Stability Verification of Nearly Periodic Linear Impulsive Systems using Reachability Analysis

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Motivation

- Embedded control technology is becoming pervasive.
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- Challenges: time varying-delays, time-varying sampling period, communication constraints...
Sampled-data control systems

\[ u(t) \rightarrow u(t_k) \rightarrow \text{CONTROLLER} \rightarrow z(t_k) \rightarrow z(t) \]

\[ t_k \rightarrow T \rightarrow \delta \rightarrow \tau_{k+1} \rightarrow t_{k+2} \]

\[ T \quad \delta \]

Motivation (cont’d)
Sampled-data control systems.

\[ \dot{z}(t) = Az(t) + Bu(t_k), \quad \forall t \in (t_k, t_{k+1}), \quad k \in \mathbb{N} \]

\[ u(t_k^+) = Kz(t_k) \]

\[ t_{k+1} - t_k \in [T, T + \delta], \]
Sampled-data control systems.

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\[t_{k+1} - t_k \in [T, T + \delta],\]

Nearly periodic linear impulsive systems (NPILS).

\[
\dot{x}(t) = A_c x(t), \quad \forall t \neq t_k.
\]

\[
x(t_{k+1}) = A_d x(t_k), \quad \forall t = t_k.
\]

\[t_{k+1} - t_k = T + \tau_k; \quad t_0 = 0; \quad \tau_k \in [0, \delta].\]
Motivation (cont’d)

- **Sampled-data control systems.**
  
  \[ \dot{z}(t) = Az(t) + Bu(t_k), \quad \forall t \in (t_k, t_{k+1}), \quad k \in \mathbb{N} \]
  \[ u(t_k^+) = Kz(t_k) \]
  
  \[ t_{k+1} - t_k \in [T, T + \delta], \]

- **Nearly periodic linear impulsive systems (NPILS).**
  
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  \[ x(t_k^+) = A_d x(t_k), \quad \forall t = t_k. \]

\[
A_c = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad A_d = \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix}, \quad x(t) = \begin{pmatrix} z(t) \\ u(t_k) \end{pmatrix}
\]
Theoretical necessary and sufficient conditions for the stability of nearly periodic systems.
Contribution

1. Theoretical necessary and sufficient conditions for the stability of nearly periodic systems.

2. Computationally oriented approach, search for a robust invariant set, using reachability analysis.
Outline of the Talk

1. Problem Formulation
2. Stability Conditions for NPILS
3. Stability Verification using Reachability Analysis
4. Examples
5. Extension to time contract design
6. Ongoing and future work
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1. **Problem Formulation**

2. Stability Conditions for NPILS

3. Stability Verification using Reachability Analysis

4. Examples

5. Extension to time contract design

6. Ongoing and future work
Problem Formulation

- Nearly periodic linear impulsive systems (NPILS).

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**Definition**

*The system is said to be GUES if there exist \( \lambda > 0 \) and \( C > 0 \) such that, for all \( x(0) \in \mathbb{R}^n \) and \( \tau_k \in [0, \delta] \) where \( k \in \mathbb{N} \),

\[ \|x(t)\| \leq Ce^{-\lambda t} \|x(0)\|. \]
Problem Formulation

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- Problem 1: The algorithmic verification of GUES.
- Problem 2: [Contract synthesis] Synthesize a set \( \Pi \) such that for all \( (T, \delta) \in \Pi \), NPILS is GUES.
Related work for Problem 1

Parametric LMIs sufficient conditions:

(a) Discrete-time and convex embedding approach [Hetel et al. (2011, 2013)].
(b) A time delay method [Liu et al. (2010); Seuret and Peet (2013)].
(c) A hybrid system formulation [Dai et al. (2010)].
(d) An Input/Output stability approach [Omran et al. (2014); Fujioka (2009)]

→ Conservatism due to only sufficient conditions.
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   (d) An Input/Output stability approach [Omran et al. (2014); Fujioka (2009)]
      →Conservatism due to only sufficient conditions.

2. Polytopic robust invariant set: [Fiacchini and Morarescu 2014].
   →Conservatism due to possibly unbounded computations.
Reachable set under continuous dynamics

\[ \mathcal{R}^A_{[t,t']} (S) = \bigcup_{\tau \in [t,t']} e^{\tau A_c} S \]

Remark

\[ \mathcal{R}^A_{[t,t']} (S) \text{ is the flow of all states belonging to the set } S \text{ under the continuous dynamics corresponding to } A_c \text{ in the interval } [t, t'] \]
Reachable set of NPILS

\[ \Phi(S) = R_{[0,\delta]}^{A_c} (e^{T A_c A_d S}) \]

Remark

\( \Phi(S) \) is the exact expression of the set of states \( x(t_{k+1}) \) where \( x(t_k) \in S \subseteq \mathbb{R}^n \) for all possible time uncertainties.

