HW 5: correction

$\boxed{\text{Exercise 1}}$

1) Find the solution to the PDE: $\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} + u = 3x$, with "boundary condition" $u(0,x) = 2x$.

From the look of the PDE, the characteristic curves $s \mapsto (t(s), x(s))$ have equation: $\begin{cases} t'(s) = 1 \\ x'(s) = x(s). \end{cases}$

i.e. up to a change in the parametrization of curves: $\begin{cases} t(s) = s \\ x(s) = Ce^{s}, \end{cases}$ for some cst C.

We now introduce the value function $z(s) := u(t(s), x(s))$, i.e. the value of u along a fixed characteristic curve (i.e. a given C).

We have: $z'(s) + z(s) = 3x(s) = 3Ce^{s}$, which is an ode we know how to study.

- the homogeneous equation has solution $z_{hom}(s) = De^{-s}$.
- we search for a sol. to the inhomogeneous equation under the form $z(s) = D(s)e^{-s}$, (method of variation of constants).

we have $D'(s)e^{-s} = 3Ce^{s}$

i.e. $D'(s) = 3Ce^{2s}$, and $D(s) = \frac{3}{2}Ce^{2s} + E$, for some constant E,

and the solution to the inhomogeneous equation is: $z(s) = D(s)e^{-s} = \frac{3}{2}Ce^{s} + Ee^{-s}$.

Coming back to our problem, E is an arbitrary function of the constant C, indicating the considered characteristic curve.

$z(s) = \frac{3Ce^{s}}{2} + f(C)e^{-s}$

Returning to the original variables (t,x), $\boxed{u(t,x) = \frac{3x}{2} + f(xe^{-t})e^{-t}.}$

If eventually: $u(0,x) = 2x$, then $\frac{3x}{2} + f(x) = 2x$, and $f(x) = \frac{x}{2}$. All in all $\boxed{u(t,x) = \frac{3x}{2} + \frac{xe^{-2t}}{2}.}$

$\boxed{\text{Exercise 2}}$

1) Simply observe that the difference $w(t,x) := u(t,x) - u(t,x)$ also satisfies the one-dimensional heat equation for $x \in (0,\ell)$, $t > 0$.

Furthermore, $w(t,x) \geq 0$ for $\begin{cases} (t,x) = (0,x), & x \in [0,\ell] \\ (t,x) = (t,0), & t > 0 \\ (t,x) = (t,\ell), & t > 0 \end{cases}$, and the conclusion follows from a use of the max. principle.

2) The difference $w = v - u$ satisfies $\frac{\partial w}{\partial t} - k\frac{\partial^2 w}{\partial x^2} = \underbrace{(g - f)}_{\geq 0}$, and is ≥ 0 for $\begin{cases} (t,x) = (0,x) \\ - - - \end{cases}$

The max. principle stays true for a positive source (which is easily checked by coming back to the proof)

$\boxed{\text{Exercise 3}}$ ~~Th~~ Assume that there exist two solutions u_1, u_2 to the considered heat equation, with the considered set of boundary conditions.

Their difference $w := u_1 - u_2$ satisfies: $\begin{cases} \frac{\partial w}{\partial t} - k\frac{\partial^2 w}{\partial x^2} = 0 \\ \frac{\partial w}{\partial x}(t,0) = \frac{\partial w}{\partial x}(t,\ell) = 0 \quad \forall t > 0 \\ \forall x \in (0,\ell),\ w(0,x) = 0 \end{cases}$

and we are brought back to showing that such a solution to the heat equation with homogeneous Neumann BC, and homogeneous IC is necessarily 0.

We use the energy method: let $E(t) := \frac{1}{2}\int_0^\ell w^2(t,x)\,dx$.

Then: $\frac{d}{dt}E(t) = \int_0^\ell w \frac{\partial w}{\partial t}\,dx$

$= \int_0^\ell w\, k \frac{\partial^2 w}{\partial x^2}\,dx$ by using (HE)

$= \underbrace{k\left[w\frac{\partial w}{\partial x}\right]_0^\ell}_{= 0 \text{ from Neumann BC.}} - k\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx$ by an integration by parts

≤ 0. Hence the energy is decreasing, that is, $\forall t > 0$, $E(t) \leq E(0) = 0$ because of the initial condition for w.

and, necessarily $\int_0^\ell w^2(t,x)\,dx = 0 \Rightarrow w = 0$ because w^2 is positive and continuous, with 0 integral.

