Some variants of stochastic homogenization

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Joint works with
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Real composite material

Courtesy M. Thomas and EADS
Line of thought

- go beyond the idealistic setting of periodic materials
- do not treat fully general random materials (fine theoretically, but too expensive to treat practically);
- consider materials that are, in a sense to be made precise, random perturbations of periodic materials;
- and adapt the modelling and the numerical approach.

Work on the simplest possible equation $-\nabla \cdot (a(\cdot) \nabla u^\varepsilon) = f$. 
Bounds
$A(x, \omega) = \sum_{k \in \mathbb{Z}^2} 1_{Q+k}(x) a_k(\omega) \mathbf{ld}_2$, \quad a_k \text{ i.i.d., } \mathbb{P}(a_k = a) = \mathbb{P}(a_k = b) = 1/2$

$A^* = a^* \mathbf{ld}_2 = \sqrt{ab} \mathbf{ld}_2$.

Set $a = 1$, and check the accuracy of some classical bounds for various $b$. 
**Hashin-Shtrikman bounds - 2**

Case $a = 1$, $b = 10$:

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<td>1.81</td>
<td>2.38</td>
<td>3.16</td>
<td>4.00</td>
<td>4.19</td>
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Error is larger than 25%.
Homogenization theory

\[-\text{div} \left( A \left( \frac{x}{\varepsilon} \right) \nabla u^{\varepsilon} \right) = f, \quad \text{a periodic}\]

\[-\text{div} \left( A \left( \frac{x}{\varepsilon}, \omega \right) \nabla u^{\varepsilon} \right) = f \quad \text{a stationary}\]

\[-\text{div} \left( A \left( \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right) \right) \nabla u^{\varepsilon} \right) = f \quad \text{a periodic, } \nabla \Phi \text{ stationary}\]
Randomness → many practical difficulties. 
Simplify the situation from the theoretical viewpoint: consider a simple scalar linear PDE (representing e.g. heat conduction, or Darcy law):

\[-\text{div} \left[ A_\varepsilon(x, \omega) \nabla u^\varepsilon \right] = f \quad \text{in the domain } \mathcal{D}, \quad u^\varepsilon \text{ fixed at the boundary}
\]

\[A_\varepsilon(x, \omega) \equiv \text{thermal conductivity tensor, with a small characteristic length } \varepsilon.\]

Outline:

- Some background materials
- A variant of stochastic homogenization.
- Dedicated approaches for weakly stochastic materials, when \( A_\varepsilon(x, \omega) \) is almost deterministic.
- Variance reduction.
Homogenization 1.0.1: the periodic setting

\[-\text{div} \left[ A_{\text{per}} \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon \right] = f \quad \text{in} \quad D, \quad u^\varepsilon = 0 \quad \text{on} \quad \partial D,\]

with $A_{\text{per}}$ symmetric and $\mathbb{Z}^d$-periodic: $A_{\text{per}}(x + k) = A_{\text{per}}(x)$ for any $k \in \mathbb{Z}^d$.

When $\varepsilon \to 0$, $u^\varepsilon$ converges to $u^*$ solution to

\[-\text{div} \left[ A^* \nabla u^* \right] = f \quad \text{in} \quad D, \quad u^* = 0 \quad \text{on} \quad \partial D.\]

The effective matrix $A^*$ is given by

\[[A^*]_{ij} = \int_Q (e_i + \nabla w_{e_i}(y))^T A_{\text{per}}(y) e_j \, dy, \quad Q = \text{unit cube} = (0, 1)^d\]

with, for any $p \in \mathbb{R}^d$, $w_p$ solves the so-called corrector problem:

\[-\text{div} \left[ A_{\text{per}}(y) (p + \nabla w_p) \right] = 0 \quad \text{in} \quad \mathbb{R}^d, \quad w_p \text{ is } \mathbb{Z}^d\text{-periodic.}\]

Note that $u_p(y) = p \cdot y + w_p(y)$ satisfies $\langle \nabla u_p \rangle = p$.

$\rightarrow$ Solve $d$ PDEs (for $p = e_i$, $1 \leq i \leq d$) on the bounded domain $Q$: easy!
Stochastic homogenization setting

We consider statistically homogeneous random materials:

\[-\text{div} \left[ A \left( \frac{x}{\varepsilon}, \omega \right) \nabla u^\varepsilon \right] = f \quad \text{in} \quad D\]

The tensor $A(x, \omega)$ is such that

- $A(x, \omega)$ and $A(x + k, \omega)$ share the same probability distribution, for any $k \in \mathbb{Z}^d$. For a given realization of the randomness, properties may be different. But, on average, they are identical: the material is statistically homogeneous.

