Some questions around the control of nanowires

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ANR project : SICOMAF

(SI'mulation et CO'ntrôlè de MA'tériau's Ferromagnétiques)

Aims

Model and control ferromagnetic materials, optimize the shape and composition of ferromagnetic particles for technological applications.

Possible applications :
- waves guides (mobile phones, ...)
- magnetic storage : hard disks, magnetic memories MRAMs
- magnetic micro particles for antiradar paints
- Behavior of magnetic components for nano-electronic sciences

Models available : micromagnetism

Description of the magnetization (vector field) :
- static configurations : micromagnetic theory of Brown (60’s)
- dynamics : Landau-Lifchitz equation
Basis of micromagnetism

Thermodynamical description of ferromagnetic materials: Micromagnetism, W.F. Brown, in the 60’s.

Magnetic domain

Open subset $\Omega \subset \mathbb{R}^3$ occupied by the ferromagnetic material.

$\Rightarrow$ spontaneous magnetization, only depending on the temperature:

Magnetization

Vector field $m$ on $\Omega$, s.t. $\|m(x)\| = 1$ a.e. in $\Omega$.

- Remanent magnetization under external field
- Critical temperature dividing linear and nonlinear behavior
- Emergence of microstructures: walls and domains
Basis of micromagnetism

Static micromagnetic model

\[
\min_{m \in H^1(\Omega, \mathbb{S}^2)} E(m)
\]

where

\[
E(m) = A \int_{\Omega} \| \nabla m \|^2 + \int_{\mathbb{R}^3} \| H_d(m) \|^2 - \int_{\Omega} m \cdot H_{\text{ext}}
\]

- \( A \int_{\Omega} \| \nabla m \|^2 \): exchange term
- \( H_d(m) \): demagnetizing field solution of

\[
\begin{cases}
\text{curl}(H_d) = 0 \\
\text{div}(H_d) = -\text{div}(m)
\end{cases}
\]

in \( \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3) \)

\( m \) extended by 0 outside \( \Omega \)

- \( H_{\text{ext}} \): Zeeman, action of an external (magnetic) field
Dynamical model

Magnetization vector: \( m(t, x) \)

Landau-Lifshitz equation

\[
\frac{\partial m}{\partial t} = -m \wedge H(m) - m \wedge (m \wedge H(m))
\]

where \( H(m) = -\frac{dE}{dm} = 2A\Delta m + H_d(m) + H_{ext} \) (effective field).

- \( H_d(m) \) (demagnetization field).
- \( H_{ext} \) (external magnetic field)

Remark

\[
\frac{d}{dt}(E(m(t))) = -\int_{\Omega} \|H(m) - (H(m) \cdot m)m\|^2
\]

Every steady-state satisfies \( m \wedge H(m) = 0 \)
Dynamical model

Magnetization vector: $m(t, x)$

Landau-Lifshitz equation

$$\frac{\partial m}{\partial t} = -m \wedge H(m) - m \wedge (m \wedge H(m))$$

where $H(m) = -\frac{dE}{dm} = 2A\Delta m + H_d(m) + H_{ext}$ (effective field).

- $H_d(m)$ (demagnetization field).
- $H_{ext}$ (external magnetic field)

Existence of global weak solutions:
Alouges-Soyeur (1992), Visintin (1985)

Strong solutions locally in time and initial data:
Carbou-Fabrie (2001)
Dynamical model

Magnetization vector: \( m(t, x) \)

Landau-Lifshitz equation

\[
\frac{\partial m}{\partial t} = -m \wedge H(m) - m \wedge (m \wedge H(m))
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where \( H(m) = -\frac{dE}{dm} = 2A\Delta m + H_d(m) + H_{\text{ext}} \) (effective field).

- \( H_d(m) \) (demagnetization field).
- \( H_{\text{ext}} \) (external magnetic field)

Modelization, homogenization:
Carbou, Fabrie, De Simone, Haddar, Santugini, Sanchez, ...

