Computing positive invariant tubes with interval analysis
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1 V-stability
Vaimos (IFREMER and ENSTA)
\[ \dot{x} = f(x) \]

\( X_1 \): outside the corridor.
\( X_2 \): inside the corridor.
**Definition.** Consider a differentiable function $V(x) : \mathbb{R}^n \to \mathbb{R}$. The system is $V$-stable if

$$
(V(x) \geq 0 \Rightarrow \dot{V}(x) < 0).
$$

Since

$$
\dot{V}(x) = \frac{\partial f}{\partial x}(x) \cdot f(x)
$$

Checking the $V$-stability can be done using interval analysis.
Non-holonomic system
A tube is a function which associates to any $t \in \mathbb{R}$ a subset of $\mathbb{R}^n$. 
In the machine a tube can be represented by two stair functions
Example of tubes

\[ f(t) = [1, 2] \cdot t + \sin ([1, 3] \cdot t) \]

\[ g(t) = [a_0] + [a_1] t + [a_2] t^2 + [a_3] t^3 \]

\[ \int_0^t [g](\tau) \, d\tau = [a_0] t + [a_1] \frac{t^2}{2} + [a_2] \frac{t^3}{3} + [a_3] \frac{t^4}{4}. \]
3 Positive invariant tubes
Consider the time dependant system

$$S : \dot{x} = f(x, t)$$

and a tube

$$G(t) \subset \mathbb{R}^n, t \in \mathbb{R}.$$
The tube $\mathcal{G}(t)$ is said to be a \textit{positive invariant} if
\[ x(t) \in \mathcal{G}(t), \tau > 0 \implies x(t + \tau) \in \mathcal{G}(t + \tau). \]
Theorem. Consider the tube

\[ G(t) = \{ x, g(x, t) \leq 0 \} \]

where \( g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^m \). If the crossed out condition

\[
\begin{align*}
\frac{\partial g_i}{\partial x}(x, t) \cdot f(x, t) + \frac{\partial g_i}{\partial t}(x, t) &\geq 0 \\
\dot{g}_i(x, t) &\leq 0 \\
g_i(x, t) &\leq 0 \\
g(x, t) &\leq 0
\end{align*}
\]

is inconsistent for all \((x, t, i)\), then \( G(t) \) is a capture tube for \( S : \dot{x} = f(x, t) \).
A software Bubbibex (using Ibex) made by students from ENSTA Bretagne for MBDA uses interval analysis to prove the inconsistency.
4 Lattice and capture tubes
Consider $S: \dot{x} = f(x, t)$.

If $\mathcal{T}$ is the set of tubes and $\mathcal{T}_c$ is the set of all capture tubes of $S$ then $(\mathcal{T}_c, \subseteq)$ is a sublattice of $(\mathcal{T}, \subseteq)$. 
We have indeed

\[ \begin{align*}
\{ \mathcal{G}_1(t) & \in T_c \} \Rightarrow \{ \mathcal{G}_1(t) \cap \mathcal{G}_2(t) \in T_c \} \\
\{ \mathcal{G}_2(t) & \in T_c \} \Rightarrow \{ \mathcal{G}_1(t) \cup \mathcal{G}_2(t) \in T_c \}
\end{align*} \]
5 Computing capture tubes
If $G(t) \in T$, define
\[
\text{capt}(G(t)) = \bigcap \{ \overline{G}(t) \in T_c \mid G(t) \subset \overline{G}(t) \}.
\]
This set is the smallest capture tube enclosing $G(t)$.
Problem. Given $G(t) \in \mathbb{T}$, compute an interval $[G^-(t), G^+(t)] \in \mathbb{IT}$ such that

$$\text{capt}(G(t)) \in [G^-(t), G^+(t)].$$
Flow. The flow associated with $S_f : \dot{x} = f(x, t)$ is a function $\phi_{t_0, t_1} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\dot{x} = f(x, t) \Rightarrow \phi_{t_0, t_1}(x(t_0)) = x(t_1).$$
Proposition. For the system $S_f : \dot{x} = f(x, t)$ and the tube $G(t)$, we have

$$\text{capt}(G(t)) = G(t) \cup \Delta G(t),$$

with

$$\Delta G(t) = \{(x, t) \mid \exists (x_0, t_0) \text{ satisfying the cross out condition } t \geq t_0, \phi_{t_0, t}(x_0) \notin G(t)\}$$

