_Y \nabla w(y)\,\mathrm{d}y=\mathbf{0} \\ \textbf{since } w\in H^1_\#(Y)/\mathbb{R}}}\big)\cdot\sigma - A^{-1}\sigma\cdot\sigma\Big)\,\mathrm{d}y \\ &= \max_{\sigma(y)\in\mathbb{R}^d} \left(2\xi\cdot\sigma - \left(\int_Y A^{-1}(y)\,\mathrm{d}y\right)\sigma\cdot\sigma\right), \end{aligned}$$

where the second line follows by taking the supremum over constant fluxes $\sigma(y)\equiv\sigma$, and not over any matrix fields $y\mapsto\sigma(y)\in\mathbb{R}^d$.

- An explicit calculation of the last maximum value yields

$$A^*\xi\cdot\xi \geq \left(\int_Y A^{-1}(y)\,\mathrm{d}y\right)\xi\cdot\xi,$$

as desired.

□

The G-closure problem (IV)

- The lemma shows, in particular, that all matrices $A^* \in G_\theta$ satisfy the Voigt-Reuss bounds:
$$\forall \xi \in \mathbb{R}^d, \quad \lambda_\theta^- |\xi|^2 \leq A^* \xi \cdot \xi \leq \lambda_\theta^+ |\xi|^2,$$
where $\lambda_\theta^-, \lambda_\theta^+$ are defined by:
$$\lambda_\theta^- = \left(\frac{1-\theta}{\alpha} + \frac{\theta}{\beta} \right)^{-1}, \text{ and } \lambda_\theta^+ = (1-\theta)\alpha + \theta\beta.$$

- Unfortunately, all the matrices satisfying Voigt-Reuss bounds are not in G_θ.

- The characterization of G_θ (e.g. by means of inequalities over eigenvalues of matrices A^*) is a difficult problem, known as the G-closure problem.

- We study one particular subset of tensors in G_θ, that of laminates. Their effective tensors can be computed explicitly, and this comes in handy in characterizing further G_θ.

Part IV

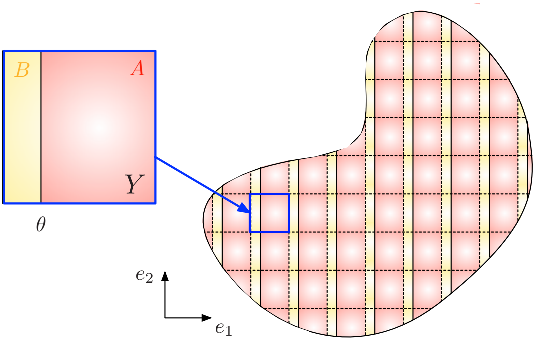
Mathematical homogenization

Rank 1 laminates (I)

- Let A and B be two (possibly anisotropic) symmetric, positive definite matrices.
- We consider the periodic homogenization induced by the following pattern in Y:

$$A(y) = \chi(y_1)B + (1 - \chi(y_1))A, \text{ where } \chi(t) = \begin{cases} 1 & \text{if } t \leq \theta, \\ 0 & \text{otherwise,} \end{cases}$$

where e_1 is the lamination direction.



Notation

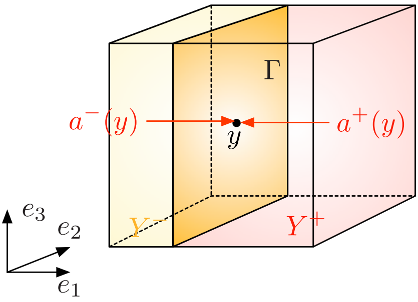
- $\Gamma := \{y = (y_1, \ldots, y_d) \in Y,\ \ y_1 = \theta\}$ is the interface between the two phases:

  $Y^- := \{y = (y_1, \ldots, y_d) \in Y,\ \ y_1 < \theta\}$ and $Y^+ := \{y = (y_1, \ldots, y_d) \in Y,\ \ y_1 > \theta\}$.

- When $a(y)$ is a discontinuous quantity across Γ, we denote by:

$$a^-(y) := \lim_{\substack{t \to 0 \\ t > 0}} a(y - te_1) \text{ and } a^+(y) := \lim_{\substack{t \to 0 \\ t > 0}} a(y + te_1)$$

  the one-sided limits of $a(y)$ across Γ.



Lemma 14.

The homogenized tensor A^* associated with the pattern $A(y)$ is given by the lamination formula:

$$A^* = \theta B + (1-\theta)A - \theta(1-\theta)\frac{(B-A)e_1 \otimes (B-A)e_1}{(1-\theta)Be_1 \cdot e_1 + \theta A e_1 \cdot e_1}.$$

Assuming that $(B-A)$ is invertible, this rewrites:

$$\theta(A^* - A)^{-1} = (B-A)^{-1} + \frac{1-\theta}{Ae_1 \cdot e_1} e_1 \otimes e_1.$$

Proof: Let $\xi \in \mathbb{R}^d$ be given; the explicit formula for A^* reads:

$$A^*\xi = \int_Y A(y)(\xi + \nabla w(y))\,\mathrm{d}y,$$

where w is the unique solution in $H^1_\#(Y)/\mathbb{R}$ to the cell problem:

$$\begin{cases} -\mathrm{div}(A(y)(\xi + \nabla w(y))) = 0 & \text{in } Y, \\ y \mapsto w(y) & \text{is } Y-\text{periodic}. \end{cases} \quad \text{(CP)}$$

Rank 1 laminates (IV)

- Letting $u(y) = \xi \cdot y + w(y)$, the particular structure of (CP) suggests to search for $u(y)$ so that it is affine in Y^- and Y^+:

$$u(y) = (b \cdot y + c_b)\chi(y_1) + (a \cdot y + c_a)(1 - \chi(y_1)),$$

  for some constants $c_a, c_b \in \mathbb{R}$ and vectors $a, b \in \mathbb{R}^d$ to be found.

- $u(y)$ complies with (CP) if anf only if it satisfies (see the transmission conditions):
  - $\mathrm{div}(A(y)\nabla u) = 0$ in both phases Y^-, Y^+.
  - $u(y)$ is continuous across Γ;
  - The flux of $u(y)$ through Γ is continuous: $B\nabla u \cdot e_1^- = A\nabla u \cdot e_1^+$;
  - The function $y \mapsto u(y) - \xi \cdot y$ is Y periodic.