- $x \mapsto \mathbb{E}(A(x, \cdot))$ is a periodic function.

- Ergodicity property: space average $\sim$ average over realizations:

  \[
  \frac{1}{|Q_N|} \int_{Q_N} A(x, \omega) \, dx \xrightarrow{N \to \infty} \mathbb{E} \left[ \int_{Q} A(x, \cdot) \, dx \right]
  \]

  with $Q = (0, 1)^d$ and $Q_N = (-N, N)^d$. 
From one cell to the other, $A(x, \omega)$ varies. However, in average, it remains the same: $\mathbb{E} [A(x, \omega)]$ is periodic.
\[
-\text{div} \left[ A \left( \frac{x}{\varepsilon}, \omega \right) \nabla u^\varepsilon \right] = f \quad \text{in} \quad D, \quad u^\varepsilon = 0 \quad \text{on} \quad \partial D.
\]

\(u^\varepsilon(\cdot, \omega)\) converges to \(u^*\) solution to

\[
-\text{div} \left[ A^* \nabla u^* \right] = f \quad \text{in} \quad D, \quad u^* = 0 \quad \text{on} \quad \partial D,
\]

where the effective matrix \(A^*\) is given by

\[
[A^*]_{ij} = \mathbb{E} \left( \int_Q (e_i + \nabla w_{e_i}(y, \cdot))^T A(y, \cdot) e_j \, dy \right),
\]

where \(w_p\) solves

\[
\begin{cases}
-\text{div} \left[ A(y, \omega) (p + \nabla w_p(y, \omega)) \right] = 0 & \text{in} \quad \mathbb{R}^d, \quad p \in \mathbb{R}^d, \\
\nabla w_p \text{ is statist. homog.}, \quad \mathbb{E} \left( \int_Q \nabla w_p(y, \cdot) \, dy \right) = 0.
\end{cases}
\]

The corrector problem is set on \(\mathbb{R}^d\). Theoretically, the RVE is infinite.
A variant of the classical homogenization setting
A variant of classical stochastic homogenization

Classical stochastic homogenization:

\[-\text{div} \left[ A \left( \frac{x}{\varepsilon}, \omega \right) \nabla u^\varepsilon (x, \omega) \right] = f(x) \text{ in } \mathcal{D}, \quad u^\varepsilon = 0 \text{ on } \partial \mathcal{D}\]

where the matrix $A$ is stationary.

We consider here a variant:

\[-\text{div} \left[ A_{\text{per}} \left( \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right) \right) \nabla u^\varepsilon (x, \omega) \right] = f(x) \text{ in } \mathcal{D}, \quad u^\varepsilon = 0 \text{ on } \partial \mathcal{D}\]

for a periodic matrix $A_{\text{per}}$ and a random diffeomorphism $\Phi$, with $\nabla \Phi$ stationary.

In general, $A_{\text{per}} \circ \Phi^{-1}$ is NOT stationary.

Random diffeomorphism

\[ \Phi(., \omega) \]

A 'real' material \( \equiv \) a random deformation of a reference periodic material
Up to (more or less) random glasses, the material is periodic!
Deformed structure

The periodic structure corresponds to identical fibers set on a $\mathbb{Z}^2$ lattice.
Discrete stationary setting

Let \((\tau_k)_{k \in \mathbb{Z}^d}\) be a group action that preserves the measure \(\mathbb{P}\) and is ergodic:

\[
\forall k \in \mathbb{Z}^d, \quad \forall B \in \mathcal{F}, \quad \mathbb{P} (\tau_k B) = \mathbb{P} (B)
\]

\[
\forall B \in \mathcal{F}, \quad (\tau_k B = B \text{ for any } k \in \mathbb{Z}^d) \implies \mathbb{P} (B) = 0 \text{ or } 1.
\]

A function \(F\) is said stationary if,

\[
\forall k \in \mathbb{Z}^d, \quad F (x + k, \omega) = F (x, \tau_k \omega) \quad \text{a.e., a.s.}
\]

Only discrete shifts are allowed.