$\boxed{\text{Exercise 4}}$ 1) We define $u(t,x) := e^{-bt}v(t,x)$

Hence: $\frac{\partial u}{\partial t}(t,x) = -b\left(e^{-bt}v(t,x)\right) + e^{-bt}\frac{\partial v}{\partial t}(t,x)$

$\frac{\partial^2 u}{\partial x^2}(t,x) = e^{-bt}\frac{\partial^2 v}{\partial x^2}(t,x)$

Hence: $\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} + bu = 0$ on the one hand, and equals: $-\cancel{bu} + e^{-bt}\frac{\partial v}{\partial t}(t,x) - ke^{-bt}\frac{\partial^2 v}{\partial x^2}(t,x) + \cancel{bu} = e^{-bt}\left(\frac{\partial v}{\partial t}(t,x) - k\frac{\partial^2 v}{\partial x^2}(t,x)\right)$:

Thus, $\boxed{\frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = 0}$

2) We use the formula from the lectures: $v(t,x)$ is solution to the diffusion equation over the real line with initial condition $v(0,x) = e^{0}u(0,x) = \phi(x)$.

Hence:

$\forall t > 0, \forall x \in \mathbb{R}$, $v(t,x) = \frac{1}{\sqrt{4\pi k t}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}}\phi(y)\,dy$.

and $u(t,x) = \frac{e^{-bt}}{\sqrt{4\pi k t}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}}\phi(y)\,dy$

3) This is very similar to the previous questions.

$\boxed{\text{Exercise 5}}$ We consider the system: $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + r\frac{\partial u}{\partial t} = 0$, where $t > 0$, $x \in (0,\ell)$, and $0 < r < \frac{2\pi c}{\ell}$.

with $u(t,0) = u(t,\ell) = 0$

and the initial conditions: $\forall x \in (0,\ell)$: $u(0,x) = \phi(x)$, $\frac{\partial u}{\partial t}(0,x) = \psi(x)$.

1) This is very similar to exercise 3.

Assume that u_1, u_2 are two solutions to the system. Then $w := u_1 - u_2$ satisfies: $\begin{cases} \frac{\partial^2 w}{\partial t^2} - c^2\frac{\partial^2 w}{\partial x^2} + r\frac{\partial w}{\partial t} = 0 \\ w(t,0) = w(t,\ell) = 0 \\ w(0,x) = \frac{\partial w}{\partial t}(0,x) = 0 \end{cases}$

We multiply $\left(\frac{\partial^2 w}{\partial t^2} - c^2\frac{\partial^2 w}{\partial x^2} + r\frac{\partial w}{\partial t} = 0\right)$ by $\frac{\partial w}{\partial t}$ and integrate by parts:

$\int_0^\ell \frac{\partial^2 w}{\partial t^2}\frac{\partial w}{\partial t}\,dx - c^2\int_0^\ell \frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial t}\,dx + r\int_0^\ell \left(\frac{\partial w}{\partial t}\right)^2 dx = 0.$

All in all, we obtain: $\frac{1}{2}\frac{d}{dt}\left(\int_0^\ell \left(\frac{\partial w}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 dx\right) = -r\int_0^\ell \left(\frac{\partial w}{\partial t}\right)^2 dx.$

and the energy $E(t) = \frac{1}{2}\int_0^\ell \left(\frac{\partial w}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 dx$ is a decreasing quantity over time: $\forall t \geq 0$, $E(t) \leq E(0)$.

But $E(0) = \frac{1}{2}\int_0^\ell \varphi(x)^2 + (\varphi'(x))^2 dx = 0.$

Hence $E(t) = 0$ $\forall t \geq 0$, and we conclude as in exercise 3.

2) For a separated solution $u(t,x) = T(t)X(x)$,

$\frac{\partial^2 u}{\partial t^2}(t,x) = T''(t)X(x)$ and $\frac{\partial^2 u}{\partial x^2}(t,x) = T(t)X''(x)$. We have, $\forall t > 0$, $\forall x \in (0,\ell)$ $T''(t)X(x) - c^2T(t)X''(x) + rT'(t)X(x) = 0$.

Hence: $\frac{T''(t) + rT'(t)}{c^2T(t)} = \frac{X''(x)}{X(x)}$, and $X(0) = X(\ell) = 0$

As the left-hand side is a quantity which only depends on t,
and the right-hand side only depends on x, they are both constant:

$\exists \lambda \in \mathbb{R}$ s.t. $-\frac{T''(t) + rT'(t)}{c^2T(t)} = -\frac{X''(x)}{X(x)} = \lambda.$

3) The equation for the spatial part reads: $\begin{cases} X''(x) + \lambda X(x) = 0 & \text{on } (0,\ell) \\ X(0) = X(\ell) = 0. \end{cases}$

