Effective slip law for general viscous flows over oscillating surface

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1. INTRODUCTION

Physics of the laminar viscous fluid flow imposes the no-slip condition \( v = 0 \), at an immobile smooth solid boundary. In fact, it is only justified where the molecular viscosity is concerned. Since the fluid cannot penetrate the solid, its normal velocity is equal to zero. This is the *non-penetration condition*. To the contrary, the absence of slip is not very intuitive but corresponds to the observed situation at a smooth solid wall (see e.g. book by Panton). This experimental fact was not always accepted in the past and Navier claimed at the wall the fluid was allowed to slip, with the slip velocity proportional to the shear stress.
Navier’s model can be confirmed, at least heuristically, by kinetic-theory calculations, but the proportionality constant in Navier’s law is proportional to the mean free path divided by the continuum length (see Panton). Hence it is zero for most practical proposes. In many cases of practical significance the boundary contains asperities. An example are complex sea bottoms of random roughness and artificial bodies with periodic distribution of small bumps. A numerical simulation of the flow problems in the presence of a rough boundary is very difficult since it requires handling of huge quantity of data. For computational purposes, an artificial non-oscillatory smooth boundary, close to the original one, is taken and the equations are solved in the new domain.
This way the rough boundary is avoided, but the boundary conditions at the artificial boundary are to be determined. It is clear that the non-penetration condition should be kept, but there are no reasons to keep the no-slip in the tangential directions. Usually it is supposed that the shear stress is some function $F$ of the tangential velocity. $F$ is determined empirically and its form varies for different problems. Such relations are called the *wall laws* and classical Navier’s condition is an example.

There is a huge literature on the asymptotic behavior of PDEs in the presence of oscillating boundaries. Here we concentrate on *flow problems over rough boundaries*. 
Early mathematical work on the Couette flow over a rough plate and applications to the drag reduction is due to Amirat, Simon and collaborators in papers from 1996 to 1997. They concentrate on small Reynolds numbers. Modeling and computational of flow problems over rough surfaces was undertaken by Pironneau and collaborators in the papers mainly published in 1998. They study the stationary incompressible flow at high Reynolds number $\text{Re} \sim \frac{1}{\varepsilon}$ over a periodic rough boundary, with the roughness period $\varepsilon$. An asymptotic expansion is constructed and effective wall laws are obtained. A numerical validation is presented, but there are no mathematically rigorous convergence results. The most well known wall law is the Navier slip condition and its formal derivations are already known in the engineering literature (see e.g. work by Luchini from 1995).
First mathematically rigorous derivation of Navier’s slip condition is due to Jäger and Mikelić in their paper from 2000. It is based on the general approach to the contact problems porous media/unconfined fluid by the same authors (paper in Ann. Sc. Norm. Super. Pisa, Cl. Sci. 1996) and the derivation of interface laws by Beavers and Joseph between porous media and an unconfined fluid flow. The auxiliary problems, used to calculate the effective coefficients for the Beavers and Joseph interface law, could be slightly modified and used in obtaining the constants in Navier’s slip law. Main effort in the work by Jäger and Mikelić was to justify of the approximation in the case of the stationary Navier-Stokes equations with flow governed by a pressure drop.
Then they studied the Couette flow around rough boundary (riblets) in *Communications in Mathematical Physics* in 2003). Rigorous estimates lead to conclusion that the riblets reduce significantly tangential drag, which may explain their presence on the skin of Nektons.
For generalization to quasi-geostrophic flows we refer to the paper by Bresch and Gérard-Varet (Comm Math Phys. 2005) and references therein. All these results were obtained for *periodic* rough boundaries. In fact realistic natural rough boundaries are *random*. For generalization of the periodic results by Jäger and Mikelić, we refer to the works of Gérard-Varet and collaborators (Comm. Pure Appl. Math. 2008 and Comm. Math. Phys. 2009).
In the above work, the results on the Navier slip condition were obtained by looking at the perturbations of Couette and Poiseuille flows. Such particular situation allows avoiding complications with the incompressibility condition. However for a general incompressible viscous flows it is not possible. We have to get pressure estimates in presence of an oscillating boundary. Goal of this talk is to present derivation of Navier’s slip condition in a general situation.

Definition of the geometry

Flow takes place in a domain $\Omega = (0, L)^3$ in $\mathbb{R}^3$ and in adjacent rough layer. Its bottom and top are denoted by
\[ \Sigma = (0, L)^2 \times \{0\}, \quad \text{and} \quad \Sigma_L = (0, L)^2 \times \{L\} \]

respectively. Adjacent to \( \Sigma \) there is the thin layer \( R^\varepsilon \), having a rapidly oscillating boundary, described in the text which follows.

Let \( \varepsilon > 0 \) be such that \( L/\varepsilon \in \mathbb{N} \). Let

\[
\Upsilon : (0, 1)^2 \to \mathbb{R}, \quad (y_1, y_2) \mapsto \Upsilon(y_1, y_2)
\]

be a Lipschitz surface, 1-periodic with respect to \( y_1 \) and \( y_2 \), and satisfying

\[ 0 \leq \Upsilon(y_1, y_2) \leq b_3, \]

for all \( (y_1, y_2) \in (0, 1)^2 \), with \( 0 < b_3 < 1 \).
We introduce the canonical cell of roughness (the canonical hump) \( Y \subset Z = (0, 1)^3 \) by

\[
Y = \{ y \in Z \mid \Upsilon(y_1, y_2) < y_3 < 1 \}
\]

and denote the corresponding infinite layer by

\[
\mathcal{L} = \bigcup_{k \in \mathbb{Z}^2} \left( Y + (k_1, k_2, -1) \right).
\]

The layer of roughness \( R^\varepsilon \) is now given by

\[
R^\varepsilon = \varepsilon \mathcal{L} \cap \left((0, L)^2 \times (-\varepsilon, 0)\right),
\]

and having the rough boundary

\[
\mathcal{B}^\varepsilon = \left\{ x \in \mathbb{R}^3 : (x_1, x_2) \in (0, L)^2, x_3 = \varepsilon \left( \Upsilon \left( \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) - 1 \right) \right\}.
\]
Thus $B^\varepsilon$ consists of a large number of periodically
distributed humps of characteristic length and amplitude of
order $\varepsilon$, small compared with the characteristic length of the
macroscopic domain. We set

$$\Omega^\varepsilon = \Omega \cup \Sigma \cup R^\varepsilon,$$

and denote by $\partial\Omega_{lat} = \partial\Omega^\varepsilon \setminus (B^\varepsilon \cup \Sigma_L)$. We will work in the
following functional space

$$V(\Omega^\varepsilon) = \{ f \in H^1(\Omega^\varepsilon) : f = 0 \text{ on } \Sigma_L \cup B^\varepsilon, \text{ } f \text{ is L-periodic in } x_1 \}$$ (4)