Ergodic theorem:

\[
F \left( \frac{x}{\varepsilon}, \omega \right) \xrightarrow{\varepsilon \to 0} \mathbb{E} \left( \int_Q F (x, \cdot) dx \right) \text{ in } L^\infty (\mathbb{R}^d), \text{ a.s.}
\]
Stochastic deformations

\[-\text{div} \left[ A_{\text{per}} \left( \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right) \right) \nabla u^\varepsilon(x, \omega) \right] = f(x) \text{ in } \mathcal{D}, \quad u^\varepsilon = 0 \text{ on } \partial\mathcal{D}\]

Assumptions:

- the map \( \Phi(\cdot, \omega) \) is almost surely a diffeomorphism from \( \mathbb{R}^d \) to \( \mathbb{R}^d \), with

\[
\text{EssInf}_{\omega \in \Omega, x \in \mathbb{R}^d} (\det(\nabla \Phi(x, \omega))) = \nu > 0,
\]

\[
\text{EssSup}_{\omega \in \Omega, x \in \mathbb{R}^d} |\nabla \Phi(x, \omega)| = M < +\infty,
\]

- \( \nabla \Phi(x, \omega) \) is stationary:

\[
\forall k \in \mathbb{Z}^d, \quad \nabla \Phi(x + k, \omega) = \nabla \Phi(x, \tau_k \omega)
\]

Why do we need \( \nabla \Phi \) to be stationary?
Consider \( b(x, \omega) = b_{\text{per}}(\Phi^{-1}(x, \omega)) \). Let us compute its average,

\[
\frac{1}{R} \int_0^R b(x, \omega) \, dx = \frac{1}{R} \int_0^R b_{\text{per}}(\Phi^{-1}(x, \omega)) \, dx
\]

With the change of variables \( y = \Phi^{-1}(x, \omega) \), we have

\[
\int_0^R b(x, \omega) \, dx \quad = \quad \frac{1}{\Phi^{-1}(R, \omega) - \Phi^{-1}(0, \omega)} \int_{\Phi^{-1}(0, \omega)}^{\Phi^{-1}(R, \omega)} b_{\text{per}}(y) \Phi'(y, \omega) \, dy
\]

Our assumptions on \( \Phi \) imply that \( \Phi^{-1}(R, \omega) - \Phi^{-1}(0, \omega) \approx M^{-1}R \) a.s.

Thus, if \( \Phi'(y, \omega) \) is stationary, this converges to

\[
\left[ \mathbb{E} \int_0^1 \Phi'(y, \omega) \, dy \right]^{-1} \quad \mathbb{E} \int_0^1 b_{\text{per}}(y) \Phi'(y, \omega) \, dy.
\]
Ergodic theorem

For any periodic function $b_{\text{per}}$,

$$
b_{\text{per}} \left( \Phi^{-1} \left( \frac{x}{\varepsilon} \right) \right) \xrightarrow{\ast} \varepsilon \to 0 \det \left( \mathbb{E} \left( \int_Q \nabla \Phi(x, \cdot) \, dx \right) \right)^{-1} \mathbb{E} \left( \int_{\Phi(Q, \cdot)} b_{\text{per}} \left( \Phi^{-1}(x, \cdot) \right) \, dx \right)
$$

in $L^\infty(\mathbb{R}^d)$, almost surely.

In the periodic case $\Phi = I_d$,

$$
g \left( \frac{x}{\varepsilon} \right) \xrightarrow{\ast} \varepsilon \to 0 \int_Q g(x) \, dx
$$
Homogenization result (XB, C. Le Bris, P.-L. Lions, 2006)

\[-\text{div} \left[ A_{\text{per}} \left( \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right) \right) \nabla u^\varepsilon (x, \omega) \right] = f(x) \text{ in } \mathcal{D}, \quad u^\varepsilon = 0 \text{ on } \partial \mathcal{D} \]

\(u^\varepsilon(\cdot, \omega)\) converges (weakly in \(H^1\) and strongly in \(L^2\)) to \(u_*\) almost surely, with

\[-\text{div} \left[ A^* \nabla u_* (x) \right] = f(x) \text{ in } \mathcal{D}, \quad u_* = 0 \text{ on } \partial \mathcal{D} \]

with the homogenized matrix

\[A^*_{ij} = \det \left[ \mathbb{E} \left[ \int_Q \nabla \Phi (y, \cdot) dy \right] \right]^{-1} \mathbb{E} \left[ \int_{\Phi(Q, \cdot)} (e_i + \nabla w_{e_i} (y, \cdot))^T A_{\text{per}} \left( \Phi^{-1} (y, \cdot) \right) e_j dy \right] \]

where, for all \(p \in \mathbb{R}^d\), \(w_p\) is the corrector defined by

\[
\begin{cases}
- \text{div} \left[ A_{\text{per}} \left( \Phi^{-1} (x, \omega) \right) (p + \nabla w_p (x, \omega)) \right] = 0 \text{ in } \mathbb{R}^d, \\
w_p (x, \omega) = \widetilde{w}_p (\Phi^{-1} (x, \omega), \omega), \quad \nabla \widetilde{w}_p \text{ is stationary,} \\
\mathbb{E} \left( \int_{\Phi(Q, \cdot)} \nabla w_p (y, \cdot) dy \right) = 0.
\end{cases}
\]
Going further
Numerical approximation

\[-\text{div} \left[ A_{\text{per}} \left( \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right) \right) \nabla u^\varepsilon(x, \omega) \right] = f(x) \quad \text{in} \ D, \quad u^\varepsilon = 0 \quad \text{on} \ \partial D\]