- With such a definition, it holds:

$$\nabla u(y) = \chi(y_1)b + (1 - \chi(y_1))a, \text{ and } A(y)\nabla u(y) = \chi(y_1)Bb + (1 - \chi(y_1))Aa.$$

Rank 1 laminates (V)

- From the form of u, it holds automatically that

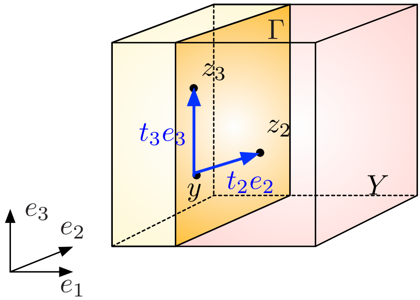
$$-\mathrm{div}(A(y)\nabla u) = 0 \text{ in } Y^- \text{ and } Y^+.$$

- The continuity of $u(y)$ across Γ imposes that:

$$\forall y, z \in \Gamma, \quad (a-b)\cdot y = (a-b)\cdot z.$$

Since for all $i = 2, \ldots, d$, there exists $t \neq 0$ and $y, z \in \Gamma$ such that $(y - z) = te_i$, the previous identity implies that there exists $t \in \mathbb{R}$ such that

$$a - b = te_1.$$



Rank 1 laminates (VI)

- The continuity of fluxes across Γ implies that:

$$Bb \cdot e_1 = Aa \cdot e_1;$$

inserting $a - b = te_1$ and rearranging yields:

$$t = \frac{(B-A)e_1 \cdot e_1}{Ae_1 \cdot e_1}, \text{ and so } a - b = \frac{(B-A)e_1 \cdot e_1}{Ae_1 \cdot e_1} e_1.$$

- On a different note, the Y-periodicity of $y \mapsto u(y) - \xi \cdot y$ imposes that:

$$\int_Y \nabla u(y)\, dy = \xi, \quad \Rightarrow \quad \theta b + (1-\theta) a = \xi.$$

- Combining both expressions yields the explicit expression of u.
- Finally,

$$A^* \xi = \int_Y A(y) \nabla u(y)\, dy = \theta Bb + (1-\theta) Aa,$$

which yields, after calculation:

$$A^* \xi = \theta B\xi + (1-\theta) A\xi - \theta(1-\theta) \frac{(B-A)\xi \cdot e_1}{(1-\theta) Be_1 \cdot e_1 + \theta Ae_1 \cdot e_1} (B-A) e_1,$$

as desired.

The previous result has a particularly nice expression when A and B are isotropic:

$$A = \alpha \mathrm{I} \text{ and } B = \beta \mathrm{I} \text{ for some } \alpha, \beta > 0.$$

Corollary 15.

The homogenized matrix A^* reads:

$$A^* = \begin{pmatrix} \lambda_\theta^- & & & \\ & \lambda_\theta^+ & & 0 \\ 0 & & \ddots & \\ & & & \lambda_\theta^+ \end{pmatrix},$$

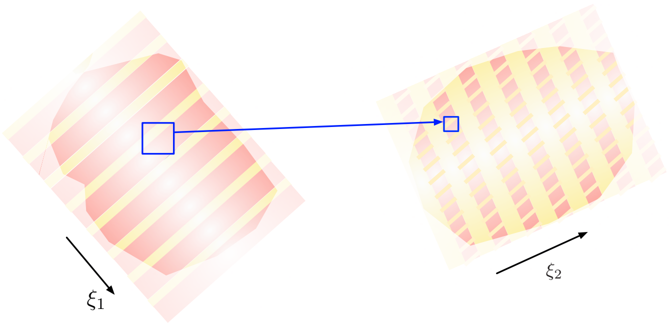
where λ_θ^- and λ_θ^+ are defined by:

$$\lambda_\theta^- = \left(\frac{\theta}{\beta} + \frac{1-\theta}{\alpha} \right)^{-1} \text{ and } \theta\beta + (1-\theta)\alpha.$$

Sequential laminates (I)

The previous lamination procedure can be iterated:

- At first, the pure phase B is mixed with A with direction ξ_1 and proportions θ_1, $(1-\theta_1)$ to give the rank 1 laminate A_1^*.
- The pure phase B is then mixed with the rank-1 laminate A_1^* with direction ξ_2 and proportions θ_2, $(1-\theta_2)$ to produce a rank 2-laminate.
- The pure phase B is mixed with a rank-2 laminate to produce a rank 3 laminate.
- ...



Sequential laminates (II)

Iterating the lamination formula yields the following conclusion.

Proposition 16.

Let ξ_i *and* θ_i, $i = 1, \ldots, p$ *be* p *unitary directions in* $\mathbb{R}^d$ *and* p *volume fractions in* $[0, 1]$. *The* *rank* p *sequential laminate* A_p^* *with inclusion* A *and matrix* B, *in respective proportions*

$$1 - \theta := \prod_{i=1}^{p} (1 - \theta_i) \text{ and } \theta,$$

resulting from the previous procedure is given by:

$$(1-\theta)(A_p^* - B)^{-1} = (A - B)^{-1} + \sum_{i=1}^{p} \left(\theta_i \prod_{j=1}^{i-1} (1 - \theta_j) \right) \frac{\xi_i \otimes \xi_i}{B\xi_i \cdot \xi_i}.$$

- The materials resulting from this procedure are called sequential laminates.
- For a given volume fraction θ of matrix B, we denote by $L_\theta \subset G_\theta$ the set of all sequential laminates.

Sequential laminates (III)

Sequential laminates are often characterized by:

- The total volume fraction θ of matrix B;
- The rank p of the material;
- The lamination directions $\xi_i \in \mathbb{R}^d$, $i = 1, \ldots, p$;
- Lamination parameters $m_i \in [0, 1]$ accounting for the volume fractions θ_i.

Proposition 17.

Let $\xi_i \in \mathbb{R}^d$ and let $\theta \in [0, 1]$ be a volume fraction. Let $m_i \in [0, 1]$, $i = 1, \ldots, p$ be lamination parameters, satisfying:

$$\sum_{i=1}^{p} m_i = 1.$$

Then there exists a rank p sequential laminate A_p^* with matrix B and inclusion A in respective proportions θ and $(1-\theta)$, such that:

$$(1-\theta)(A_p^* - B)^{-1} = (A - B)^{-1} + \theta \sum_{i=1}^{p} m_i \frac{\xi_i \otimes \xi_i}{B\xi_i \cdot \xi_i}.$$

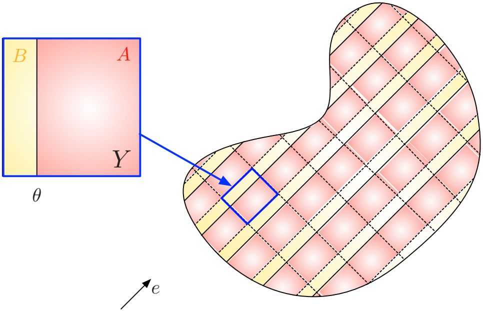
Laminated structures in elasticity (I)

- Let A and B be two isotropic Hooke's tensors:

$$\forall \xi \in \mathbb{R}^{d\times d}_s, \quad A\xi = 2\mu_A \xi + \lambda_A \mathrm{tr}(\xi)\mathrm{I}, \quad B\xi = 2\mu_B \xi + \lambda_B \mathrm{tr}(\xi)\mathrm{I},$$

and let $\kappa_A = \lambda_A + \frac{2}{d}\mu_A$, $\kappa_B = \lambda_B + \frac{2}{d}\mu_B$ be the bulk moduli of A and B.