Recall the cross out condition:

$$\begin{cases} 
\frac{\partial g_i}{\partial x}(x, t) \cdot f(x, t) + \frac{\partial g_i}{\partial t}(x, t) \geq 0 \\
g_i(x, t) = 0 \\
g(x, t) \leq 0 
\end{cases}$$
6 Pendulum
Pendulum:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\sin x_1 - 0.15 \cdot x_2
\end{align*}
\]
The energy

\[ E(x) = \frac{1}{2} \dot{x}_1^2 - \cos x_1 + 1 = \frac{1}{2} x_2^2 - \cos x_1 + 1 \]

allows us to find candidate for the positive invariant tube:

\[ g(x, t) = E(x) - 1 = \frac{1}{2} x_2^2 - \cos x_1. \]
The cross-out conditions

\[
\begin{cases}
\text{(i)} & (\sin x_1 \ x_2) \begin{pmatrix} x_2 \\ -\sin x_1 - 0.15 \cdot x_2 \end{pmatrix} = -0.15 \cdot x_2^2 \geq 0, \\
\text{(ii)} & \frac{1}{2} x_2^2 - \cos x_1 = 0.
\end{cases}
\]

has two solutions: \( x = (\pm \frac{\pi}{2}, 0) \).
Without considering the energy, we consider, as an candidate tube:

\[ g(x, t) = x_1^2 + x_2^2 - 1. \]
(a) boxes which enclose the points satisfying the cross-out condition; (b) guaranteed integration $\Delta G$ of these boxes; (c) inner approximation $C_{\text{Capt}}(G)$; (d) outer approximation $C^+$ of $\text{Capt}(G)$
7 Dubin’s car
Consider the Dubin’s car

\[
\begin{align*}
\dot{x} &= \cos \theta \\
\dot{y} &= \sin \theta \\
\dot{\theta} &= u
\end{align*}
\]

where \( u \in [-2, 2] \).
To move toward the target \((x_d, y_d)\), we take the controller:

\[
\begin{aligned}
\mathbf{n} &= \frac{1}{\sqrt{(x_d-x)^2+(y_d-y)^2}} \begin{pmatrix} x_d-x \\ y_d-y \end{pmatrix} + \frac{2}{\sqrt{x_d^2+y_d^2}} \begin{pmatrix} \dot{x}_d \\ \dot{y}_d \end{pmatrix} \\
\theta_d &= \text{atan2} (\mathbf{n}) \\
u &= -2 \cdot \sin (\theta - \theta_d) .
\end{aligned}
\]
\[
\begin{align*}
  x_d(t) &= \rho_x \cos t \\
  y_d(t) &= \rho_y \sin t.
\end{align*}
\]

For the derivative, we get
\[
\begin{align*}
  \dot{x}_d(t) &= -\rho_x \sin t \\
  \dot{y}_d(t) &= \rho_y \cos t.
\end{align*}
\]
**Target tube.** We want the robot to stay inside the set

\[ \mathbb{G}(t) = \{ x \mid g(x, t) \leq 0 \}, \]

with

\[
\begin{align*}
    g_1(x, t) &= (x - x_d)^2 + (y - y_d)^2 - \rho^2 \\
    g_2(x, t) &= \left( \cos \theta - \frac{n_x}{\|n\|} \right)^2 + \left( \sin \theta - \frac{n_y}{\|n\|} \right)^2 - \alpha^2.
\end{align*}
\]
Resolution. We used the solver Bubbibex.
The tube is proved to be unsafe.
Bubbibex is able to compute the margin (i.e., width$\left(\left[ G^- (t), G^+ (t) \right]\right)$).
Question?