Every element of $V(\Omega^\varepsilon)$ is extended by zero to
$(0, L)^2 \times (-\varepsilon, 0) \setminus R^\varepsilon$. 
The microscopic equations

Let \( f \in C^\infty(\overline{\Omega}^\varepsilon \times [0, T])^3 \), \( \text{supp } f \subset \overline{\Omega}^\varepsilon \times (0, T] \), periodic in \((x_1, x_2)\) with period \( L \). We consider the following problem:

\[
\frac{\partial u^\varepsilon}{\partial t} + \text{div} \ (u^\varepsilon \otimes u^\varepsilon) - \frac{1}{\text{Re}} \ \text{div} \ (2D(u^\varepsilon) - p^\varepsilon I) = f \quad \text{in } \Omega^\varepsilon \times (0, T),
\]

\( u^\varepsilon = 0 \) \quad \text{on } (B^\varepsilon \cup \Sigma_L) \times (0, T),

\( u^\varepsilon \) is periodic in \((x_1, x_2)\) with period \( L \),

\( u^\varepsilon|_{t=0} = 0 \) \quad \text{on } \Omega^\varepsilon.

Let

\[
\mathcal{W}^\varepsilon = \{ \varphi \in V(\Omega^\varepsilon)^3 : \text{div } \varphi = 0 \text{ in } \Omega^\varepsilon \} \quad (9)
\]

with the space \( V(\Omega^\varepsilon) \) given by (4). From the theory of the Navier-Stokes equations, it is known that the problem
(5)-(8) admits a solution in \( u^\varepsilon \in L^2(0, T; \mathcal{W}^\varepsilon) \),
\( \partial_t u^\varepsilon \in L^{4/3}(0, T; (\mathcal{W}^\varepsilon)') \). The pressure \( p^\varepsilon \) is determined from \( u^\varepsilon \) using De Rham’s theorem.

**STRATEGY: ZERO ORDER APPROXIMATION AND THEN CORRECTION TO GET A GOOD APPROXIMATION ON \( \Omega \)**

Let \( Q_T = \Omega \times (0, T) \). We consider the problem

\[
\frac{\partial u^0}{\partial t} + \text{div} \ (u^0 \otimes u^0) - \frac{1}{\text{Re}} \text{div} \ (2D(u^0) - p^0 I) = f \text{ in } Q_{(10)}
\]

\[
u^0 = 0 \quad \text{on } (\Sigma \cup \Sigma_L) \times (0, T), \tag{11}
\]

\( u^0 \) is periodic in \((x_1, x_2)\) with period \( L \), \tag{12}

\[
u^0|_{t=0} = 0 \quad \text{on } \Omega. \tag{13}
\]
Obviously problem (10)-(13) admits a solution $u^0 \in L^2(0, T; V(\Omega)^3)$, $\partial_t u^0 \in L^{4/3}(0, T; (\mathcal{W})')$, $\text{div} \ u^0 = 0$, and, after extension by zero to $(0, L)^2 \times (-\varepsilon, 0)$, it is also an element of $L^2(0, T; \mathcal{W}^\varepsilon)$. Again $p^0$ is determined using De Rham’s theorem. $p^0$ is extended by its value on $\Sigma$ to $(0, L)^2 \times (-\varepsilon, 0) \times (0, T)$, i.e.

$$p^0(x_1, x_2, x_3, t) = p^0(x_1, x_2, 0, t), \quad x_3 < 0. \quad (14)$$

Our aim is to prove that there are solutions to (5)-(8) close to $u^0$. In this scope, we use a perturbation argument, which requires the following hypothesis on $u^0$:

(H1) The problem (10)-(13) admits a solution $u^0$ such that $\nabla u^0 \in C^2(\bar{Q}_T)^9$. 

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Obviously a solution satisfying hypothesis (H1) is unique. We notice, that this hypothesis is always fulfilled in 2D and for stationary problems.

Before stating the result on existence of solutions to (5)-(8) close to \( u^0 \), we formulate the well known result about \( L^2 \)-estimates for functions on thin domains.

**Lemma 1** Let \( \varphi \in V(\Omega^\varepsilon) \). Then we have

\[
\| \varphi \|_{L^2(\Sigma)} \leq C \varepsilon^{1/2} \| \nabla_x \varphi \|_{L^2(\Omega^\varepsilon \setminus \Omega)}^3, \tag{15}
\]

\[
\| \varphi \|_{L^2(\Omega^\varepsilon \setminus \Omega)} \leq C \varepsilon \| \nabla_x \varphi \|_{L^2(\Omega^\varepsilon \setminus \Omega)}^3. \tag{16}
\]
Proposition 1 Let us suppose the hypothesis (H1) and let $f \in C^\infty(\Omega^\varepsilon \times [0, T))^3$, be such that $\text{supp } f \subset \Omega^\varepsilon \times (0, T]$, and it is periodic in $(x_1, x_2)$ with period $L$. Then the problem (5)-(8) has a solution
$$\{u^\varepsilon, p^\varepsilon\} \in L^2(0, T; \mathcal{W}^\varepsilon \cap H^2(\Omega^\varepsilon)^3) \times L^2(0, T; H^1(\Omega^\varepsilon))$$
such that $\{\partial_t u^\varepsilon, \partial_t p^\varepsilon\} \in L^2(0, T; H^1(\Omega^\varepsilon)^3) \times L^2(0, T; L^2(\Omega^\varepsilon))$, and satisfying

$$\int_0^T \|\nabla(u^\varepsilon - u^0)(t)\|_{L^2(\Omega^\varepsilon)}^2 \, dt + \int_0^T \|\nabla \partial_t(u^\varepsilon - u^0)(t)\|_{L^2(\Omega^\varepsilon)}^2 \, dt \leq C\varepsilon$$

(17)

$$\max_{0 \leq t \leq T} \left(\|u^\varepsilon - u^0(t)\|_{L^2(\Omega^\varepsilon)}^2 + \|\partial_t(u^\varepsilon - u^0)(t)\|_{L^2(\Omega^\varepsilon)}^2\right) \leq C\varepsilon$$

(18)
\( u^\varepsilon \) is unique and \( p^\varepsilon \) is unique up to a constant.