As in the lectures (it is actually the same equation as in the lectures), we obtain that there are only positive eigenvalues
$\boxed{\lambda_n = \left(\frac{n\pi}{\ell}\right)^2,}$ $n = 1, \ldots$, and the associated eigenfunctions read: $\boxed{X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right).}$

4) We now solve for the temporal part: $T''(t) + rT'(t) + \left(\frac{n\pi c}{\ell}\right)^2 T(t) = 0$, $n = 1, \ldots$

To this end, we solve the characteristic equation
$\alpha^2 + r\alpha + \left(\frac{n\pi c}{\ell}\right)^2 = 0.$, whose discriminant is: $\Delta = r^2 - 4\left(\frac{n\pi c}{\ell}\right)^2$, which is negative because of the assumption $r < \frac{2\pi c}{\ell}$. This equation has roots $\frac{-r \pm i\sqrt{4\left(\frac{n\pi c}{\ell}\right)^2 - r^2}}{2}$.

Thus, the solution to this ode is of the form: $\boxed{T(t) = e^{-\frac{rt}{2}}\left(a_n \cos\left(\frac{1}{2}\sqrt{4\left(\frac{n\pi c}{\ell}\right)^2 - r^2}\, t\right) + b_n \sin\left(\frac{1}{2}\sqrt{4\left(\frac{n\pi c}{\ell}\right)^2 - r^2}\, t\right)\right).}$

5) The general solution writes: $u(t,x) = e^{-\frac{rt}{2}} \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{1}{2}\sqrt{4\left(\frac{n\pi c}{\ell}\right)^2 - r^2}\, t\right) + b_n \sin\left(\frac{1}{2}\sqrt{4\left(\frac{n\pi c}{\ell}\right)^2 - r^2}\, t\right)\right)\sin\left(\frac{n\pi x}{\ell}\right).$

Deriving the associated initial conditions is up to you!

$\boxed{\text{Exercise 6}}$ The study is very similar to that of Exercise 5.
The equations for a separated solution write: $\begin{cases} T'(t) + k\lambda T(t) = 0 \\ X''(x) + \lambda X(x) = 0 \end{cases}$, $\begin{matrix} X(0) = X(\ell) \\ X'(0) = X'(\ell). \end{matrix}$

1) We study the spatial problem.

- There is no negative eigenvalue: indeed, if $\lambda = -\beta^2$, then $X(x)$ is of the form: $X(x) = Ae^{\beta x} + Be^{-\beta x}$.
But $X(0) = X(\ell) \Rightarrow A + B = Ae^{\beta\ell} + Be^{-\beta\ell}$
$X'(0) = X'(\ell) \Rightarrow \beta A - \beta B = \beta Ae^{\beta\ell} - \beta Be^{-\beta\ell}$ $\Big\}$ thus $\begin{matrix} 2A = 2Ae^{\beta\ell} \\ 2B = 2Be^{-\beta\ell} \end{matrix}$ and $A = B = 0$.

- Is 0 an eigenvalue? We solve $X''(x) = 0$,
i.e. $X(x) = Ax + B$, and $X(0) = X(\ell) \Rightarrow B = A\ell + B$, i.e. $A = 0$
$X'(0) = X'(\ell) \Rightarrow A = A.$
Hence, 0 is an eigenvalue, and an eigenfunction is $\boxed{X_0(x) = 1.}$

- Search for a positive eigenvalue $\lambda = \beta^2$. Then $X(x)$ must be of the form: $X(x) = A\cos\beta x + B\sin\beta x$.
$X(0) = X(\ell) \Rightarrow A = A\cos\beta\ell + B\sin\beta\ell$.
$X'(0) = X'(\ell) \Rightarrow \beta B = -\beta A\sin(\beta\ell) + \beta B\cos(\beta\ell)$.

which we can rewrite: $\begin{cases} A = A\cos\beta\ell + B\sin\beta\ell \\ B = B\cos\beta\ell - A\sin\beta\ell \end{cases}$ (since $\beta > 0$).

By multiplying the first equation by B, the second one by A, and subtracting:
$\cancel{AB} - \cancel{AB} = AB\cos\beta\ell - AB\cos\beta\ell + (A^2 + B^2)\sin\beta\ell$, thus $(A^2+B^2)\sin\beta\ell$.
and for X to be $\neq 0$, one must have $\sin\beta\ell = 0$, i.e. β is of the form $\beta = \frac{n\pi}{\ell}$, $n = 1, \ldots$

Conversely, if $\beta = \frac{n\pi}{\ell}$, for some $n = 1, \ldots$, then, there are two associated eigenfunctions to this eigenvalue,
namely: $X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right)$
and $X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right)$ (i.e. they are linearly independent, and both satisfy the necessary form for $X_\beta(x)$).

2) This is as in the lectures.