The corrector problem is set on \( \mathbb{R}^d \):

\[
\begin{aligned}
\mathcal{L}(w_p) &= 0 \quad \text{in} \ \mathbb{R}^d, \\
\nabla w_p \text{ is stationary}
\end{aligned}
\]

In practice, need to introduce truncation: introduce \( Q_N = (-N, N)^d \) and approximate \( w_p \) by \( w_p^N \) with

\[
\begin{aligned}
\mathcal{L}(w_p^N) &= 0 \quad \text{on} \ Q_N, \\
w_p^N(\cdot, \omega) \text{ is } Q_N\text{-periodic}
\end{aligned}
\]

and compute \( A_N^*(\omega) \) from there: large domain \( (N \gg 1) \), random output, . . .

Very expensive!
Assume the diffeomorphism $\Phi$ is close to the identity:

$$\Phi(x, \omega) = x + \eta \Psi(x, \omega) + O(\eta^2),$$

for $\eta$ small. Then

$$w_p(x, \omega) = w_p^0(x) + \eta w_p^1(x, \omega) + O(\eta^2),$$

with

$$-\text{div} \left( A_{\text{per}} (p + \nabla w_p^0) \right) = 0, \quad w_p^0 \text{ is } \mathbb{Z}^d\text{-periodic},$$

and

$$\begin{cases} 
-\text{div} \left[ A_{\text{per}} (\nabla w_p^1 - \nabla \Psi \nabla w_p^0) + (\nabla \Psi^T - (\text{div} \Psi) \text{Id}) A_{\text{per}} (p + \nabla w_p^0) \right] = 0, \\
\mathbb{E} \left( \int_Q \nabla w_p^1 \right) = \mathbb{E} \left( \int_Q (\nabla \Psi - (\text{div} \Psi) \text{Id}) \nabla w_p^0 \right), \quad \nabla w_p^1 \text{ stationary.}
\end{cases}$$

However, to compute $A^*$, only the expectation of $w_p^1$ is needed!

$$A_{ij}^* \sim \mathbb{E} \left[ \int (e_i + \nabla w_{e_i}(y, \cdot))^T A_{\text{per}} (\Phi^{-1} (y, \cdot)) e_j \, dy \right]$$
Taking the expectation and setting $\bar{w}_p^1 = \mathbb{E}(w_p^1)$,

\[
- \text{div} \left[ A_{\text{per}} \nabla \bar{w}_p^1 \right] = \text{RHS} \left( A_{\text{per}}, \mathbb{E}(\nabla \Psi), \nabla w_p^0 \right),
\]

\[
\int_Q \nabla \bar{w}_p^1 = \int_Q \left( \mathbb{E}(\nabla \Psi) - \mathbb{E}(\text{div } \Psi) \text{Id} \right) \nabla w_p^0, \quad \nabla \bar{w}_p^1 \text{ periodic.}
\]

Eventually,

\[
A^* = A^0 + \eta A^1 + O(\eta^2),
\]

with

\[
A_{ij}^0 = \int_Q (e_i + \nabla w_{e_i}^0)^T A_{\text{per}} e_j
\]

\[
A_{ij}^1 = \int_Q \text{fct} \left[ \mathbb{E}(\nabla \Psi), A^0, \nabla w^0, A_{\text{per}} \right] + \int_Q \left( \nabla \bar{w}_{e_i}^1 - \mathbb{E}(\nabla \Psi) \nabla w_{e_i}^0 \right)^T A_{\text{per}} e_j.
\]

Two periodic computations instead of an expensive stochastic one.
In practice, the corrector problem is solved on $Q_N = (-N, N)^d$, using e.g. FEM.

→ We do not compute $A^*$, but some $A_{N,h}^*(\omega)$.

Yet, a similar result holds (R. Costaouec, C. Le Bris, F. Legoll, CRAS 2010):

$$A_{N,h}^*(\omega) = A^0_h + \eta A_{N,h}^1(\omega) + O(\eta^2)$$

where

- $A^0_h$ is the Finite Element approximation of $A^0$, obtained by periodic homogenization of $A_{per}$.