Numerical analysis:
Alouges, Labbé, Merlet, Santugini, ...
Dynamical model

Magnetization vector: \( m(t, x) \)

Landau-Lifshitz equation

\[
\frac{\partial m}{\partial t} = -m \wedge H(m) - m \wedge (m \wedge H(m))
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where \( H(m) = -\frac{dE}{dm} = 2A\Delta m + H_d(m) + H_{ext} \) (effective field).

- \( H_d(m) \) (demagnetization field).
- \( H_{ext} \) (external magnetic field)

Controllability issues (control: external magnetic field \( t \mapsto H_{ext}(t) \))

- Carbou, Labbé, Prieur, Privat, Trélat (ANR project, 2006–2010)
- Alouges, Beauchard, Sigalotti (2009)
Loop-shaped nanowire

Loop-shaped 1D nanowire submitted to an external magnetic field:

First possibility:

Second possibility:

→ Landau-Lifshitz equation on $S^1$?
Micromagnetism

The model

Steady states

From a 3-D model to a 1-D model

The magnetic moment \( m : \mathbb{R} \times \Omega_\varepsilon \rightarrow S^2 \) of the nanowire (represented by \( \Omega_\varepsilon \subset \mathbb{R}^2 \), torus of thickness \( \varepsilon \)) verifies the Landau-Lifshitz equation:

\[
\frac{\partial m}{\partial t} = -m \wedge H(m) - m \wedge (m \wedge H(m)).
\]

where \( H(m) = \Delta m + H_d(m) + H_{\text{ext}}(m) \) (effective field)

As \( \varepsilon \downarrow 0 \):

- \((e_r, e_\theta, z)\) : cylindrical basis
- \(\Delta m \xrightarrow[\varepsilon \downarrow 0]{\varepsilon} \frac{\partial^2 m}{\partial x^2}\)
- \(\Gamma\)-convergence result:
  \[
  H_d(m) \xrightarrow[\varepsilon \rightarrow 0]{\varepsilon} h_d(m) = (m \cdot e_\theta) e_\theta.
  \]
A simpler model: 1-D model with periodic boundary conditions

Landau-Lifshitz PDE on a network of ferromagnetic nanowires

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -u \wedge h_\delta(u) - u \wedge (u \wedge h_\delta(u)) & x \in (0, L), \ t > 0 \\
u(t, 0) &= u(t, L) & t > 0 \\
\frac{\partial u}{\partial x}(t, 0) &= \frac{\partial u}{\partial x}(t, L) & t > 0 \\
\|u(t, x)\| &= 1
\end{align*}
\]

- \( u : \mathbb{R}^+ \times [0, L] \rightarrow S^2 \): magnetic moment
- \( L \): length of the nanowire
- \( x \): position in local charts
- \( h_\delta(u) := \partial_{xx}^2 u - (u.e_r)e_r - (u.e_3)e_3 + \delta \vec{d} \)

\( \delta(t) \): external field applied to the nanowire \( \rightarrow \) control
A steady state (with $\delta \equiv 0$) must satisfy $u \wedge h_0(u) = 0$.

Family of steady states (with $\delta \equiv 0$)

$$\begin{cases} 
  u_1(x) = \cos \theta(x) \\
  u_2(x) = \cos \omega \sin \theta(x) \\
  u_3(x) = \sin \omega \sin \theta(x) 
\end{cases}$$

where

- $\omega \in [0, 2\pi]$ (rotation constant parameter around the nanowire)
- $\theta$ is solution of the *pendulum equation*

\[
\begin{cases} 
  \theta''(x) = \sin \theta(x) \cos \theta(x) & x \in (0, L) \\
  \theta(0) = \theta(L) & [2\pi] \\
  \theta'(0) = \theta'(L)
\end{cases}
\]
A steady state family