**Proposition 2** For the solution to (5)-(8), constructed in Proposition 1, the following estimates hold:

\[
\| u^\varepsilon \|_{L^2(\Omega^\varepsilon \setminus \Omega \times (0,T))} + \| \partial_t u^\varepsilon \|_{L^2(\Omega^\varepsilon \setminus \Omega \times (0,T))} \leq C \varepsilon \sqrt{\varepsilon} \quad (19)
\]
\[
\| u^\varepsilon \|_{L^2(\Sigma \times (0,T))} + \| \partial_t u^\varepsilon \|_{L^2(\Sigma \times (0,T))} \leq C \varepsilon \quad (20)
\]
\[
\| u^\varepsilon - u^0 \|_{L^2(Q_T)} + \| \partial_t (u^\varepsilon - u^0) \|_{L^2(Q_T)} \leq C \varepsilon \quad (21)
\]
\[
\| p^\varepsilon - p^0 \|_{L^2(Q_T)} + \| \partial_t (p^\varepsilon - p^0) \|_{L^2(Q_T)} \leq C \sqrt{\varepsilon} \quad (22)
\]
First order approximation

In the above considerations, we have obtained the uniform a priori estimates for \( \{u^\varepsilon, p^\varepsilon\} \). Moreover, we have found that the flow equations with truncated asperities (10)-(13), posed in \( \Omega \), give an \( O(\varepsilon) L^2 \)-approximation for \( u^\varepsilon \). At \( \Sigma \) it is an \( O(\varepsilon) L^2 \)-approximation. Consequently, in the 1st approximation the free flow doesn’t see the rough boundary.

In the estimate (18) the principal contribution was coming from the surface term \( \int_\Sigma \varphi \cdot \tau \) and the Navier law will come with order \( \varepsilon \).

We will explain the main ideas on an 1D example.
Let \( \Omega_1 = (-\infty, 0) \) and \( \Omega_2 = (0, \infty) \). Interface between \( \Omega_1 \) and \( \Omega_2 \) is the point \( \Sigma = \{0\} \). Let \( Y = (0, 1) \) and \( Z^* = (0, a), \) \( 0 < a < 1 \). Then the "fluid" part of \( \Omega_1 \) is given by \( \Omega_{1F}^\varepsilon = \bigcup_{k=1}^{\infty} \varepsilon (a - k, 1 - k) \). The 1D "pore space" is now \( \Omega^\varepsilon = \Omega_{1F}^\varepsilon \cup \Sigma \cup \Omega_2 \).

Let \( f \in C_0^\infty(\mathbb{R}) \) be a given function. We consider the problem

\[
\begin{aligned}
- \frac{d^2 u^\varepsilon}{dx^2} &= f(x), \quad \text{in } \Omega^\varepsilon \\

u^\varepsilon &= 0 \quad \text{on } \partial\Omega^\varepsilon, \\
\lim_{|x| \to +\infty} \frac{du^\varepsilon}{dx} &= 0.
\end{aligned}
\]  

(23)
Example 1D 2

As in the derivation of Darcy’s law, using the 2-scales expansions, we have the following expansion for $u^\varepsilon$:

\[
\begin{align*}
    u^\varepsilon &= -\varepsilon^2 f(x) \left( \frac{x}{\varepsilon} + k \right) \left( \frac{x}{\varepsilon} + k + 1 - a \right) + O(\varepsilon^3), \\
    \text{for} & \quad -k + a - 1 \leq \frac{x}{\varepsilon} \leq -k, \quad k = 0, 1, \ldots \\
    u^\varepsilon &= \int_0^x t f(t) \, dt + x \int_x^\infty f(t) \, dt + C_\varepsilon, \quad \text{in} \quad \Omega_2,
\end{align*}
\]

(24)

where $C_\varepsilon$ is an unknown constant. The corresponding "permeability" is $k = \varepsilon^2 (1 - a)^3 / 12$. Two domains are linked through the interface $\Sigma = \{0\}$. Without an interface condition, the approximation in $\Omega_2$ is not determined.
We search for effective interface conditions at $\Sigma$, leading to a good approximation of $u^\varepsilon$ by some $u^{\text{eff}}$.

Classical way of finding interface conditions is by using matched asymptotic expansions (MMAE). A recent reference in asymptotic methods and boundary layers in fluid mechanics is the recent book by Zeytounian and for the detailed explications, we invite reader to consult it and references therein.

In the language of the MMAE, expansions in $\Omega_1$ and $\Omega_2$ give us the outer expansions. We should supplement it by an (local) inner expansion in which the independent variable is stretched out in order to capture the behavior in the neighborhood of the interface.

The MMAE approach uses the limit matching rule, by which asymptotic behavior of the outer expansion
in the neighborhood of the interface has to be equal to asymptotic behavior of the inner expansion outside interface. The stretched variable is $\xi = \frac{x}{\varepsilon^\alpha}$, $\alpha > 0$. The geometry of $\Omega_1^\varepsilon$ obliges us to take $\alpha \geq 1$. Then the zero order term in the expansion is linear in $\xi$ and the limit matching rule implies that, at the leading order,

$$u^0 = 0 \quad \text{at the interface} \quad \Sigma = \{0\}. \quad (25)$$

In $\Omega_1$ we have $u^0 = 0$. In $\Omega_2$

$$- \frac{d^2 u^0}{dx^2} = f \quad \text{and} \quad \frac{du^0}{dx} \to 0 \quad \text{when} \quad x \to +\infty. \quad (26)$$
Example 1D 5

The system \((25)-(26)\) determines \(u^0\).