- $A_{h}^1 := \mathbb{E}\left[ A_{N,h}^1 \right]$ is independent of $N$, and is easy to compute.
Numerical illustration (2D)

\[
\Phi_\eta(x, \omega) = x + \eta \Psi(x, \omega)
\]

with

\[
\Psi(x, \omega) = \begin{pmatrix}
\psi_A(x_1, \omega) \\
\psi_B(x_2, \omega)
\end{pmatrix}
\]

with

\[
\psi_A(x, \omega) = \sum_{k \in \mathbb{Z}} 1_{[k, k+1)}(x) \left( \sum_{q=0}^{k-1} A_q(\omega) + 2A_k(\omega) \int_k^x \sin^2(2\pi t) dt \right)
\]

where \((A_k)_{k \in \mathbb{Z}}\) and \((B_k)_{k \in \mathbb{Z}}\) are all i.i.d. uniform random variables.

Take as periodic reference structure

\[
A_{\text{per}}(x) = a_{\text{per}}(x) \text{ Id}
\]

with

\[
a_{\text{per}}(x) = \beta + (\alpha - \beta) \sin^2(\pi x) \sin^2(\pi y) \in C^\infty(\mathbb{R}^2)
\]
Deformed structure in $Q_N$, with $N = 5$ and $\eta = 0.05$
Error at order $\eta^2$

Relative error: \[ \frac{A_{N,h}^*(\omega) - A_{h}^0 - \eta A_{N,h}^1(\omega)}{\eta^2} \] for $h = 1/3$, $N = 20$
Numerical illustration (R. Costaouec, CLB, F. Legoll, CRAS 2010)

$h = 1/3, N = 20$

\[ A_{N,h}^{*}(\omega) = A_{h}^{0} + \eta A_{N,h}^{1}(\omega) + O(\eta^2) \quad \text{and} \quad \bar{A}_{h}^{1} := \mathbb{E} \left[ A_{N,h}^{1} \right] \]
What if the material is “fully” stochastic?

We go back to the standard setting:

\[-\text{div} \left[ A \left( \frac{x}{\varepsilon}, \omega \right) \nabla u^\varepsilon \right] = f \quad \text{in} \quad D, \quad u^\varepsilon = 0 \quad \text{on} \quad \partial D\]

with \( A \) stationary.

XB, R. Costaouec, C. Le Bris, F. Legoll, MPRF, 2012.
We go back to the standard setting:

\[ -\text{div} \left[ A \left( \frac{x}{\varepsilon}, \omega \right) \nabla u^\varepsilon \right] = f \quad \text{in} \quad \mathcal{D}, \quad u^\varepsilon = 0 \quad \text{on} \quad \partial \mathcal{D} \]

with \( A \) stationary. Then the effective matrix \( A^* \) is given by

\[ [A^*]_{ij} = \mathbb{E} \left( \int_{Q} (e_i + \nabla w_{e_i}(y, \cdot))^T A(y, \cdot) e_j \, dy \right) , \]

where \( w_p \) solves

\[
\begin{cases}
    -\text{div} [ A(y, \omega) (p + \nabla w_p(y, \omega))] = 0 & \text{in} \quad \mathbb{R}^d, \\
    \mathbb{E} \left( \int_{Q} \nabla w_p(y, \cdot) \, dy \right) = 0, \quad \nabla w_p \quad \text{stationary}
\end{cases}
\]

This corrector problem is set on \( \mathbb{R}^d \). Some approximation is in order to get to a tractable problem.
Solve the corrector problem on a truncated domain:

\[
\begin{cases}
-\text{div} \left[ A(y, \omega) \left( p + \nabla w^N_p(y, \omega) \right) \right] = 0 \quad \text{in} \quad \mathbb{R}^d, \\
w^N_p \text{ is } Q_N\text{-periodic}, \quad Q_N = (-N, N)^d
\end{cases}
\]

This yields an approximate effective matrix

\[
[A^*_N]_{ij}(\omega) = \frac{1}{|Q_N|} \int_{Q_N} \left( e_i + \nabla w^N_{e_i}(y, \cdot) \right)^T A(y, \omega) e_j \, dy
\]

Due to numerical truncation, $A^*_N$ is random!

When $N \to \infty$, we have $A^*_N \to A^*$ a.s. (Bourgeat/Piatnitski, 2004).
Reducing the statistical error

\[ [A_N^*]_{ij}(\omega) = \frac{1}{|Q_N|} \int_{Q_N} (e_i + \nabla w_{e_i}^N(y, \omega))^T A(y, \omega) e_j \, dy \]

At fixed \( N \),

\[ A^* - A_N^*(\omega) = A^* - \mathbb{E}[A_N^*] + \mathbb{E}[A_N^*] - A_N^*(\omega) \]

Can we reduce the statistical error? Can we compute more accurately \( \mathbb{E}[A_N^*] \)?