Pendulum equation

$$\theta''(x) = \sin \theta(x) \cos \theta(x), \quad x \in (0, L)$$

→ elliptic functions, period:

$$4K(k)$$ (Jacobi function)

with $$\theta'^2(x) + \cos^2 \theta(x) = \text{Cste} = k^2$$

$$0 \leq k \leq 1$$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$ (Jacobi elliptic integral of the first kind)
A steady state family

**Pendulum equation**
\[ \theta''(x) = \sin \theta(x) \cos \theta(x), \quad x \in (0, L) \]

\[ \rightarrow \text{elliptic functions, period :} \]
\[ 4K(k) \quad \text{(Jacobi function)} \]

with \( \theta'^2(x) + \cos^2 \theta(x) = \text{Cste} = k^2 \)

**Boundary conditions :**
\[ \theta(0) = \theta(L) \quad [2\pi], \quad \theta'(0) = \theta'(L) \]

This imposes \( 4K(k) = L/n \) with \( n \in \mathbb{N} \)
(and hence \( L \geq 2\pi \))
A steady state family

The states of energy are quantified

The set of values of the energy $\mathcal{E}$ of steady-states is discrete and finite: there exist $N_0 = \left\lfloor \frac{L}{2\pi} \right\rfloor$ compatible levels of energy

$$\theta'^2 + \cos^2 \theta = k_n^2 = E_n$$

$$0 \leq E_n \leq 1$$
Computation of all periodic steady states

Periodic steady-state of Landau-Lifchitz equation

A $L$-periodic steady-state is a function $u \in C^2(\mathbb{R}, S^2)$ such that

$$
\begin{align*}
    u \wedge h(u) &= 0 \quad \text{on} \ (0, L), \\
    u(0) &= u(L), \quad u'(0) = u'(L).
\end{align*}
$$

We set for every $x \in \mathbb{R},$

$$
\begin{align*}
    u_1(x) &= \cos \theta_\alpha(x), \\
    u_2(x) &= \cos \omega_\alpha(x) \sin \theta_\alpha(x), \\
    u_3(x) &= \sin \omega_\alpha(x) \sin \theta_\alpha(x),
\end{align*}
$$
Computation of all periodic steady states

Then, we infer from (1) that

\[ \omega'_\alpha \sin^2 \theta_\alpha = \alpha, \]
\[ -\theta''_\alpha + \frac{1}{2} (\omega'^2_\alpha + 1) \sin(2\theta_\alpha) = 0, \]
\[ \theta_\alpha(0) = \theta_\alpha(L) [2\pi], \theta'_\alpha(0) = \theta'_\alpha(L), \]
\[ \omega_\alpha(0) = \omega_\alpha(L) [2\pi], \omega'_\alpha(0) = \omega'_\alpha(L). \]

\[ \text{The previous steady-states family corresponds to } \alpha = 0. \]

Theorem

There exists no steady-state in the case \( \alpha \neq 0. \)
Computation of all periodic steady states

Then, we infer from (1) that

\[
\omega_\alpha' \sin^2 \theta_\alpha = \alpha, \\
- \theta_\alpha'' + \frac{1}{2} (\omega_\alpha'^2 + 1) \sin(2\theta_\alpha) = 0,
\]

(2)

\[
\theta_\alpha(0) = \theta_\alpha(L) [2\pi], \quad \theta_\alpha'(0) = \theta_\alpha'(L), \\
\omega_\alpha(0) = \omega_\alpha(L) [2\pi], \quad \omega_\alpha'(0) = \omega_\alpha'(L).
\]

The previous steady-states family corresponds to \( \alpha = 0 \).

Theorem

There exists no steady-state in the case \( \alpha \neq 0 \).
Computation of all periodic steady states

Idea of the proof

Let \( \theta_\alpha \) be a solution of (2).