- We consider the periodic homogenization of A and B in proportions $(1-\theta)$, θ, in the lamination direction $e \in \mathbb{R}^d$.



Laminated structures in elasticity (II)

- A similar (yet more technical) calculation to that conducted in the conductivity setting yields the effective tensor A^* obtained by homogenization of this pattern:

$$(1-\theta)(A^{*-1} - B^{-1})^{-1} = (A^{-1} - B^{-1})^{-1} + \theta f_B(e),$$

where the symmetric bilinear form $f_B(e)$ over matrices is defined by:

$$\forall \xi \in \mathbb{R}_s^{d\times d}, \quad f_B(e)\xi : \xi = B\xi : \xi - \frac{1}{\mu_B}|B\xi e|^2 + \frac{\mu_B + \lambda_B}{\mu_B(2\mu_B + \lambda_B)}((B\xi)e\cdot e)^2.$$

- Likewise, the rank p sequential laminate A^* obtained by mixing the matrix B with inclusions A in proportions θ and $(1-\theta)$, with lamination directions e_i and lamination parameters m_i, $i = 1, \dots, p$, reads:

$$(1-\theta)(A^{*-1} - B^{-1})^{-1} = (A^{-1} - B^{-1})^{-1} + \theta \sum_{i=1}^{p} m_i f_B(e_i).$$

Part IV

Mathematical homogenization

Optimal bounds in the two-phase conductivity case (I)

In the two-phase conductivity setting, the set G_θ can be characterized explicitly; see [Allh], Th. 2.2.3.1 for a proof.

Theorem 18.

The set G_θ of all composites obtained by a mixture of α and β in proportions $(1-\theta)$ and θ is the set of all symmetric $d \times d$ matrices whose eigenvalues $\lambda_1, \ldots, \lambda_d$ satisfy:

$$\alpha \leq \lambda_\theta^- \leq \lambda_i \leq \lambda_\theta^+ \leq \beta, \quad i = 1, \ldots, d,$$

$$\sum_{i=1}^{d} \frac{1}{\lambda_i - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha},$$

and

$$\sum_{i=1}^{d} \frac{1}{\beta - \lambda_i} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

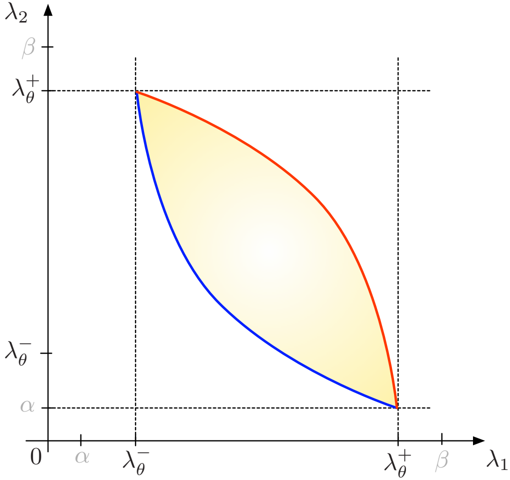
where

$$\lambda_\theta^- = \left(\frac{1-\theta}{\alpha} + \frac{\theta}{\beta} \right)^{-1} \text{ and } \lambda_\theta^+ = (1-\theta)\alpha + \theta\beta.$$

These bounds are optimal, as that they are realized by a rank d laminate.

Optimal bounds in the case of conductivity (II)

The first set of inequalities describes lower bounds for the λ_i, while the second accounts for upper bounds.



The situation in linear elasticity

- Unfortunately, no such exact characterization of the set G_θ is available in the context of linearized elasticity.

- Only bounds over tensors $A^* \in G_\theta$ are available, such as the Hashin-Shtrikman bounds over isotropic tensors in G_θ.

Part IV

Mathematical homogenization

Justification of relaxation by homogenization (I)

Let us recall the shape optimization problem of interest.

- The problem reads

$$\min_{\Omega \subset D} J(\Omega), \text{ where } J(\Omega) := \int_D j(u_\Omega)\,\mathrm{d}x + \ell \mathrm{Vol}(\Omega), \tag{SO}$$

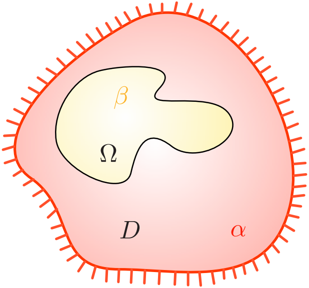
  and $j : \mathbb{R} \to \mathbb{R}$ is a given, smooth function.

- The temperature $u_\Omega \in H^1_0(D)$ is the solution to the two-phase conductivity equation:

$$\begin{cases} -\mathrm{div}(\gamma_\Omega \nabla u_\Omega) = f & \text{in } D, \\ u_\Omega = 0 & \text{on } \partial D, \end{cases}$$

  where γ_Ω reads:

$$\gamma_\Omega(x) = \alpha + \chi_\Omega(x)(\beta - \alpha), \quad x \in D.$$



Justification of relaxation by homogenization (II)

The relaxed formulation of this problem reads:

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*), \qquad \text{(H)}$$

where

- $\mathcal{CD}$ is the set of composite designs:

$$\mathcal{CD} := \left\{ (\theta, A^*) \in L^\infty(D, [0,1]) \times L^\infty(D, \mathcal{M}_{\alpha,\beta}), \;\; A^*(x) \in G_{\theta(x)} \text{ a.e. } \; x \in D \right\};$$

- The relaxed functional $J(\theta, A^*)$ reads:

$$J(\theta, A^*) = \int_D j(u_{\theta,A^*})\, \mathrm{d}x + \ell \int_D \theta(x)\, \mathrm{d}x,$$

- $u_{\theta,A^*} \in H^1_0(D)$ is the unique solution to the homogenized equation:

$$\left\{ \begin{array}{rll} -\mathrm{div}(A^*(x) \nabla u_{\theta,A^*}) = f & \text{in } D, \\ u_{\theta,A^*} = 0 & \text{on } \partial D. \end{array} \right.$$

Justification of relaxation by homogenization (III)

The following result is Th.3.2.1.1 in [Allh].

Theorem 19.