It is easy to find out that

\[
\begin{align*}
    u^0(x) &= \begin{cases} 
        \int_0^x t f(t) \, dt + x \int_x^\infty f(t) \, dt, & x \geq 0; \\
        u^0 = 0, & \text{in } \Omega_1;
    \end{cases}
\end{align*}
\]

(27)

and

\[
\begin{align*}
    u^\varepsilon(x) &= \int_{\varepsilon(a-1)}^{x} (t + \varepsilon(1 - a)) f(t) \, dt + (x + \varepsilon(1 - a)) \int_x^\infty f(t) \, dt, \\
    \text{for } x \geq -\varepsilon(1 - a);
\end{align*}
\]

(28)
Example 1D 6

\[ u^\varepsilon(x) = \int_{\varepsilon(a-1)-k\varepsilon}^{x} (t + \varepsilon(1 - a + k))f(t) \, dt + (x + \varepsilon(k + 1 - a)). \]

\[ \left( \int_{x}^{-\varepsilon k} f(t) \, dt - \frac{1}{\varepsilon(1 - a)} \int_{\varepsilon(a-1)}^{-\varepsilon k} (t + \varepsilon(1 - a))f(t) \, dt \right), \]

for \(-k\varepsilon \geq x \geq \varepsilon(a - 1) - k\varepsilon, \ k = 1, 2, \ldots\) (29)

Now we see that

\[ u^\varepsilon(x) = u^0(x) + O(\varepsilon) \quad \text{in} \quad \Omega_1. \] (30)

Nevertheless in the neighborhood of the interface \(\Sigma = \{0\}\) approximation for \(\frac{d u^\varepsilon}{d x}\) is not good and it differs at order \(O(1)\).
Why the approximation deteriorates around the interface? It is due to the fact that the MMAE method, as it is used in classical textbooks, does not suit interface problems. It matches only the function values at the interface, but not the values of the normal derivative. This difficulty is not easy to circumvent because imposing matching of the values of the function and its normal derivative leads to an ill posed problem for our 2nd order equation.

In order to circumvent the difficulty, we propose the following strategy, introduced in the papers Jäger et Mikelić (1996, 1998, 1999a,b, 2000) and Jäger et al (2001).

1. STEP: We match the function values, as when using the MMAE method. In our particular example this means
that the first approximation \( u^{0,\text{eff}} \) is given by the problem (25)-(26).

2. STEP: At \( \Sigma = \{0\} \) we have the derivative jump equal to

\[
\frac{du^0}{dx} = \int_{0}^{+\infty} f(t) \, dt.
\]

Natural stretching variable is given by the geometry and reads \( y = \frac{x}{\varepsilon} \). Therefore, the correction \( w \) is given by

\[
-d^2w \frac{dy^2}{dy^2} = 0 \text{ in } (0, +\infty); \quad [w]_{\Sigma} = w(+0) - w(-0) = 0; \quad (31)
\]

\[
\left[ \frac{dw}{dy} \right]_{\Sigma} = \frac{dw}{dy}(+0) - \frac{dw}{dy}(-0) = -\frac{du^0}{dx}(+0), \quad \text{on } \Sigma \quad (32)
\]
\[ -\frac{d^2 w}{dy^2} = 0 \quad \text{in} \quad (a-1, 0); \quad w(a-1) = 0; \quad \frac{dw}{dy} \to 0, \quad y \to +\infty. \quad (33) \]

Hence

\[
w(y) = \begin{cases} 
\frac{du^0}{dx}(+0)(1 - a), & \text{for} \quad y > 0; \\
\frac{du^0}{dx}(+0)(1 - a + y), & \text{for} \quad a - 1 < y \leq 0. \\
0, & \text{pour} \quad y \leq -1.
\end{cases}
\quad (34)
\]

We add this correction to \( u^0 \) and obtain

\[
\quad u^1,\text{eff}(x) = u^0(x) + \varepsilon w\left(\frac{x}{\varepsilon}\right). \quad \text{It is easy to see that}
\]
\[
\begin{align*}
\epsilon^e(x) &= u^0(x) + \epsilon w(\frac{x}{\epsilon}) + O(\epsilon^2); \\
\frac{d\epsilon^e}{dx}(x) &= \frac{du^0}{dx}(x) + \frac{dw}{dy}(\frac{x}{\epsilon}) + O(\epsilon). 
\end{align*}
\] (35)

Next we find out that \(u^0(+0) + \epsilon w(+0) = \epsilon (1 - a) \frac{du^0}{dx}(+0)\) and \(\frac{d\epsilon^0}{dx}(+0) + \frac{dw}{dy}(+0) = \frac{d\epsilon^0}{dx}(+0)\). Consequently, we impose the following effective interface condition:

\[
\epsilon^{eff}(+0) = \epsilon (1 - a) \frac{d\epsilon^{eff}}{dx}(+0) = \sqrt{\frac{12k}{1 - a}} \frac{d\epsilon^{eff}}{dx}(+0). 
\] (36)
Example 1D 11

In \((0, +\infty)\), \(u^{\text{eff}}\) satisfies the original PDE:

\[-\frac{d^2 u^{\text{eff}}}{dx^2} = f, \text{ dans } \Omega_2; \frac{du^{\text{eff}}}{dx} \to 0, \text{ quand } x \to +\infty.\]  \hfill (37)

By easy direct calculation, we calculate the solution \(u^{\text{eff}}\) for \((36)-(37)\) and find out that

\[\left\| u^\varepsilon - u^{\text{eff}} \right\|_{L^\infty(0, +\infty)} = \sup_{x \geq 0} |u^\varepsilon(x) - u^{\text{eff}}(x)| \leq C\varepsilon^2.\]  \hfill (38)

Clearly, in the case of a porous medium things are more complicated and \(w\) should be calculated using the corresponding boundary layer problem.
The correction order $\varepsilon$ and the Navier law

In the estimate (18) the principal contribution was coming from the surface term $\int_{\Sigma} \varphi \cdot \tau$. In order to eliminate this term, we use the functions

$$
\beta^{j,\varepsilon}(x) = \varepsilon \beta^{j} \left( \frac{x}{\varepsilon} \right) \quad \text{and} \quad \omega^{j,\varepsilon}(x) = \omega^{j} \left( \frac{x}{\varepsilon} \right), \quad x \in \Omega^{\varepsilon},
$$

(39)

where $\{\beta^{j}, \omega^{j}\}, \, j = 1, 2$ are given by the problems (40)-(44) with $\lambda = e^{j}$.