Then, its period is

\[
T_\alpha(\mathcal{E}_\alpha(\theta_\alpha)) = \frac{4\sqrt{2}}{\sqrt{d_\alpha}} K \left( \frac{2\sqrt{\mathcal{E}_\alpha(\theta_\alpha)} - \alpha^2}{d_\alpha} \right),
\]

where

\[
d_\alpha = \mathcal{E}_\alpha(\theta_\alpha) + 1 + \sqrt{(1 - \mathcal{E}_\alpha(\theta_\alpha))^2 + 4\alpha^2}.
\]
Computation of all periodic steady states

Idea of the proof

\[ \eta \mapsto T_\alpha(\alpha^2 + \eta) \text{ for } \alpha \in \{0, 0.1, 0.5, \frac{\sqrt{2}}{2}, 1, 3\} \]
Computation of all periodic steady states

Idea of the proof

- $\theta_\alpha$ is $L$-periodic $\implies$ quantification property of the energy $E_\alpha(\theta_\alpha)$.

An additional constraint.

The fact that $\omega_\alpha$ is periodic leads to an overdetermined system:

$$\alpha \int_0^L \frac{dx}{\sin^2 \theta_\alpha(x)} = 0 \ [2\pi].$$

(4)

- Assume the existence of a steady-state $\theta_{\alpha_0}$, for $\alpha_0 \neq 0$;
- $L$ must be an integer multiple of the period,
  $$\exists n \in \mathbb{N}^* \mid L = nT_{\alpha_0}(E_{\alpha_0}(\theta_{\alpha_0})) ;$$
- We make $\alpha$ vary and follow a path of solutions $\theta_\alpha$;
Computation of all periodic steady states

Idea of the proof

- We follow a path of solutions \( \theta_{\alpha} \) satisfying (2).
- There holds
  \[
  T_{\alpha}(\alpha^2) = \frac{2\pi}{\sqrt{\alpha^2 + 1}}.
  \]
- It is possible to increase \( \alpha \) up to a value \( \alpha_1 \) satisfying \( T_{\alpha_1}(\alpha_1^2) = L/n \).
- For \( \alpha < \alpha_1 \), \( \alpha \) close to \( \alpha_1 \), we set \( E_{\alpha}(\theta_{\alpha}) = \alpha^2 + \eta \), with \( \eta > 0 \) small.
- At the first order in \( \eta \),
  \[
  \theta(x) - \frac{\pi}{2} \approx \sqrt{\frac{\eta}{1 + \alpha^2}} \sin(\sqrt{1 + \alpha^2} x).
  \]
- From (4) and \( T_{\alpha}(\alpha^2 + \eta) = L/n \), we get
  \[
  \alpha L + \frac{\alpha}{2\pi} \frac{\alpha^2 + 1}{1 - 2\alpha^2} \left( \frac{L}{n} - \frac{2\pi}{\sqrt{\alpha^2 + 1}} \right) \left( 2\sqrt{1 + \alpha^2} L - \sin(2\sqrt{1 + \alpha^2} L) \right) = 2k\pi,
  \]
  for every \( \alpha < \alpha_1 \) sufficiently close to \( \alpha_1 \).
Local stability properties of these steady states
Linearization around a steady state

\[ \frac{\partial u}{\partial t} = -u \wedge h_\delta(u) - u \wedge (u \wedge h_\delta(u)), \quad x \in (0, L), \quad t > 0 \]

- \( M_0(x) = \begin{pmatrix} \cos \theta(x) \\ \sin \theta(x) \\ 0 \end{pmatrix} \) steady state of energy \( E_n \).