The problem (H) is a relaxation of the shape optimization problem (SO), i.e.:

1. *The problem (H) has at least one global minimizer.*
2. *For every minimizing sequence Ω_n of classical designs for (SO), there exists a subsequence (still labeled by $_n$) such that*
   - *The functions χ_{Ω_n} converge weakly * to a density $\theta \in L^\infty(D,[0,1])$,*
   - *The conductivity $\Big(\alpha\chi_{\Omega_n} + (1-\chi_{\Omega_n})\beta\Big)\mathrm{I}$ H-converges to a matrix $A^*(x)$,*

   and $(\theta, A^*) \in \mathcal{CD}$ is a minimizer for (H).
3. *Conversely, for every minimizer $(\theta, A^*) \in \mathcal{CD}$ of (H), there exists a sequence Ω_n of shapes such that:*
   - *$\chi_{\Omega_n} \to \theta$ weakly * in $L^\infty(D,[0,1])$;*
   - *$\alpha\chi_{\Omega_n} + (1-\chi_{\Omega_n})\beta$ H-converges to A^*;*
   - *Ω_n is a minimizing sequence for $J(\Omega)$.*

Part IV

Mathematical homogenization

Numerical homogenization: the conductivity case (I)

- In the two-phase conductivity setting, the set G_θ is characterized exactly by a set of inequalities over matrix eigenvalues.

- In practice, it is not trivial to use this characterization in the resolution of the problem
$$\min_{(\theta, A^*) \in \mathcal{CD}} J(\theta, A^*),$$
since the constraint that $A^*(x)$ should belong to $G_{\theta(x)}$ for a.e. $x \in D$ is quite difficult to enforce.

- Fortunately, the following result, which is very particular to the conductivity context, allows for a simpler parametrization of the set $\mathcal{CD}$.

Theorem 20.

Let $(\theta, A^*) \in \mathcal{CD}$ be a minimizer of (H). Then there exists another minimizer $(\tilde{\theta}, \tilde{A}^*) \in \mathcal{CD}$ of (H) such that

$$\tilde{A}^*(x) \text{ is a rank one laminate in } G_{\tilde{\theta}(x)} \quad \text{for a.e. } x \in D.$$

Sketch of proof:

- The derivative of the partial mapping $A^* \mapsto J(\theta, A^*)$ reads:

$$\forall H \in \mathbb{R}_s^{d \times d}, \quad \frac{\partial J}{\partial A^*}(\theta, A^*)(H) = \int_D H \nabla u_{\theta, A^*} \cdot \nabla p_{\theta, A^*} \, dx,$$

  where the adjoint state $p_{\theta, A^*} \in H_0^1(D)$ is the solution to

$$\begin{cases} -\text{div}(A^* \nabla p_{\theta, A^*}) = -j'(u_{\theta, A^*}) & \text{in } D, \\ p_{\theta, A^*} = 0 & \text{on } \partial D. \end{cases}$$

- The set G_θ is convex, and so, for fixed $\theta \in L^\infty(D, [0,1])$, the optimality condition for A^* reads:

$$\forall A^0(x) \in L^\infty(D, \mathcal{M}_{\alpha,\beta}) \text{ s.t. } A^0(x) \in G_{\theta(x)} \text{ a.e. } x \in D,$$

$$\int_D (A^0 - A^*) \nabla u_{\theta, A^*} \cdot \nabla p_{\theta, A^*} \, dx \geq 0.$$

Numerical homogenization: the conductivity case (III)

- Considering perturbations $A^0(x)$ of $A^*(x)$ about any point $x \in D$, it follows that $A^*(x)$ necessarily satisfies, for a.e. $x \in D$:

$$(A^*\nabla u_{\theta,A^*} \cdot \nabla p_{\theta,A^*})(x) = \min_{A^0 \in G_\theta(x)} (A^0 \nabla u_{\theta,A^*} \cdot \nabla p_{\theta,A^*})(x).$$

  We now extract information about A^* from this requirement.

- Let $x \in D$ be fixed. We assume that $\nabla u_{\theta,A^*}(x) \neq 0$ and $\nabla p_{\theta,A^*}(x) \neq 0$; see [Allh], Th. 3.2.2.3 for the proof in the general case.

- Denoting $e = \frac{\nabla u_{\theta,A^*}(x)}{|\nabla u_{\theta,A^*}(x)|}$ and $e' = \frac{\nabla p_{\theta,A^*}(x)}{|\nabla p_{\theta,A^*}(x)|}$, it holds:

$$\forall A^0 \in G_{\theta(x)}, \quad 4A^0 e \cdot e' = A^0(e+e')\cdot(e+e') - A^0(e-e')\cdot(e-e').$$

- Hence, we obtain the lower bound:

$$\begin{aligned} \min_{A^0 \in G_{\theta(x)}} 4A^0 e \cdot e' &\geq \min_{A^0 \in G_{\theta(x)}} A^0(e+e')\cdot(e+e') - \max_{A^0 \in G_{\theta(x)}} A^0(e-e')\cdot(e-e') \\ &= \lambda^-_{\theta(x)}|e+e'|^2 - \lambda^-_{\theta(x)}|e-e'|^2. \end{aligned}$$

Numerical homogenization: the conductivity case (IV)

- Conversely, since $(e+e')$ and $(e-e')$ are orthogonal, there exists a rank 1 laminate $A^1 \equiv A^1(x) \in G_{\theta(x)}$ in direction $(e+e')$ satisfying:

$$A^1(e+e') = \lambda^-_{\theta(x)}(e+e'), \text{ and } A^1(e-e') = \lambda^+_{\theta(x)}(e-e').$$

- Thus, we have proved that:

$$\min_{A^0 \in G_{\theta(x)}} 4A^0 e\cdot e' = 4A^1 e\cdot e' = \lambda^-_{\theta(x)}|e+e'|^2 - \lambda^-_{\theta(x)}|e-e'|^2.$$

- Actually, from the Voigt-Reuss bounds on homogenized tensors, any tensor $A^0 \in G_{\theta(x)}$ realizing the above minimum necessarily satisfies:

$$A^0(e+e') = A^1(e+e') = \lambda^-_{\theta(x)}(e+e') \text{ and } A^0(e-e') = A^1(e-e') = \lambda^+_{\theta(x)}(e-e').$$

- Hence, the matrix field $A1(x)$ is such that:

$$A^*\nabla u_{\theta,A^*} = A^1\nabla u_{\theta,A^*} \text{ and } A^*\nabla p_{\theta,A^*} = A^1\nabla p_{\theta,A^*}, \text{ a.e. } x \in D.$$

Numerical homogenization: the conductivity case (V)

- The functions u_{θ,A^*} and $p_{\theta,A^*} \in H^1_0(D)$ satisfy:

$$\begin{cases} -\text{div}(A^1(x)\nabla u_{\theta,A^*}) = f & \text{in } D, \\ u_{\theta,A^*} = 0 & \text{on } \partial D, \end{cases}$$

  and

$$\begin{cases} -\text{div}(A^1(x)\nabla p_{\theta,A^*}) = -j'(u_{\theta,A^*}) & \text{in } D, \\ p_{\theta,A^*} = 0 & \text{on } \partial D. \end{cases}$$

  From the well-posedness of both problems, we infer:

$$u_{\theta,A^1} = u_{\theta,A^*} \text{ and } p_{\theta,A^1} = p_{\theta,A^*} \text{ on } D.$$

- As a result, the matrix field $A^1(x) \in G_{\theta(x)}$ satisfies:

$$J(\theta, A^1) = J(\theta, A^*) = \int_D j(u_{\theta,A^*})\,\mathrm{d}x,$$

  and so $(\theta, A^1) \in \mathcal{CD}$ is also optimal for the problem (H).