The crucial role is played by an auxiliary problem. It reads as follows:
For a given constant vector $\lambda \in \mathbb{R}^2$, find $\{\beta^\lambda, \omega^\lambda\}$ that solve

$$-\triangle_y \beta^\lambda + \nabla_y \omega^\lambda = 0 \quad \text{in } Z^+ \cup (Y - e^3) \quad (40)$$

$$\text{div}_y \beta^\lambda = 0 \quad \text{in } Z_{bl} \quad (41)$$

$$[\beta^\lambda]_S(\cdot, 0) = 0 \quad \text{on } S \quad (42)$$

$$[\{\nabla_y \beta^\lambda - \omega^\lambda I\}e^3]_S(\cdot, 0) =$$

$$[\{2D_y(\beta^\lambda) - \omega^\lambda I\}e^3]_S(\cdot, 0) = (\lambda, 0) \quad \text{on } S \quad (43)$$

$$\beta^\lambda = 0 \quad \text{on } (Y - e^3), \quad \{\beta^\lambda, \omega^\lambda\} \text{ is } y' = (y_1, y_2) - \text{periodic,} \quad (44)$$

where $S = (0, 1) \times \{0\}$, $Z^+ = (0, 1) \times (0, +\infty)$ and $Z_{bl} = Z^+ \cup S \cup (Y - e^3)$. 


Let $V = \{ z \in L^2_{loc}(Z_{bl})^3 : \nabla_y z \in L^2(Z_{bl})^9; \ z = 0 \text{ on } (\mathcal{Y} - \mathbf{e}^3); \ \text{div}_y z = 0 \text{ in } Z_{bl} \text{ and } z \text{ is } y' = (y_1, y_2)\text{-periodic } \}. \text{ Then, by the Lax-Milgram lemma, there is a unique } \beta^\lambda \in V \text{ satisfying}

\begin{align*}
\int_{Z_{bl}} \nabla \beta^\lambda \nabla \varphi \ dy &= -\int_{S} \varphi \lambda \ dy_1 dy_2, \quad \forall \varphi \in V. \tag{45}
\end{align*}

Using De Rham’s theorem we obtain a function $\omega^\lambda \in L^2_{loc}(Z_{bl})$, unique up to a constant and satisfying (40). By the elliptic theory, 

$\{\beta^\lambda, \omega^\lambda\} \in V \cap C^\infty(Z^+ \cup (Y - \mathbf{e}^3))^3 \times C^\infty(Z^+ \cup (Y - \mathbf{e}^3))$, for any solution to (40)-(44).

In the neighborhood of $S$ we have $\beta^\lambda - (\lambda_1, \lambda_2, 0)(y_3 - y_3^2/2)e^{-y_3} H(y_3) \in W^{2,q}$ and $\omega^\lambda \in W^{1,q}$, for all $q \in [1, \infty)$. 
Lemma 4 For any positive $a, a_1$ and $a_2$, $a_1 > a_2$, the solution $\{\beta^\lambda, \omega^\lambda\}$ satisfies

\[
\begin{align*}
C^{3, bl}_\lambda &= \int_0^1 \int_0^1 \beta^\lambda_3(y_1, y_2, a) \, dy_1 dy_2 = 0, \\
\int_0^1 \int_0^1 \omega^\lambda(y_1, y_2, a_1) \, dy_1 dy_2 &= \int_0^1 \int_0^1 \omega^\lambda(y_1, y_2, a_2) \, dy_1 dy_2, \\
C^{j, bl}_\lambda &= \int_0^1 \int_0^1 \beta^\lambda_j(y_1, y_2, a_1) \, dy_1 dy_2 = \int_0^1 \int_0^1 \beta^\lambda_j(y_1, y_2, a_2) \, dy_1 dy_2, \quad j \\
C^{bl}_\lambda &= \sum_{j=1}^{2} C^{j, bl}_\lambda \lambda_j = \int_S \beta^\lambda \lambda \, dy_1 dy_2 = -\int_{Z^{bl}} |\nabla \beta^\lambda(y)|^2 \, dy < 0.
\end{align*}
\]
Lemma 6 Let $a > 0$ and let $\beta^{a,\lambda}$ be the solution for (40)-(44) with $S$ replaced by $S_a = (0, 1)^2 \times \{a\}$ and $Z^+$ by $Z^+_a = (0, 1)^2 \times (a, +\infty)$. Then we have

$$C_{\lambda}^{a,bl} = \int_0^1 \int_0^1 \beta^{a,\lambda}(y_1, y_2, a) \lambda \, dy_1 = C_{\lambda}^{bl} - a \left| \lambda \right|^2. \quad (47)$$

Lemma 7 (see Jäger - Mikelić) Let $\{\beta^j, \omega^j\}$ be as above and let $M_{ij} = \frac{1}{|S|} \int_S \beta^j_i \, dy_1 dy_2$ be Navier’s matrix. Then the matrix $M$ is symmetric negatively definite.
Lemma 9 Let \( \{ \beta^j, \omega^j \} \), \( j = 1, 2 \), be as above. Then we have

\[
\begin{align*}
\left| D^\alpha \text{curl}_y \beta^j(y) \right| &\leq Ce^{-2\pi y_3}, \quad y_3 > 0, \quad \alpha \in \mathbb{N}^2 \cup (0, 0) \\
\left| \beta^j(y) - (M_{1j}, M_{2j}, 0) \right| &\leq C(\delta)e^{-\delta y_3}, \quad y_3 > 0, \quad \forall \delta < 2\pi \\
\left| D^\alpha \beta^j(y) \right| &\leq C(\delta)e^{-\delta y_3}, \quad y_3 > 0, \quad \alpha \in \mathbb{N}^2, \quad \forall \delta < 2\pi \\
\left| \omega^j(y) \right| &\leq Ce^{-2\pi y_3}, \quad y_3 > 0.
\end{align*}
\]

Correcting the compressibility effects:

We define \( Q^{j,k,bl} \), \( j, k = 1, 2 \), by

\[
\begin{align*}
\text{div}_y Q^{j,k,bl}(y) = \beta^j_k(y) - M_{jk}H(y_3) \quad &\text{in} \quad \mathbb{Z}^+ \cup (\mathcal{Y} - e^3), \quad (49) \\
Q^{j,k,bl} = 0 \quad &\text{on} \quad \partial \mathcal{Y} - e^3, \quad \text{Q}^{j,k,bl} \quad &\text{is 1-periodic in} \quad (y_1, y_2)
\end{align*}
\]
\[ [Q^{j,k,bl}]_S = -\frac{e^3}{|S|} \int_{Y-e^3} \beta^j_k(y) \, dy. \] (51)

**Proposition 10** Problem (49)-(51) has at least one solution \( Q^{j,k,bl} \in H^1(Z^+ \cup (\Gamma - e^3))^3 \cap C^\infty_{loc}(Z^+ \cup (\Gamma - e^3))^3 \).