- Mobile frame \((M_0(x), M_1(x), M_2)\):
  \[ M_1(x) = \begin{pmatrix} -\sin \theta(x) \\ \cos \theta(x) \\ 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

- \( u \in S^2 \Rightarrow \) new coordinates \( r = (r_1, r_2) \) s.t.

\[ u(t, x) = \sqrt{1 - r_1^2(t, x) - r_2^2(t, x)} \ M_0(x) + r_1(t, x) \ M_1(x) + r_2(t, x) \ M_2 \]

\( \rightarrow \) Projection of the PDE on \((M_1, M_2)\).
Linearization around a steady state

In these new coordinates, for $\|r\|$ small:

\[
\begin{align*}
\text{Landau-Lifshitz equation} \\
\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \\
\frac{\partial \mathbf{r}}{\partial t} = \mathcal{A} \mathbf{r} + \delta U(x, r) + R(x, r, r_x, r_{xx})
\end{align*}
\]

- $\mathcal{A} = \begin{pmatrix} A + \text{Id} & A + E_n \text{Id} \\ -(A + \text{Id}) & A + E_n \text{Id} \end{pmatrix}$ (not diagonalizable)

  - with
  - $A = \partial_{xx}^2 - 2 \cos^2 \theta(x) \text{Id}$ on $D(A) = H^2_{\text{per}}(0, L)$

- $\exists C > 0$ s.t. for all $(p, q) \in (\mathbb{R}^2)^2$,

\[
|R(x, r, p, q)| \leq C(|r|^2|q| + |r|.|p| + |r|.|p|^2 + |r|^2).
\]
Spectral study of \( A = \partial_{xx}^2 - 2 \cos^2 \theta(x) \, \text{Id} \)

- \( A + E_n \, \text{Id} = -\ell^* \ell \leq 0 \)
  with \( \ell = \partial_x + \theta'(x) \cotan \theta(x) \, \text{Id} \) on \( D(\ell) = H^1_{\text{per}}(0, L) \).

- \( A \sin \theta = -E_n \sin \theta \)
  \( A \theta' = -\theta' \)
  \( A \cos \theta = -(1 + E_n) \cos \theta \)

There exists a hilbertian basis \((e_k)_{k \in \mathbb{N}}\) of \( L^2(0, L) \), consisting of eigenvectors of \( A \), associated with eigenvalues \( \lambda_k \) that are at most double, with

\[-\infty < \cdots \leq \lambda_k \leq \cdots \leq \lambda_1 \leq \lambda_0.\]

Moreover:
- there cannot be two successive equalities;
- \( x \mapsto e_k(x) \) vanishes \( k - 1 \) or \( k \) or \( k + 1 \) times on \([0, L]\).
- \( \lambda_0 = -E_n \), associated with \( e_0 = \sin \theta \).
Spectral study of $A = \partial_{xx}^2 - 2 \cos^2 \theta(x) \Id$

- $A + E_n \Id = -\ell^* \ell \leq 0$
  with $\ell = \partial_x + \theta'(x) \cotan \theta(x) \Id$ on $D(\ell) = H^1_{per}(0, L)$.

- $A \sin \theta = -E_n \sin \theta$
  $A \theta' = -\theta'$
  $A \cos \theta = -(1 + E_n) \cos \theta$

There exists a hilbertian basis $(e_k)_{k \in \mathbb{N}}$ of $L^2(0, L)$, consisting of eigenvectors of $A$, associated with eigenvalues $\lambda_k$ that are at most double, with

$$-\infty < \cdots \leq \lambda_k \leq \cdots \leq \lambda_1 \leq \lambda_0.$$  

Moreover :

- there cannot be two successive equalities ;
- $x \mapsto e_k(x)$ vanishes $k - 1$ or $k$ or $k + 1$ times on $[0, L]$.
- $\lambda_0 = -E_n$, associated with $e_0 = \sin \theta$. 

Spectral study of $A = \partial_{xx}^2 - 2 \cos^2 \theta(x) \text{Id}$

- $A + E_n \text{Id} = -\ell^*\ell \leq 0$
  
  with $\ell = \partial_x + \theta'(x)\cotan \theta(x) \text{Id}$ on $D(\ell) = H^1_{\text{per}}(0, L)$.