□

Numerical homogenization: the conductivity case (VI)

- By virtue of this theorem, the minimization problem (H) is equivalent to:

$$\min_{(\theta,A)\in\mathcal{L}^1\mathcal{D}} J(\theta,A), \text{ where } J(\theta,A) = \int_D j(u_{\theta,A})\,\mathrm{d}x + \ell \int_D \theta(x)\,\mathrm{d}x,$$

where $\mathcal{L}^1\mathcal{D} \subset \mathcal{CD}$ is the set of rank 1 laminated composites:

$$\mathcal{L}^1\mathcal{D} := \Big\{(\theta,A^*) \in L^\infty(D,[0,1])\times L^\infty(D,\mathcal{M}_{\alpha,\beta}),\ A^*(x) \text{ is a rank 1 laminate of } \alpha \text{ and } \beta \text{ in proportions } (1-\theta(x)) \text{ and } \theta(x) \text{ a.e. } \ x\in D\Big\}.$$

- In 2d, for any $\theta\in[0,1]$, any matrix $A^*\in G_\theta$ can be written as:

$$A^* \equiv A^*(\theta,\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \lambda_\theta^- & 0 \\ 0 & \lambda_\theta^+ \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

- Then, in 2d, (H) rewrites as a parametric optimization problem:

$$\min_{(\theta,\phi)\in\mathcal{U}_{\mathrm{ad}}} J(\theta,\phi), \quad J(\theta,\phi) = \int_D j(u_{\theta,\phi})\,\mathrm{d}x + \ell \int_D \theta(x)\,\mathrm{d}x,$$

where $u_{\theta,\phi} \in H^1_0(D)$ is the solution to

$$\begin{cases} -\mathrm{div}(A(\theta(x),\phi(x))\nabla u_{\theta,\phi}) = f & \text{in } D, \\ u_{\theta,\phi} = 0 & \text{on } \partial D. \end{cases}$$

and

$$\mathcal{U}_{\mathrm{ad}} = \Big\{(\theta,\phi) \in L^\infty(D,[0,1]) \times L^\infty(D,[0,\pi])\Big\}.$$

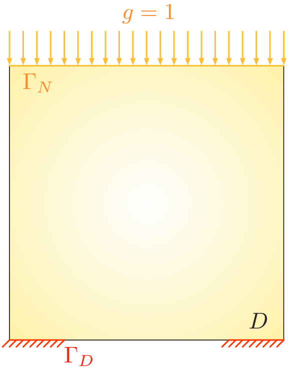
- All the numerical methods developed in the context of parametric optimization (gradient algorithm, etc.) can be readily applied to this problem.

Numerical example: the optimal radiator (I)

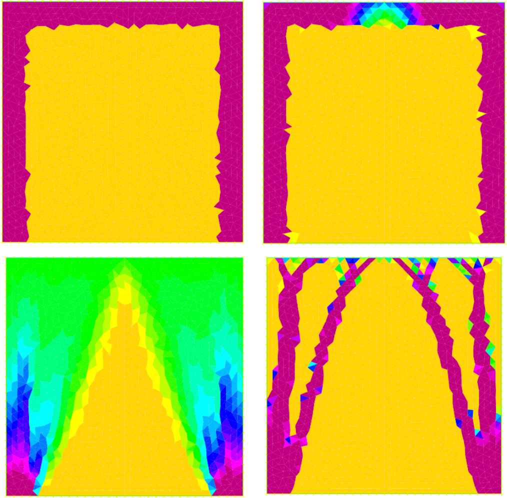
- We minimize the compliance of a thermal chamber D, which equals in this case the mean temperature where heating occurs:

$$\min_{(\theta,A^*)\in\mathcal{CD}} J(u_{\theta,A^*}), \text{ where } J(u_{\theta,A^*}) = \int_{\Gamma_N} u_{\theta,A^*} \, \mathrm{d}s.$$

- A volume constraint is added by means of a fixed Lagrange multiplier.



Numerical example: the optimal radiator (II)



(From left to right, top to bottom) Initialization and several iterations of the optimization of the compliance in a thermal chamber; reprinted from [Allc].

Numerical homogenization: the linear elasticity case

- We consider the homogenized problem:

$$\min_{(\theta,A^*)\in\mathcal{CD}} J(\theta,A^*), \text{ where } J(\theta,A^*) = \int_D j(u_{\theta,A^*})\,\mathrm{d}x + \ell \int_D \theta(x)\,\mathrm{d}x, \qquad (\mathrm{H})$$

  and $u_{\theta,A^*} \in H^1_0(D)^d$ is the unique solution to the homogenized elasticity system:

$$\begin{cases} -\mathrm{div}\,(A^* e(u_{\theta,A^*})) = f & \text{in } D, \\ u_{\theta,A^*} = 0 & \text{on } \partial D, \end{cases}$$

- The set G_θ (and that of composite desgin $\mathcal{CD}$) is not explicitly known!
- Fortunately, when the objective function is the compliance (plus volume), i.e.

$$J(\theta,A^*) = \int_D f\cdot u_{\theta,A^*}\,\mathrm{d}x + \ell\int_D \theta(x)\,\mathrm{d}x,$$

  the following result allows to cast the resolution of (H) within a set which is explicitly parametrized.

Theorem 21.

When the minimized function $J(\theta,A^*)$ is the compliance, the problem (H) has one global minimizer (θ,A^*) in which is a rank d laminated composite.

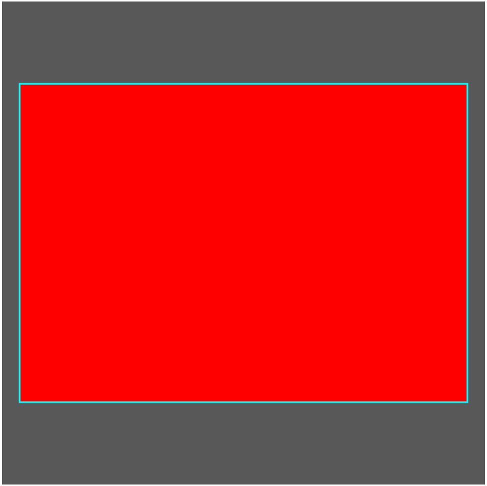
Minimization of the compliance of a 2d cantilever (I)

- We minimize the compliance of a 2d cantilever:

$$\min_{(\theta, A^*) \in \mathcal{CD}} J(u_{\theta,A^*}), \text{ where } J(u_{\theta,A^*}) = \int_{\Gamma_N} g \cdot u_{\theta,A^*} \, \mathrm{d}s.$$

- A volume constraint is added by means of a fixed Lagrange multiplier.