Furthermore, \( Q^{j,k,bl} \in W^{1,q}(Z^+)^3 \), \( Q^{j,k,bl} \in W^{1,q}(\Gamma - e^3)^3 \), for all \( q \in [1, +\infty) \) and there exists \( \gamma_0 > 0 \) such that

\[ e^{\gamma_0 y_3} Q^{j,k,bl} \in H^1(Z^+)^3. \] (52)
Now we introduce the " 2-scale " velocity by

$$u(\varepsilon) = u^0 + \varepsilon \sum_{j=1}^{2} \left( \beta^j \left( \frac{x}{\varepsilon} \right) - (M_{j1}, M_{j2}, 0)H(x_3) \right) \frac{\partial u^0_j}{\partial x_3} \bigg|_{\Sigma} +$$

$$\varepsilon H(x_3) \sum_{j,k=1}^{2} M_{kj} g^{kj}(x, t) + O(\varepsilon^2)$$

(53)

where the boundary layer functions $\beta^j$ are given by (40)-(44), with $\lambda = e^j$ and the last term corresponds to the counterflow generated by stabilization of the boundary layer functions $\beta^j$. It is given by the system
\[
\begin{aligned}
\frac{\partial g^{kj}}{\partial t} + \text{div} \left( g^{kj} \otimes u^0 + g^{kj} \otimes u^0 \right) - \frac{1}{Re} \text{div} \left( 2D(g^{kj}) - p_g^{kj} I \right) &= 0 \quad \text{in} \quad Q_T; \\
\text{div} \; g^{kj} &= 0 \quad \text{in} \quad Q_T; \\
\end{aligned}
\]

\[
\begin{aligned}
g^{kj}(x, 0) &= 0 \quad \text{in} \quad \Omega; \\
g^{kj} &= \frac{\partial u^0_j}{\partial x_3} e^k \quad \text{on} \quad \Sigma \times (0, T), \quad g^{kj} = 0 \quad \text{on} \quad \Sigma_L \times (0, T); \\
\end{aligned}
\]

\[
\left\{ g^{kj}, p_g^{kj} \right\} \text{, is } L\text{-periodic in } (x_1, x_2)
\]

(54)
On the interface $\Sigma$

\[
\frac{\partial u_i}{\partial x_3} = \frac{\partial u_i^0}{\partial x_3} + \sum_{j=1}^{2} \frac{\partial \beta_j}{\partial y_3} \left( \frac{x}{\varepsilon} \right) \frac{\partial u_j^0}{\partial x_3} + O(\varepsilon) \text{ and } \frac{1}{\varepsilon} u_i = \sum_{j=1}^{2} \beta_j \left( \frac{x}{\varepsilon} \right) \frac{\partial u_j^0}{\partial x_3} + O(\varepsilon)
\]

After averaging we obtain the familiar form of the Navier’s slip condition

\[
u_{i}^{\text{eff}} = \varepsilon \sum_{j=1}^{2} M_{ji} \frac{\partial u_j^{\text{eff}}}{\partial x_3} \text{ on } \Sigma,
\]

where $u_{i}^{\text{eff}}$ is the average over the impurities. The higher order terms are neglected. We’ll rigorously justify (NFC).
Now we define the "2-scale" pressure \( p(\varepsilon) \) by

\[
p(\varepsilon) = p^0 + \sum_{j=1}^{2} \omega^{j,\varepsilon} \frac{\partial u_j^0}{\partial x_3} |\Sigma| + H(x_3)\varepsilon \sum_{j,k=1}^{2} M_{kj} p_{g}^{kj}
\]

and

\[
\mathcal{U}_0^\varepsilon(x, t) = u^\varepsilon - u(\varepsilon) \quad \text{and} \quad \mathcal{P}_0^\varepsilon = p^\varepsilon - p(\varepsilon).
\]

**Proposition 3** We have \( \mathcal{U}_0^\varepsilon \in H^1(\Omega^\varepsilon \times (0, T)) \) and \( \{\mathcal{U}_0^\varepsilon, \mathcal{P}_0^\varepsilon\} \) satisfies the variational equation

\[
\int_{\Omega^\varepsilon} \left( \frac{\partial \mathcal{U}_0^\varepsilon}{\partial t} + \text{div} \left( \mathcal{U}_0^\varepsilon \otimes u^\varepsilon + u^0 \otimes \mathcal{U}_0^\varepsilon \right) \right) \varphi \, dx +
\]
\[
\frac{1}{\text{Re}} \int_{\Omega^\varepsilon} (2D(U_0^\varepsilon) - P_0^\varepsilon I) : D(\varphi) \, dx = \int_{\Omega^\varepsilon \setminus \Omega} f \cdot \varphi \, dx - \int_{\Omega^\varepsilon} \text{div} ((U_0^\varepsilon - \varepsilon \omega) \otimes \omega) \varphi \, dx - \sum_{j=1}^{2} \left< \Phi_{1,j}^1(t), \varphi \right>_{V'((\Omega^\varepsilon)')} = \left< \Phi_1^1(t), \varphi \right>_{V'((\Omega^\varepsilon)')},
\]
\[U_0^\varepsilon(x, 0) = 0 \quad \text{on} \quad \Omega^\varepsilon. \quad (56)\]

Finally, we have
\[
\text{div} \, U_0^\varepsilon = - \sum_{j=1}^{2} ((\beta_j^j - \varepsilon M^j H(x_3)) \cdot \nabla r_j. \quad (58)
\]
Proposition 4 We have

\[ \sqrt{\int_0^T \left| < \Phi^1(t), \varphi > V'(\Omega^\varepsilon), V(\Omega^\varepsilon) \right|^2 dt} \leq C \varepsilon^{3/2} \| \nabla \varphi \|_{L^2(\Omega T)}, \]  

(59)

\[ \| \text{div} \mathcal{U}_0^\varepsilon \|_{L^2(\Omega^\varepsilon \times (0,T))} \leq C \varepsilon^{3/2}. \]  

(60)

Next we need to diminish the compressibility effects for the velocity error. It is achieved by adding a particular correction at order \( O(\varepsilon^2) \).