- $A \sin \theta = -E_n \sin \theta$
  
  $A \theta' = -\theta'$

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There exists a hilbertian basis $(e_k)_{k \in \mathbb{N}}$ of $L^2(0, L)$, consisting of eigenvectors of $A$, associated with eigenvalues $\lambda_k$ that are at most double, with

$$-\infty < \cdots \leq \lambda_k \leq \cdots \leq \lambda_1 \leq \lambda_0.$$

Moreover:

- there cannot be two successive equalities;
- $x \mapsto e_k(x)$ vanishes $k - 1$ or $k$ or $k + 1$ times on $[0, L]$.
- $\lambda_0 = -E_n$, associated with $e_0 = \sin \theta$. 
Spectral study of $A = \partial_{xx}^2 - 2 \cos^2 \theta(x) \, \text{Id}$

For $L \simeq 2\pi N_0$ and $E_n \simeq 0$, one has $A \simeq \partial_{xx}^2$ and hence

$$\lambda_k \simeq -\left(\frac{2k\pi}{L}\right)^2, \quad k \in \mathbb{N} \quad \text{(simple eigenvalues)}$$

Setting $u = v \sin \theta$, one has

$$(A + E_n \text{Id})u = \lambda u \iff -\partial_x (\sin^2 \theta \, \partial_x v) = -\lambda \sin^2 \theta \, v$$

Sturm-Liouville operator with real coupled boundary conditions

Lemma

$$\lambda'(L) = -\sin^2 \theta(L) \left(v'(L)^2 + \lambda(L)v(L)^2\right)$$

Consequence

$L \mapsto \lambda(L)$ decreasing and analytic

Spectral study of \( A = \partial_{xx}^2 - 2 \cos^2 \theta(x) \text{ Id} \)

- For \( L \simeq 2\pi N_0 \) and \( E_n \simeq 0 \), one has \( A \simeq \partial_{xx}^2 \) and hence
  \[ \lambda_k \simeq -\left( \frac{2k\pi}{L} \right)^2, \quad k \in \mathbb{N} \quad \text{(simple eigenvalues)} \]

- Setting \( u = v \sin \theta \), one has
  \[ (A + E_n \text{ Id})u = \lambda u \ \Leftrightarrow \ -\partial_x (\sin^2 \theta \partial_x v) = -\lambda \sin^2 \theta \ v \]

Sturm-Liouville operator with real coupled boundary conditions

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Consequence

\( L \mapsto \lambda(L) \) decreasing and analytic

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Around the control of nanowires

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Spectral study of \( A = \partial_{xx}^2 - 2 \cos^2 \theta(x) \, \text{Id} \)

- Characterization of double eigenvalues:
  
  \[
  (A + E_n \, \text{Id}) u = \lambda u \quad \Leftrightarrow \quad -\partial_x (\sin^2 \theta \, \partial_x v) = -\lambda \sin^2 \theta \, v
  \]

  \[
  \Leftrightarrow \quad Y' = \begin{pmatrix}
  0 & 1/\sin^2 \theta \\
  \lambda \sin^2 \theta & 0
  \end{pmatrix} Y
  \quad \text{with} \quad Y = \begin{pmatrix}
  v \\
  \sin^2 \theta \, \partial_x v
  \end{pmatrix}
  \]

  \[
  Y(0) = Y(L)
  \]

  Let \( \Phi(x, 0; \lambda) \) the resolvent. Then:

  1. \( \lambda \) eigenvalue \( \Leftrightarrow \det(\Phi(L, 0; \lambda) - \text{Id}) = 0 \) (transcendental equation)
  2. \( \lambda \) double eigenvalue \( \Leftrightarrow \Phi(L, 0; \lambda) - \text{Id} = 0 \)

Consequence

Every eigenvalue of \( A \) is simple except for certain (isolated) values of \( L \).
Spectral study of $A = \partial_{xx}^2 - 2 \cos^2 \theta(x) \text{Id}$