Minimization of the compliance of a 2d cantilever (II)



Credits: [Allaire2]

Minimization of the compliance of a 2d cantilever (III)

Credits: [Allaire2]

Beyond compliance: partial relaxation

- When the objective function $J(\theta, A^*)$ is no longer the compliance, the previous strategy can no longer be employed.
- Nevertheless, one may perform a formal partial relaxation of the original shape optimization problem.
- This amounts to searching for the minimizer of $J(\theta, A^*)$ over the subset $\mathcal{LD} \subset \mathcal{CD}$ of laminated composites:

$$\min_{(\theta, A^*) \in \mathcal{LD}} J(\theta, A^*), \text{ where } J(\theta, A^*) = \int_D j(u_{\theta, A^*}) \, \mathrm{d}x + \ell \int_D \theta(x) \, \mathrm{d}x, \qquad \text{(PR)}$$

and the set $\mathcal{LD}$ is defined by:

$$\mathcal{LD} := \Big\{ (\theta, A^*) \in L^\infty(D, [0,1]) \times L^\infty(D, \mathcal{M}_{\alpha,\beta}), \;\; A^*(x) \text{ is a laminate of } A \text{ and } B \text{ in proportions } (1-\theta(x)) \text{ and } \theta(x) \text{ a.e. } \; x \in D \Big\}.$$

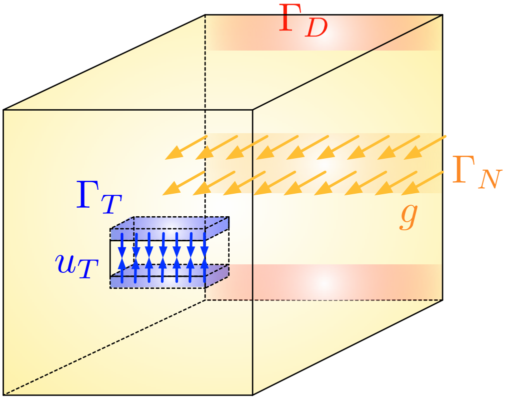
- Laminated composites $(\theta, A^*) \in \mathcal{LD}$ can be parametrized explicitly.
- Of course, there is no guarantee that solving (PR) will yield the global minimizer of (H).

Example: optimization of a 3d grip (II)

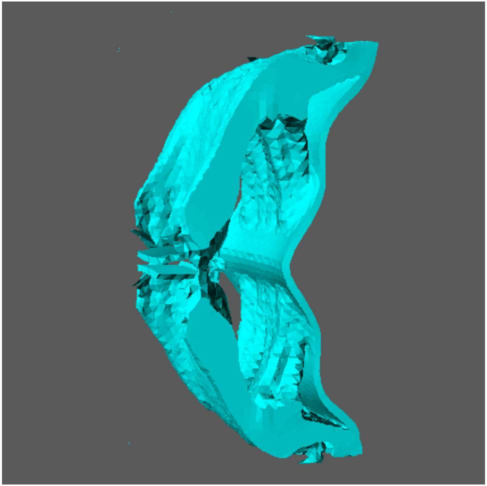
We minimize the least-square criterion:

$$\min_{(\theta, A^*) \in \mathcal{CD}} J(u_{\theta,A^*}), \text{ where } J(u_{\theta,A^*}) = \int_{\Gamma_T} |u_{\theta,A^*} - u_T|^2 \, \mathrm{d}s,$$

where $u_T : D \to \mathbb{R}^d$ is a target displacement.



Example: optimization of a 3d grip (II)



Credits: [Allaire2]

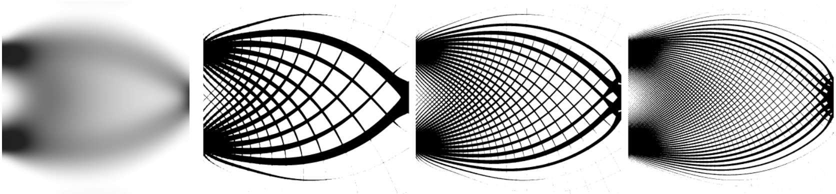
Post-processing and deshomogenization

- From the optimal composite design $(\theta^*, A^*) \in \mathcal{CD}$, a true, "black-and-white" shape Ω is easily recovered by thresholding:

$$\Omega := \{x \in D, \ \theta^*(x) > c\},$$

where $c \in [0,1]$ is chosen so that, e.g. Ω satisfies a desired volume constraint.

- More elaborate strategies are available, which do use the optimal microstructure tensor A^* to generate minimizing sequences Ω^n for $J(\Omega)$; see for instance the deshomogenization method from [PanTra, GroSig].



The "deshomogenization" method allows to infer minimizing sequences for the shape functional $J(\Omega)$ from the datum of the optimal composite design (θ, A^*) (picture from [?]).

Appendix

The dual space

Let V be a Banach space, equipped with the norm $||\cdot||$.

Definition 3.

The dual space V^* of V is the Banach space of linear continuous forms $\varphi : V \to \mathbb{R}$ of V, that is:

$$\exists C > 0, \quad \forall v \in V, \ |\langle \varphi, v \rangle| \le C||v||.$$

The norm $||\varphi||$ of an element $\varphi \in V^*$ is:

$$||\varphi|| = \sup_{v \in V, v \neq 0} \frac{|\langle \varphi, v \rangle|}{||v||}.$$

Examples

- For $1 < p < \infty$, the dual space of $L^p(\Omega)$ is $L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$:
- The dual space of $L^1(\Omega)$ can be identified with $L^\infty(\Omega)$.

Strong and weak convergence in Banach spaces

Let V be a Banach space, equipped with the norm $||\cdot||$.

- A sequence of elements $u_n \in V$ converges strongly to some $u \in V$ if:
$$||u_n - u|| \xrightarrow{n\to\infty} 0.$$
- A sequence of elements $u_n \in V$ converges weakly to some $u \in V$ if:
$$\text{For all } \varphi \in V^*, \quad \langle \varphi, u_n \rangle \xrightarrow{n\to\infty} \langle \varphi, u \rangle.$$

Of course, strong convergence implies weak convergence.

Weak * convergence in Banach spaces

Let V be a Banach space, with dual space V^*; we denote the duality pairing by

$$\langle \varphi, v \rangle_{V^*,V} \text{ or simply } \langle \varphi, v \rangle,$$

when the context is clear.

One third notion of convergence is available in V^*.

Definition 4.

*A sequence φ_n in the dual space V^* converges weakly * to some element $\varphi \in V^*$ if:*

$$\text{For all } u \in V, \quad \langle \varphi_n, u \rangle \xrightarrow{n\to\infty} \langle \varphi, u \rangle.$$

The following fundamental result is a consequence of the Banach-Alaoglu theorem.

Theorem 22.

Let φ_n be a bounded sequence in V^*; then there exists a subsequence φ_{n_k} of φ_n and an element $\varphi \in V^*$ such that:

$$\varphi_{n_k} \xrightarrow{k\to\infty} \varphi \text{ weakly * in } V^*.$$

Weak * convergence: an instructive example

Let $D \subset \mathbb{R}^d$ be a bounded domain.