Let \( Q^{j,k,bl} \) be given by (49)-(51). Furthermore let \( Q^{j,k,bl,\varepsilon}(x) = \varepsilon^2 Q^{j,k,bl}(x/\varepsilon) \) and let \( w^{k,j} \) be defined by
\[
\frac{\partial w^{kj}}{\partial t} + \text{div} \left( w^{kj} \otimes u^0 + w^{kj} \otimes u^0 \right) - \frac{1}{\text{Re}} \text{div} \left( 2D(w^{kj}) - p_{wj} I \right) = 0 \quad \text{in } Q_T;
\]

\[
\text{div } w^{kj} = \frac{1}{L^3} \int_{\Sigma} \frac{\partial}{\partial x_k} \frac{\partial u^0_j}{\partial x_3} \, dS \quad \text{in } Q_T;
\]

\[
w^{kj}(x, 0) = 0 \quad \text{in } \Omega;
\]

\[
w^{kj} = -\frac{\partial}{\partial x_k} \frac{\partial u^0_j}{\partial x_3} e^3 \quad \text{on } \Sigma \times (0, T), \quad w^{kj} = 0 \quad \text{on } \Sigma_L \times (0, T);
\]

\[
\{w^{kj}, p_{wj}\}, \quad \text{is L-periodic in } (x_1, x_2)
\]

We introduce the following error functions, where the compressibility effects are reduced to the next order:
We summarize now properties of the new error functions:

\[ \mathcal{U}^\varepsilon(x, t) = \mathcal{U}_0^\varepsilon(x, t) + \sum_{j,k=1}^{2} Q^{j,k,bl,\varepsilon}(x) \frac{\partial}{\partial x_k} \frac{\partial u_0^j}{\partial x_3}(x, t) \mid_S - \varepsilon^2 H(x_3) \sum_{j,k=1}^{2} \frac{1}{|S|} \left( \int_{Y-e^3} \beta_k^j(y)\,dy \right) \mathbf{w}^kj(x, t) \]  

(62)

\[ \mathcal{P}^\varepsilon(x, t) = \mathcal{P}_0^\varepsilon(x, t) - \varepsilon^2 H(x_3) \sum_{j,k=1}^{2} \frac{1}{|S|} \left( \int_{Y-e^3} \beta_k^j(y)\,dy \right) p_{kj}^w(x, t). \]  

(63)
Proposition 5 Let \( \{U^\varepsilon, P^\varepsilon\} \) be given by (62)-(63). Then we have \( U^\varepsilon \in H^1(\Omega^\varepsilon \times (0,T)) \) and \( \{U^\varepsilon, P^\varepsilon\} \) satisfies the variational equation

\[
\int_{\Omega^\varepsilon} \left( \frac{\partial U^\varepsilon}{\partial t} + \text{div} \ (U^\varepsilon \otimes u^\varepsilon + u^0 \otimes U^\varepsilon) \right) \varphi \ dx + \frac{1}{\text{Re}} \int_{\Omega^\varepsilon} (2D(U^\varepsilon) - P^\varepsilon I) : D(\varphi) \ dx = < \Phi(t), \varphi >_{V'(\Omega^\varepsilon),V(\Omega^\varepsilon)},
\]

(64)

\[
U^\varepsilon(x, 0) = 0 \quad \text{on} \quad \Omega^\varepsilon,
\]

(65)
with

$$\sqrt{\int_0^T < \Phi(t), \varphi >_{V'(\Omega^\varepsilon), V(\Omega^\varepsilon)}^2} \, dt \leq C\varepsilon^{3/2}||\nabla \varphi||_{L^2(Q_T)}.$$  \hfill (66)

Finally, we have

$$\text{div} \ U^\varepsilon = \sum_{j,k=1}^{2} Q_{j,k,bl,\varepsilon}(x) \cdot \nabla \frac{\partial}{\partial x_k} \frac{\partial u_j^0}{\partial x_3} (x, t) +$$

$$\varepsilon^2 H(x_3) \sum_{j,k=1}^{2} \frac{1}{L^3} \frac{1}{|S|} \left( \int_{Y - e^3} \beta^j_k(y) \, dy \right) \int \frac{\partial}{\partial x_k} \frac{\partial u_j^0}{\partial x_3} dS = g^\varepsilon \hfill (67)$$

$$||\text{div} \ U^\varepsilon||_{L^2(\Omega^\varepsilon \times (0,T))} \leq C\varepsilon^2. \hfill (68)$$
Elimination of the pressure field requires solenoidal test functions. We prove that there exists \( z^\varepsilon \in H^1(0, T; V(\Omega^\varepsilon)) \) such that \( \text{div } z^\varepsilon = g^\varepsilon \) and
\[
\| z^\varepsilon \|_{H^1(0,T;V(\Omega^\varepsilon))} \leq C\varepsilon^{3/2}.
\] (69)

**Proposition 6** Let us suppose the hypothesis (H1) and let \( f \in C^\infty(\overline{\Omega^\varepsilon} \times [0, T]) \), be such that \( \text{supp } f \subset \overline{\Omega^\varepsilon} \times (0, T) \), and it is periodic in \((x_1, x_2)\) with period \( L \). Then we have
\[
\int_0^T \left( \| \nabla (U^\varepsilon(t) - z^\varepsilon(t)) \|_{L^2(\Omega^\varepsilon)}^2 + \| \nabla \partial_t (U^\varepsilon(t) - z^\varepsilon(t)) \|_{L^2(\Omega^\varepsilon)}^2 \right) \, dt \leq C\varepsilon^3
\] (70)
\[
\max_{0 \leq t \leq T} \left( \| (U^\varepsilon(t) - z^\varepsilon(t)) \|^2_{L^2(\Omega^\varepsilon)} + \| \partial_t (U^\varepsilon(t) - z^\varepsilon(t)) \|^2_{L^2(\Omega^\varepsilon)} \right) \leq C\varepsilon_3^3
\]  
(71)

\[
\| U^\varepsilon - z^\varepsilon \|_{L^2(\Omega^\varepsilon \setminus \Omega \times (0,T))}^3 + \| \partial_t (U^\varepsilon - z^\varepsilon) \|_{L^2(\Omega^\varepsilon \setminus \Omega \times (0,T))}^3 \leq C\varepsilon^{5/2}
\]  
(72)

\[
\| U^\varepsilon - z^\varepsilon \|_{L^2(\Sigma \times (0,T))}^3 + \| \partial_t (U^\varepsilon - z^\varepsilon) \|_{L^2(\Sigma \times (0,T))}^3 \leq C\varepsilon^2
\]  
(73)