[Related works by Gloria and Otto]
Understanding the truncation in the 1D case

Assume that 
\[ a(x, \omega) = \sum_{k \in \mathbb{Z}} 1_{[k-1,k)}(x) \ a_k(\omega) \quad \text{with } a_k \text{ i.i.d.} \]

The effective coefficient is the harmonic average:

\[ a^* = \left[ \mathbb{E} \left( \int_0^1 a^{-1}(x, \cdot) \, dx \right) \right]^{-1} = \left[ \mathbb{E} \left( \frac{1}{a_0} \right) \right]^{-1} \]

After truncation, we obtain the approximation

\[ a^*_N(\omega) = \left[ \frac{1}{N} \int_0^N a^{-1}(x, \omega) \, dx \right]^{-1} = \left[ \frac{1}{N} \sum_{k=1}^N \frac{1}{a_k(\omega)} \right]^{-1} \]

By law of large numbers, \( \lim_{N \to \infty} a^*_N(\omega) = a^* \).

But, at any \( N \), \( \mathbb{E} (a^*_N) \neq a^* \) (bias), and \( a^*_N \) is random (variance).

For finite \( N \), the approximation \( a^*_N(\omega) \) is random.
$A(x, \omega) = \sum_{k \in \mathbb{Z}^2} 1_{Q+k}(x) \ a_k(\omega) \ \text{Id}_2, \quad a_k \text{ independent identically distributed}$

$a_k = \alpha \text{ or } \beta \text{ with equal probability.}$
Monte Carlo approximation

- Consider $2M$ realizations $A^m(y, \omega)$, compute for each of these
  - the corrector $w^N_p, m$, solution to
    
    $$
    -\text{div} \left[ A^m(y, \omega) \left( p + \nabla w^N_p, m(y, \omega) \right) \right] = 0, \quad w^N_p, m \text{ is } Q_N\text{-periodic},
    $$
  - and the approximate homogenized matrix
    
    $$
    [A^*_N, m]_{ij}(\omega) = \frac{1}{|Q_N|} \int_{Q_N} \left( e_i + \nabla w^N_p, m(y, \cdot) \right)^T A^m(y, \omega) e_j \, dy.
    $$

- Approximate $\mathbb{E}(A^*_N)$ by
  
  $$
  I_{2M} = \frac{1}{2M} \sum_{m=1}^{2M} A^*_N, m(\omega).
  $$

Classical confidence interval:

$$
\left| \mathbb{E}([A^*_N]_{ij}) - [I_{2M}]_{ij} \right| \leq 1.96 \sqrt{\text{Var}([A^*_N]_{ij})} \sqrt{2M}.
$$

The accuracy of $I_{2M}$ is directly linked with the variance of $A^*_N$. 
In practice, on a 2D example . . .

$I_{2M} \approx \mathbb{E}([A_N^*]_{11})$ (along with confidence intervals) for a given number $2M$ of realizations, and several sizes for $Q_N$.

For moderate $N$, the statistical error $\gg$ systematic error

Our aim: compute $\mathbb{E}(A_N^*)$ more efficiently, for any given $N$. 

Antithetic variables

Goal: compute $\mathbb{E}(f(U))$, with $U$ a random variable uniformly distributed in $[0, 1]$.

- Basic Monte Carlo method: using $2M$ independent realizations of $U(\omega)$,

$$I_{2M}(\omega) = \frac{1}{2M} \sum_{m=1}^{2M} f(U_m(\omega))$$

- Alternative approximation:

$$\overline{I}_{2M}(\omega) = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{2} \left( f(U_m(\omega)) + f(1 - U_m(\omega)) \right)$$

$1 - U(\omega)$ has the same law as $U(\omega)$:

$I_{2M}$ and $\overline{I}_{2M}$ both converge to $\mathbb{E}(f(U))$

At fixed $M$,

- both estimators have the same cost (same number of evaluations of $f$)
- accuracy?
When does it work?

\[ I_{2M}(\omega) = \frac{1}{2M} \sum_{m=1}^{2M} f(U_m(\omega)) \]

\[ \overline{I}_{2M}(\omega) = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{2} \left( f(U_m(\omega)) + f(1 - U_m(\omega)) \right) \]

Let's compare the variance:

\[ \text{Var} \overline{I}_{2M} = \text{Var} I_{2M} + \frac{1}{2M} \text{Cov}(f(U), f(1 - U)) \]

\( \overline{I}_{2M} \) is a better estimator than \( I_{2M} \) \iff \( \text{Cov}(f(U), f(1 - U)) \leq 0 \)

Lemma: assume that \( f : [0, 1] \mapsto \mathbb{R} \) is non-decreasing. Then
\( \text{Cov}(f(U), f(1 - U)) \leq 0. \)
Back to the homogenization context

We will apply the exact same idea to homogenization, with

\[
\text{input } \equiv U(\omega) \sim A(x, \omega) \mid_{x \in Q_N}, \quad \text{output } \equiv f(U) \sim A_N^*(\omega)
\]

Any time a random structure is considered, we will also make the computations with the antithetic structure.