- As we have just seen, $L^\infty(D)$ is the dual of $L^1(D)$.
- Let χ_n be a sequence of characteristic functions of subset Ω_n:

$$\chi_n(x) = \begin{cases} 1 & \text{if } x \in \Omega_n, \\ 0 & \text{if } x \in D \setminus \Omega_n. \end{cases}$$

- Since χ_n is bounded (by 1) in $L^\infty(D)$, a subsequence χ_{n_k} converges weakly * to some function $\theta \in L^\infty(D)$.
- It is easy to prove that θ is a density function, i.e. $\theta(x) \in [0, 1]$ for a.e. $x \in D$, but θ is not a characteristic function.
- Actually, every density function $\theta \in L^\infty(D, [0, 1])$ can be realized as the weak * limit of a sequence χ_n of characteristic functions.

Reflexive Banach spaces and Hilbert spaces

Let V be a Banach space. There is a canonical injection $J : V \to V^{**}$: for any $x \in V$, $J(x)$ is the element in V^{**} defined by:

$$\langle J(x), \varphi \rangle_{V^{**},V^*} := \langle \varphi, x \rangle_{V^*,V}.$$

This mapping is injective as a corollary of the Hahn-Banach theorem, and it allows to see V as a subspace of V^{**}.

Definition 5.

The Banach space V is reflexive if the mapping J is an isomorphism.

In reflexive Banach spaces, weak and weak* convergence are identical notions.

Proposition 23.

Let v_n be a bounded sequence in a reflexive Banach space; then there exists a subsequence v_{n_k} which converges weakly to some $v \in V$.

In particular, Hilbert spaces are reflexive.
It follows that every bounded sequence in a Hilbert space converges weakly.

Proposition 24.

Let u_n and v_n be two sequences in a Hilbert space H such that:

$$u_n \xrightarrow{n\to\infty} u \text{ strongly in } H, \text{ and } v_n \xrightarrow{n\to\infty} v \text{ weakly in } H.$$

Then

$$\langle u_n, v_n \rangle \xrightarrow{n\to\infty} \langle u, v \rangle.$$

A useful lemma in periodic homogenization

The following lemma characterizes the weak convergence of a sequence of functions obtained by ε-rescaling and periodization of a function $f \in L^2(Y)$.

Lemma 25.

Let $f \in L^2_\#(Y)$, and let f_ε be the sequence in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ defined by:

$$f_\varepsilon(x) := f\left(\frac{x}{\varepsilon}\right), \text{ a.e. } x \in \mathbb{R}^d.$$

Then for any bounded set $D \subset \mathbb{R}^d$, f_ε converges weakly in $L^2(D)$ to the average $m(f) := \int_Y f(y)\,\mathrm{d}y$.

$$\forall g \in L^2(D), \quad \int_D f_\varepsilon(x) g(x)\,\mathrm{d}x \xrightarrow{\varepsilon \to 0} m(f) \int_D g(x)\,\mathrm{d}x.$$

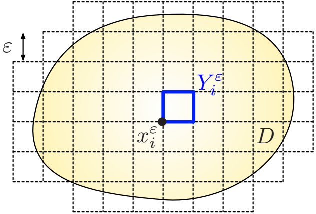
<u>*Proof:*</u> The proof proceeds within two steps.

A useful lemma

Step 1. We prove that the sequence f_ε is bounded in $L^2(D)$.

- Let $\{Y_i^\varepsilon\}_{i=1,\dots,n(\varepsilon)}$ be the covering of D defined by ε-rescaling and translation of the unit cell Y:

$$Y_i^\varepsilon = x_i^\varepsilon + \varepsilon Y, \quad i = 1, \dots, n(\varepsilon).$$



- The number $n(\varepsilon)$ of such cells is:

$$n(\varepsilon) = \frac{|D|}{\varepsilon^d}(1 + \mathrm{o}(1)).$$

- For each $i = 1, \dots, n(\varepsilon)$ a change of variables yields:

$$\int_{Y_i^\varepsilon} f_\varepsilon^2(x)\,\mathrm{d}x = \varepsilon^d \int_Y f^2(y)\,\mathrm{d}y.$$

- It follows that:

$$||f_\varepsilon||^2_{L^2(D)} = \sum_{i=1}^{n(\varepsilon)} \int_{Y_i^\varepsilon \cap D} f_\varepsilon^2(x)\,\mathrm{d}x = |D| \left(\int_Y f^2(y)\,\mathrm{d}y \right)(1 + \mathrm{o}(1)),$$

and so f_ε is indeed a bounded sequence in $L^2(D)$.

A useful lemma

Step 2. Thanks to the density of $\mathcal{C}_c^\infty(D)$ in $L^2(D)$, it is enough to prove that:

$$\forall \phi \in \mathcal{C}_c^\infty(D), \quad \int_D f_\varepsilon(x)\phi(x)\,\mathrm{d}x \xrightarrow{\varepsilon \to 0} m(f) \int_D \phi(x)\,\mathrm{d}x.$$

- For a given function $\phi \in \mathcal{C}_c^\infty(D)$ and for each cell $i = 1, \dots, n(\varepsilon)$, it holds:

$$\left| \int_{Y_i^\varepsilon} f_\varepsilon(x)\phi(x)\,\mathrm{d}x - m(f)\varepsilon^d \phi(x_i^\varepsilon) \right| \leq \varepsilon^d m(|f|) \max_{x,x' \in Y_i^\varepsilon} |\phi(x) - \phi(x')|.$$

- Thanks to the triangle inequality, this entails:

$$\left| \int_D f_\varepsilon(x)\phi(x)\,\mathrm{d}x - m(f)\varepsilon^d \sum_{i=1}^{n(\varepsilon)} \phi(x_i^\varepsilon) \right| \leq \underbrace{n(\varepsilon)\varepsilon^d}_{\leq Cste} \underbrace{m(|f|) \max_{\substack{x,x' \in D \\ |x-x'| \leq d\varepsilon}} |\phi(x) - \phi(x')|}_{= \mathrm{o}(1) \text{ since } \phi \in \mathcal{C}_c^\infty(D)}.$$

- In addition, since $\phi \in \mathcal{C}_c^\infty(D)$, the Riemann sum theory yields:

$$\int_D \phi(x)\,\mathrm{d}x = \varepsilon^d \sum_{i=1}^{n(\varepsilon)} \phi(x_i^\varepsilon) + \mathrm{o}(1).$$

- Combining both estimates allows to conclude.

Theorem 26.