\[
\| U^\varepsilon - z^\varepsilon \|_{L^2(Q_T)}^3 + \| \partial_t (U^\varepsilon - z^\varepsilon) \|_{L^2(Q_T)}^3 \leq C\varepsilon^2
\]  
(74)

\[
\| P^\varepsilon \|_{L^2_0(Q_T)} + \| \partial_t P^\varepsilon \|_{L^2_0(Q_T)} \leq C\varepsilon^{3/2}.
\]  
(75)
Effective problem

\[
\frac{\partial \mathbf{u}^{\text{eff}}}{\partial t} + \text{div} \left( \mathbf{u}^{\text{eff}} \otimes \mathbf{u}^{\text{eff}} \right) - \frac{1}{\text{Re}} \text{div} \left( 2D(\mathbf{u}^{\text{eff}}) - p^{\text{eff}} \mathbf{I} \right) = \mathbf{f} \quad \text{in} \quad Q_T, \quad (76)
\]

\[u_3^{\text{eff}} = 0 \quad \text{on} \quad \Sigma_T \quad \text{and} \quad (77)\]

\[u_k^{\text{eff}} = \varepsilon \sum_{j=1}^{2} M_{kj} \frac{\partial}{\partial x_3} u_j^{\text{eff}}, \quad k = 1, 2, \quad \text{on} \quad \Sigma_T, \quad (77)\]

\[\mathbf{u}^{\text{eff}} \quad \text{is periodic in} \quad (x_1, x_2) \quad \text{with period} \quad L, \quad (78)\]

\[\mathbf{u}^{\text{eff}} = 0 \quad \text{on} \quad \Sigma_L \times (0, T), \quad \mathbf{u}^{\text{eff}}|_{t=0} = 0 \quad \text{on} \quad \Omega. \quad (79)\]
Theorem

Let us suppose the hypothesis \((H1)\) on the solution to the Navier-Stokes equations for the problem with no rugosities. Then we have

\[
\max_{0 \leq t \leq T} \| \nabla (u^\varepsilon - u^{\text{eff}})(t) \|_{L^1(\Omega)}^2 \leq C\varepsilon
\]  

(80)

\[
\max_{0 \leq t \leq T} \| (u^\varepsilon - u^{\text{eff}})(t) \|_{L^2(\Omega)^3}^2 \leq C\varepsilon^{3/2}
\]  

(81)

\[
\max_{0 \leq t \leq T} \| (p^\varepsilon - p^{\text{eff}})(t) \|_{L^2_{\text{loc}}(\Omega)^3}^2 \leq C\varepsilon.
\]  

(82)

How to prove it?

Let \(\Sigma_T = \Sigma \times (0, T)\) and let
\[ V^{\text{eff}}(\Omega) = \{ b \in H^1(\Omega)^3 : \ b = 0 \ \text{on} \ \Sigma_L, \ b_3 = 0 \ \text{on} \ \Sigma, \ b \ \text{is L-per} \} \] (83)

and \[ \mathcal{W}^{\text{eff}} = \{ \varphi \in V^{\text{eff}}(\Omega) : \ \text{div} \ \varphi = 0 \ \text{in} \ \Omega \}. \] (84)

The problem (76)-(79) which admits a solution \[ u^{\text{eff}} \in L^2(0,T; \mathcal{W}^{\text{eff}}), \ \partial_t u^{\text{eff}} \in L^{4/3}(0,T; (\mathcal{W}^{\text{eff}})'), \ \text{div} \ u^{\text{eff}} = 0. \] Again \( p^{\text{eff}} \) is determined using De Rham's theorem.

Our first aim is to prove that there are smooth solutions to (76)-(79) close to \( u^0 \). In this scope, we use a perturbation argument, which requires the hypothesis (H1) on \( u^0 \).
proposition 7 Let us suppose the hypothesis (H1) and let $f \in C^\infty(\overline{\Omega} \times [0, T])^3$, be such that $\text{supp } f \subset \overline{\Omega} \times (0, T]$, and it is periodic in $(x_1, x_2)$ with period $L$. Then the problem (76)-(79) has a solution $\{u^{eff}, p^{eff}\} \in L^2(0, T; W^{eff} \cap H^2(\Omega)^3) \times L^2(0, T; H^1(\Omega))$, such that $\{\partial_t u^{eff}, \partial_t p^{eff}\} \in L^2(0, T; H^1(\Omega)^3) \times L^2(0, T; L^2(\Omega))$. $u^{eff}$ is unique and $p^{eff}$ is unique up to a constant.

Next we prove
Proposition 8 We have

\[ \int_0^T \| \nabla (u^{eff} - u^0 - \varepsilon \sum_{j,k=1}^{2} M_{kj} g^{kj}) (t) \|^2_{L^2(\Omega)^9} \, dt + \]

\[ \int_0^T \| \nabla \partial_t (u^{eff} - u^0 - \varepsilon \sum_{j,k=1}^{2} M_{kj} g^{kj}) (t) \|^2_{L^2(\Omega)^9} \, dt \leq C \varepsilon^{3/2} \]  

\( (85) \)

\[ \max_{0 \leq t \leq T} \left( \| (u^{eff} - u^0 - \varepsilon \sum_{j,k=1}^{2} M_{kj} g^{kj}) (t) \|^2_{L^2(\Omega)^3} + \right) \]

\[ \| \partial_t (u^\varepsilon - u^0 - \varepsilon \sum_{j,k=1}^{2} M_{kj} g^{kj}) (t) \|^2_{L^2(\Omega)^3} \right) \leq C \varepsilon^{3/2} \]  

\( (86) \)
\[
\max_{0 \leq t \leq T} \left( \| (p_v - p^0 - \varepsilon \sum_{j,k=1}^2 M_{kj} p_{g,j}^k(t)) \|_{L^2(\Omega \times T)}^2 + \| \partial_t (p_v - p^0 - \varepsilon \sum_{j,k=1}^2 M_{kj} p_{g,j}^k(t)) \|_{L^2(\Omega \times T)}^2 \right) \leq C \varepsilon^{3/2}
\] (87)

\[\Rightarrow\text{ statement on the Theorem}\]

OPEN PROBLEMS:

a) could we simplify proofs using 2-scale convergence of boundary layers (Allaire, Capdebosq, Conca).

b) Turbulent flows?