Example: each time we see

\[
A(x, \omega) = \sum_{k \in \mathbb{Z}^d} 1_{Q+k}(x) \ a_k(\omega) \ \text{Id}, \quad a_k(\omega) \text{ are i.i.d.}
\]

we also do the computations with the antithetic field:

\[
B(x, \omega) = \sum_{k \in \mathbb{Z}^d} 1_{Q+k}(x) \ b_k(\omega) \ \text{Id}
\]

where \(b_k(\omega)\) is antithetic to \(a_k(\omega)\).
At each point \(x\), we replace the local microstructure by the antithetic microstructure.
Antithetic materials

\[ A(x, \omega) = \sum_{k \in \mathbb{Z}^d} 1_{Q+k}(x) \, a_k(\omega) \, \text{Id} \quad \rightarrow \quad B(x, \omega) = \sum_{k \in \mathbb{Z}^d} 1_{Q+k}(x) \, b_k(\omega) \, \text{Id} \]

If \( a_k = \alpha \) or \( \beta \) with equal probability, then set \( b_k(\omega) = \beta \) whenever \( a_k(\omega) = \alpha \).

If \( a_k \) is uniformly distributed in \([\alpha, \beta]\), then set \( b_k(\omega) = \alpha + \beta - a_k(\omega) \).
consider $2M$ independent realizations $A^m(x, \omega)$, the associated correctors $w_p^{N,m}$ and effective matrices $A^*_{N,m}(\omega)$, and the estimator

$$I_{2M} = \frac{1}{2M} \sum_{m=1}^{2M} A^*_{N,m}(\omega).$$

consider $M$ independent realizations $A^m(x, \omega)$,

- build the $M$ antithetic fields $B^m(x, \omega)$,
- for each of these $B^m(x, \omega)$, compute the associated corrector

$$-\text{div} \left[ B^m(y, \omega) \left( p + \nabla v_p^{N,m}(y, \omega) \right) \right] = 0 \text{ in } \mathbb{R}^d, \quad v_p^{N,m} \text{ is } Q_N\text{-periodic.}$$

Set $B^*_{N,m}(\omega) = \frac{1}{|Q_N|} \int_{Q_N} \left( e_i + \nabla v_{e_i}^{N,m}(y, \cdot) \right)^T B^m(y, \omega) e_j \, dy.$

$$\overline{I}_{2M} = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{2} \left( A^*_{N,m}(\omega) + B^*_{N,m}(\omega) \right).$$
Efficiency comparison

\[ I_{2M} = \frac{1}{2M} \sum_{m=1}^{2M} A^*_{N,m}(\omega) \]

\[ \bar{I}_{2M} = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{2} \left( A^*_{N,m}(\omega) + B^*_{N,m}(\omega) \right) \]

- **Equal cost**: for both estimators, \(2M\) corrector problems need to be solved.

- **Convergent**: \(I_{2M}\) and \(\bar{I}_{2M}\) both converge to \(\mathbb{E}(A^*_N)\)

- **Efficiency**: let’s see numerically!
Numerical experiments

\[ A(x, \omega) = \sum_{k \in \mathbb{Z}^2} 1_{Q+k}(x) a_k(\omega) \text{Id}_2, \quad a_k \text{ i.i.d.,} \quad a_k \sim \mathcal{U}[\alpha, \beta] \]

\[ I_{2M} \text{ and } \overline{I}_{2M} \text{ (and confidence interval), } \alpha = 3, \beta = 20. \]

Accuracy gain \( \geq \sqrt{6} \) (e.g. CPU time gain of 6 for equal accuracy).