- Characterization of double eigenvalues:

  $$(A + E_n \text{Id})u = \lambda u \iff -\partial_x (\sin^2 \theta \partial_x v) = -\lambda \sin^2 \theta v$$

  $$\iff Y' = \begin{pmatrix} 0 & 1/\sin^2 \theta \\ \lambda \sin^2 \theta & 0 \end{pmatrix} Y \quad \text{with} \quad Y = \begin{pmatrix} v \\ \sin^2 \theta \partial_x v \end{pmatrix}$$

  $$Y(0) = Y(L)$$

Let $\Phi(x, 0; \lambda)$ the resolvent. Then:

1. $\lambda$ eigenvalue $\iff \det(\Phi(L, 0; \lambda) - \text{Id}) = 0$ (transcendental equation)

2. $\lambda$ double eigenvalue $\iff \Phi(L, 0; \lambda) - \text{Id} = 0$

Consequence

Every eigenvalue of $A$ is simple except for certain (isolated) values of $L$. 
Spectral study of $A = \partial_{xx}^2 - 2 \cos^2 \theta(x) \, \text{Id}$

- Characterization of double eigenvalues:
  
  \[
  (A + E_n \text{Id})u = \lambda u \iff -\partial_x(\sin^2 \theta \partial_x v) = -\lambda \sin^2 \theta \, v
  \]
  
  \[
  \iff Y' = \begin{pmatrix} 0 & 1 / \sin^2 \theta \\ \lambda \sin^2 \theta & 0 \end{pmatrix} Y
  \text{ with } Y = \begin{pmatrix} v \\ \sin^2 \theta \partial_x v \end{pmatrix}
  \]
  
  $Y(0) = Y(L)$

Let $\Phi(x, 0; \lambda)$ the resolvent. Then:

1. $\lambda$ eigenvalue $\iff \det(\Phi(L, 0; \lambda) - \text{Id}) = 0$ (transcendental equation)
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Consequence:

Every eigenvalue of $A$ is simple except for certain (isolated) values of $L$. 
Spectral study of $A = \partial_{xx}^2 - 2 \cos^2 \theta(x) \text{Id}$

- Characterization of double eigenvalues:

$$(A + E_n \text{Id})u = \lambda u \iff -\partial_x (\sin^2 \theta \partial_x v) = -\lambda \sin^2 \theta \ v$$

$$\iff Y' = \begin{pmatrix} 0 & 1/\sin^2 \theta \\ \lambda \sin^2 \theta & 0 \end{pmatrix} Y \quad \text{with} \quad Y = \begin{pmatrix} v \\ \sin^2 \theta \partial_x v \end{pmatrix}$$

$Y(0) = Y(L)$

Let $\Phi(x, 0; \lambda)$ the resolvent. Then:

1. $\lambda$ eigenvalue $\iff \det(\Phi(L, 0; \lambda) - \text{Id}) = 0$ (transcendental equation)
2. $\lambda$ double eigenvalue $\iff \Phi(L, 0; \lambda) - \text{Id} = 0$

Consequence

Every eigenvalue of $A$ is simple except for certain (isolated) values of $L$. 
Importance of the simplicity of the spectrum for controllability issues:

\[
\begin{align*}
\dot{x} &= x + u \\
\dot{y} &= y + 2u \\
\end{align*}
\]

→ not controllable: \( \frac{d}{dt} (y - 2x) = y - 2x \)
Linearization around a steady state

Consequences for $\mathcal{A} = \begin{pmatrix} A + \text{Id} & A + E_n \text{Id} \\ -(A + \text{Id}) & A + E_n \text{Id} \end{pmatrix}$

Diagonalization into blocks of $2 \times 2$ linear systems, $k \in \mathbb{N}$:

$\dot{r}_{1k} = (\lambda_k + 1)r_{1k} + (\lambda_k + E_n)r_{2k}$

$\dot{r}_{2k} = -(\lambda_k + 1)r_{1k} + (\lambda_k + E_n)r_{2k}$

$A_k = \begin{pmatrix} \lambda_k + 1 & \lambda_k + E_n \\ -(\lambda_k + 1) & \lambda_k + E_n \end{pmatrix}$