Let $D \subset \mathbb{R}^d$ be a bounded, Lipschitz domain. Then the injection $H^1(D) \subset L^2(D)$ is compact, i.e. for every sequence $v_n \in H^1(D)$,

$$\text{If } v_n \xrightarrow{n\to\infty} v \text{ weakly in } H^1(D), \text{ then } v_n \xrightarrow{n\to\infty} v \text{ strongly in } L^2(D).$$

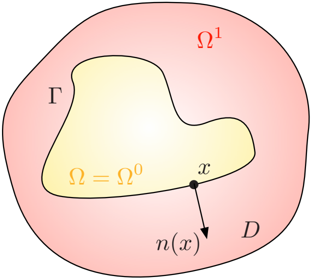
Transmission boundary conditions (I)

- Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain.

- A Lipschitz subdomain $\Omega \Subset D$ delimits two phases within D:

$$\Omega_0 := \Omega, \text{ and } \Omega_1 := D \setminus \overline{\Omega},$$

separated by the interface $\Gamma = \partial\Omega$.



- n is the unit normal vector to Γ, pointing outward $\partial\Omega$.

- We consider the two-phase conductivity equation:

$$\left\{ \begin{array}{rl} -\mathrm{div}(A(x)\nabla u) = f & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{array} \right. \quad \text{where } A(x) := \left\{ \begin{array}{ll} \alpha_0 & \text{in } \Omega_0, \\ \alpha_1 & \text{in } \Omega_1, \end{array} \right. \qquad \text{(TP)}$$

Theorem 27 (Transmission conditions).

- *Let u be the unique solution to (TP) in $H^1_0(D)$, and let the restrictions*

$$u_0 := u|_{\Omega_0} \in H^1(\Omega_0) \text{ and } u|_{\Omega_1} \in H^1(\Omega_1).$$

Then, u_0 and u_1 are solutions to the coupled system

$$-\mathrm{div}(\alpha_0 \nabla u_0) = f \quad \text{in} \quad \Omega_0, \quad \text{and} \quad \left\{ \begin{array}{ll} -\mathrm{div}(\alpha_1 \nabla u_1) = f & \text{in } D, \\ u_1 = 0 & \text{on } \partial D, \end{array} \right. \tag{C}$$

supplemented with the transmission conditions at the interface Γ:

$$u_0 = u_1 \quad \text{and} \quad \alpha_0 \frac{\partial u_0}{\partial n} = \alpha_1 \frac{\partial u_1}{\partial n} \quad \text{on } \Gamma. \tag{TC}$$

- *Conversely, if $u_0 \in H^1(\Omega_0)$ and $u_1 \in H^1(\Omega_1)$ are solutions to (C) (TC), then the function*

$$u(x) = \left\{ \begin{array}{ll} u_0(x) & \text{for } x \in \Omega_0, \\ u_1(x) & \text{for } x \in \Omega_1, \end{array} \right.$$

is the unique solution to (TP) in $H^1_0(D)$.

Transmission boundary conditions (III)

Hint of the proof:

We only prove the implication "(TP) $\Rightarrow$ (C) + (TC)" (the converse being analogous).

- Since $u \in H_0^1(D)$, the trace theorem directly implies the first condition in (TC):

$$u_0 = u_1 \text{ on } \Gamma.$$

- The variational formulation for u reads: for any $\phi \in \mathcal{C}_c^\infty(D)$,

$$\begin{aligned} \int_D f\phi \,\mathrm{d}x &= \int_D A(x)\nabla u \cdot \nabla \phi \,\mathrm{d}x \\ &= \int_{\Omega_0} \alpha_0 \nabla u_0 \cdot \nabla \phi \,\mathrm{d}x + \int_{\Omega_1} \alpha_1 \nabla u_1 \cdot \nabla \phi \,\mathrm{d}x. \end{aligned}$$

- Applying Green's formula to both integrals in the above right-hand side yields:

$$\int_D f\phi \,\mathrm{d}x = -\int_{\Omega_0} \mathrm{div}(\alpha_0 \nabla u_0)\phi \,\mathrm{d}x - \int_{\Omega_1} \mathrm{div}(\alpha_1 \nabla u_1)\phi \,\mathrm{d}x$$
$$+ \int_\Gamma \left(\frac{\partial u_0}{\partial n} - \frac{\partial u_1}{\partial n} \right) \phi \,\mathrm{d}s,$$

where the - sign in front of $\frac{\partial u_1}{\partial n}$ follows from the fact that the unit normal vector to Γ pointing outward Ω_1 is $-n$.

Transmission boundary conditions (IV)

- Taking arbitrary $\phi \in \mathcal{C}_c^\infty(\Omega_0)$ with support inside Ω_0 yields:

$$-\mathrm{div}(\alpha_0 \nabla u_0) = f \text{ in } \Omega_0,$$

  and likewise:

$$-\mathrm{div}(\alpha_1 \nabla u_1) = f \text{ in } \Omega_1.$$

- There remains: for any $\phi \in \mathcal{C}_c^\infty(D)$,

$$\int_\Gamma \left(\frac{\partial u_0}{\partial n} - \frac{\partial u_1}{\partial n} \right) \phi \, \mathrm{d}s = 0.$$

  Since the trace of ϕ on Γ is arbitrary, the second transmission condition follows:

$$\alpha_0 \frac{\partial u_0}{\partial n} = \alpha_1 \frac{\partial u_1}{\partial n} \text{ on } \Gamma.$$

$\square$

Remark This principle extends to many other physical situations, such as that of linearized elasticity.

The "uniqueness of the limit argument" (I)

In a very general Hausdorff topological space X,

- Assume that we aim to prove that a sequence $x_n \in X$ converges to some element $\ell \in X$.
- In practice, it often happens that we can only prove that a subsequence x_{n_p} converges to ℓ.
- Actually, quite often, what we are able to prove is that

"From any subsequence x_{n_p} of x_n we can extract a further subsequence $x_{n_{p_q}}$ of x_{n_p} which converges to ℓ".

- Then, a simple argument reveals that the whole sequence x_n converges to ℓ.

The "uniqueness of the limit argument" (II)

Lemma 28.

Let X be a Hausdorff topological space, and let $x_n \in X$ be a sequence. Assume that there exists $\ell \in X$ such that

> *For all subsequence $\left\{x_{n_p}\right\}_{p\in\mathbb{N}}$, there exists a further subsequence $\left\{x_{n_{p_q}}\right\}_{q\in\mathbb{N}}$ converging to ℓ.*

Then the whole sequence x_n converges to ℓ as $n \to \infty$.

<u>Proof:</u> Assume that x_n does not converge to ℓ. Then there exists an open subset $U \subset X$ containing ℓ and a subsequence $\left\{x_{n_p}\right\}_{p\in\mathbb{N}}$ such that:

$$\forall p \in \mathbb{N}, \quad x_{n_p} \in X \setminus U.$$

From the assumption, one may then extract a subsequence $\left\{x_{n_{p_q}}\right\}_{q\in\mathbb{N}}$ from $\left\{x_{n_p}\right\}_{p\in\mathbb{N}}$ such that:

$$x_{n_{p_q}} \xrightarrow{q\to\infty} \ell.$$

In particular, there exists $p \in \mathbb{N}$ large enough such that $x_{n_p} \in U$, which is a contradiction.

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