Approach efficient even if \( N \) is not large!
Different realizations: no systematic bias

\[ A(x, \omega) = \sum_{k \in \mathbb{Z}^2} 1_{Q+k}(x) a_k(\omega) \mathbf{1}_d, \quad a_k \text{ i.i.d.,} \quad a_k \sim \mathcal{U}[\alpha, \beta] \]

In addition, the CPU gain is roughly insensitive to the size of \( Q_N \).
The CPU time gain (ratio of variances) is roughly insensitive to the size of $Q_N$. 

$$A(x, \omega) = \sum_{k \in \mathbb{Z}^2} 1_{Q+k}(x) a_k(\omega) \text{Id}_2, \quad a_k \text{ i.i.d., } \quad a_k \sim \mathcal{U}[\alpha, \beta]$$
Numerical experiments - 2: non equidistributed case

\[ A(x, \omega) = \sum_{k \in \mathbb{Z}^2} 1_{Q+k}(x) a_k(\omega) \text{Id}_2, \quad a_k \text{ i.i.d., } \mathbb{P}(a_k = \alpha) = 1/3, \mathbb{P}(a_k = \beta) = 2/3 \]

In practice: \( a_k = \phi(u_k) = \alpha \mathbf{1}_{0 \leq u_k \leq 1/3} + \beta \mathbf{1}_{1/3 \leq u_k \leq 1} \), with \( u_k \sim \mathcal{U}[0, 1] \).

Take \( b_k = \phi(1 - u_k) \).
Robustness of the numerical results - 1

Good variance reduction on $A_N^*$ for many input fields:

- **Correlated fields:** $A(x, \omega)|_{Q+k}$ is possibly correlated with $A(x, \omega)|_Q$. The underlying uncorrelated structure is known:

$$A(x, \omega) = \sum_{k \in \mathbb{Z}^2} 1_{Q+k}(x) \left[ \frac{1}{2} a_k(\omega) + \frac{1}{16} \sum_{|q|=1} a_{k+q}(\omega) \right] \text{Id}_2$$

with $a_k$ i.i.d. and $a_k \sim \mathcal{U}[\alpha, \beta]$.

$$B(x, \omega) = \sum_{k \in \mathbb{Z}^2} 1_{Q+k}(x) \left[ \frac{1}{2} b_k(\omega) + \frac{1}{16} \sum_{|q|=1} b_{k+q}(\omega) \right] \text{Id}_2$$

with $b_k$ antithetic to $a_k$: here, $b_k(\omega) = \alpha + \beta - a_k(\omega)$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variance reduction on $[A_N^*(\omega)]_{11}$</td>
<td>31.17</td>
<td>39.89</td>
<td>25.43</td>
<td>45.52</td>
</tr>
</tbody>
</table>
Robustness of the numerical results - 2

Good variance reduction on $A^*_N$ for many input fields:

- **Anisotropic fields:**

  \[
  A(x, \omega) = \sum_{k \in \mathbb{Z}^2} 1_{Q+k}(x) A_k(\omega) \text{ with } A_k(\omega) = \begin{pmatrix} \alpha_k(\omega) & \gamma_k(\omega) \\ \gamma_k(\omega) & \beta_k(\omega) \end{pmatrix},
  \]

  where $\{\alpha_k\}_{k \in \mathbb{Z}^2}$, $\{\beta_k\}_{k \in \mathbb{Z}^2}$ and $\{\gamma_k\}_{k \in \mathbb{Z}^2}$ are three independent families of independent random variables.

  \[
  B(x, \omega) = \sum_{k \in \mathbb{Z}^2} 1_{Q+k}(x) B_k(\omega) \text{ with } B_k(\omega) = \begin{pmatrix} \bar{\alpha}_k(\omega) & \bar{\gamma}_k(\omega) \\ \bar{\gamma}_k(\omega) & \bar{\beta}_k(\omega) \end{pmatrix},
  \]

  where $\bar{\alpha}_k$ is antithetic to $\alpha_k$, . . .

<table>
<thead>
<tr>
<th>$N$</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variance reduction on $[A^*<em>N(\omega)]</em>{11}$</td>
<td>49.44</td>
<td>30.54</td>
<td>34.39</td>
<td>28.55</td>
</tr>
</tbody>
</table>
Robustness of the numerical results - 3

Variance reduction not only for $A_N^*$, but also for

- its eigenvalues
- the eigenvalues $\lambda_k$ of the operator ($\sim$ vibration frequencies):

$$-\text{div} \left[ A_N^*(\omega) \nabla u_k(\omega) \right] = \lambda_k(\omega) u_k(\omega)$$

- the solution of the (approximated) homogenized problem:
  $A(x, \omega) \rightarrow$ homogenized matrix $A_N^*(\omega)$, and then solve

$$-\text{div} \left[ A_N^*(\omega) \nabla u_N^*(\omega) \right] = f$$

$A(x, \omega) \rightarrow$ antithetic field $B(x, \omega), \ldots, -\text{div} \left[ B_N^*(\omega) \nabla v_N^*(\omega) \right] = f$

$$\inf_{x \in D} \frac{\text{Var} u_N^*(x)}{\text{Var} \left[ \frac{1}{2}(u_N^*(x) + v_N^*(x)) \right]} \approx 9 = \text{CPU time gain at equal accuracy}$$