$A_k$ is Hurwitz $\iff \lambda_k < -1$

Recall that

$A \sin \theta = -E_n \sin \theta$

$A \theta' = -\theta' \quad \{\text{unstable modes}\}$

$A \cos \theta = -(1 + E_n) \cos \theta$

$\{\text{stable modes}\}$
Linearization around a steady state

Consequences for $\mathcal{A} = \begin{pmatrix} A + \text{Id} & A + E_n\text{Id} \\ -(A + \text{Id}) & A + E_n\text{Id} \end{pmatrix}$

Diagonalization into blocks of $2 \times 2$ linear systems, $k \in \mathbb{N}$:

For $\epsilon_k$ and $\epsilon_2$, we have:

$\dot{r}_1 = (\lambda_k + 1)r_1 + (\lambda_k + E_n)r_2_k$

$\dot{r}_2 = -(\lambda_k + 1)r_1 + (\lambda_k + E_n)r_2_k$

$\Rightarrow A_k = \begin{pmatrix} \lambda_k + 1 & \lambda_k + E_n \\ -(\lambda_k + 1) & \lambda_k + E_n \end{pmatrix}$

$A_k$ is Hurwitz $\iff \lambda_k < -1$

Consequence

- There always exists at least one unstable mode.
- If $L$ is large, then there exist $N_0$ unstable modes.
Control

Landau-Lifshitz equation

\[ \frac{\partial r}{\partial t} = \mathcal{A}r + \delta U(x, r) + R(x, r, r_x, r_{xx}) \]

with

\[ \mathcal{A} = \begin{pmatrix} A + \text{Id} & A + E_n\text{Id} \\ -(A + \text{Id}) & A + E_n\text{Id} \end{pmatrix} \]

Using the spectral decomposition \( \Rightarrow \) linear systems:

\[ \frac{d}{dt} \begin{pmatrix} r_{1k} \\ r_{2k} \end{pmatrix} = A_k \begin{pmatrix} r_{1k} \\ r_{2k} \end{pmatrix} + \delta \begin{pmatrix} U_{1k} \\ U_{2k} \end{pmatrix} \quad k \in \mathbb{N} \]
Control

First possibility: external magnetic field applied to the whole nanowire:

\[ \frac{d}{dt} \begin{pmatrix} r_{1k} \\ r_{2k} \end{pmatrix} = A_k \begin{pmatrix} r_{1k} \\ r_{2k} \end{pmatrix} + \delta \begin{pmatrix} U_{1k} \\ U_{2k} \end{pmatrix} \]

\[ k \in \mathbb{N} \]

\[ \Rightarrow U(x, r) \sim \sin \theta(x) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow U_{1k} = U_{2k} = 0, \ \forall k \geq 1 \]

→ not stabilizable, not controllable
Control

Second possibility: external magnetic field generated by an inductance rolling around the nanowire:

\[ H_{\text{ext}}(t) = \frac{d}{dt} \begin{pmatrix} r_{1k} \\ r_{2k} \end{pmatrix} = A_k \begin{pmatrix} r_{1k} \\ r_{2k} \end{pmatrix} + \delta \begin{pmatrix} U_{1k} \\ U_{2k} \end{pmatrix} \]

For every \( N \in \mathbb{N} \), Kalman condition satisfied for the \( N \) first systems, except for certain values of the length of the solenoid and of the perimeter \( L \) of the nanowire.
Control

Consequences

1. It is possible to design explicitly a feedback control stabilizing any steady state.

2. It is possible to design explicitly a feedback control passing approximately from any steady state to any other one having the same level of energy.

Open questions

1. Is it possible to jump to other levels of energy?

2. 2-D case: analysis, control and simulation of the Landau-Lifshitz equation on a thin layer.

3. Shape optimization problems.

(ongoing work)