Iterates of \( \Phi \): \( \Phi^0(S) = S \) for all \( S \subseteq \mathbb{R}^n \), and \( \Phi^{k+1} = \Phi \circ \Phi^k \) for all \( k \in \mathbb{N} \).
Theorem

For all proper compact sets $S \subset \mathbb{R}^n$ the following are equivalent:

1. **The system is** GUES.
2. **There exists a triplet** $(k, j, \rho) \in \mathbb{N}_{\geq 1} \times \mathbb{N}[0,k-1] \times \mathbb{R}[0,1)$ **such that**
   $$\Phi^k(S) \subseteq \rho \Phi^j(S).$$
3. **There exists a pair** $(k, \rho) \in \mathbb{N}_{\geq 1} \times \mathbb{R}[0,1)$ **such that**
   $$\Phi^k(S) \subseteq \rho \bigcup_{j=0}^{k-1} \Phi^j(S).$$
**Proof**

- *It is obvious that* $(2) \implies (3)$. *Hence, it is sufficient to prove that* $(1) \implies (2)$ *and* $(3) \implies (1)$.

- $(1) \implies (2)$: *We will prove that* $\Phi^k(S) \subseteq \rho S$.

\[
\|x(t_k)\| \leq Ce^{-\lambda t_k} \|x(0)\| \leq Ce^{-\lambda kT} \|x(0)\|
\]

*which can be rewritten as*

\[
\Phi^k(S) \subseteq Ce^{-\lambda kT} \|x(0)\|B.
\]

- *But* $cB \subseteq S \subseteq \bar{c}B$. *Then,*

\[
\Phi^k(S) \subseteq Ce^{-\lambda kT}cB \subseteq \frac{Ce^{-\lambda kT}c}{c} S.
\]

M. AL-Khatib (LJK-UJF)
Proof

(3) $\implies$ (1): Let $\gamma = \frac{1}{\rho^k}$; then for $0 \leq k - 1$, $\rho \leq \gamma^{k-j}$ and

$$\Phi^k(S) \subseteq \rho \bigcup_{j=0}^{k-1} \Phi^j(S) \subseteq \bigcup_{j=0}^{k-1} \gamma^{k-j} \Phi^j(S).$$

Let $S' = \bigcup_{j=0}^{k-1} \gamma^{-j} \Phi^j(S)$, then using properties of the map $\Phi$:

$$\Phi(S') = \bigcup_{j=0}^{k-1} \gamma^{-j} \Phi^{j+1}(S) = \left( \bigcup_{j=0}^{k-2} \gamma^{-j} \Phi^{j+1}(S) \right) \cup \gamma^{-k+1} \Phi^k(S).$$

Making a change of index in the union and using (1) yield

$$\Phi(S') \subseteq \gamma S'.$$
Stability conditions for NPILS

Necessary and Sufficient Theoretical Conditions

Proof

1. \( c'B \subseteq S' \subseteq \overline{c'}B \). then \( x(0) \in \|x(0)\|B \subseteq \|x(0)\|_{c'} S' \) and

\[
x(t_i) \in \Phi^i \left( \|x(0)\|_{c'} S' \right) \subseteq \|x(0)\|_{c'} \gamma^i S' \subseteq \|x(0)\|_{c'} \gamma^i \overline{c'} B.
\]

\[\Rightarrow \|x(t_i)\| \leq \frac{\gamma^i \overline{c'}}{c'} \|x(0)\| \ \forall i \in \mathbb{N}\]

2. \( t \in (t_i, t_{i+1}] \), then \( t - t_i \leq T + \delta \) and \( i \geq t/(T + \delta) \) and

\[
\|x(t)\| \leq e^{\|A_c\|(T+\delta)} \|A_d\| \frac{\gamma^i \overline{c'}}{c'} \|x(0)\|
\]

\[\leq e^{\|A_c\|(T+\delta)} \|A_d\| \overline{c'} \frac{\ln(\gamma)}{T+\delta} t \|x(0)\|.
\]
Φ is hard to compute, so we use \( \overline{\Phi} \) satisfying the following assumption:

**Assumption**

For all compact sets \( S \), \( \Phi(S) \subseteq \overline{\Phi}(S) \).

Iterates of \( \overline{\Phi} \): \( \overline{\Phi}^0(S) = S \) for all \( S \subseteq \mathbb{R}^n \), and \( \overline{\Phi}^{k+1} = \overline{\Phi} \circ \overline{\Phi}^k \) for all \( k \in \mathbb{N} \).
Corollary

If there exists a proper compact set $S \subset \mathbb{R}^n$ and $(k, \rho) \in \mathbb{N}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that $\Phi^k(S) \subseteq \rho \bigcup_{j=0}^{k-1} \Phi^j(S)$, then the system is GUES.
Proof

- let \( S' = \bigcup_{j=0}^{k-1} \gamma^{-j} \Phi^j(S) \) where \( \gamma = \frac{1}{\rho_k} \).

- Then

\[
\Phi(S') = \Phi \left( \bigcup_{j=0}^{k-1} \gamma^{-j} \Phi^j(S) \right) = \bigcup_{j=0}^{k-1} \gamma^{-j} \Phi(\Phi^j(S))
\]

\[
\subseteq \bigcup_{j=0}^{k-1} \gamma^{-j} \Phi(\Phi^j(S)) = \bigcup_{j=0}^{k-1} \gamma^{-j} \Phi^{j+1}(S).
\]

- \( \Phi(S') \subseteq \gamma S' \).
Reachability Analysis

Recall: $\Phi(S) = R_{[0,\delta]}^{Ac}(e^{T_{Ac}A_dS}) = R_{[0,\delta]}^{Ac}(D)$
\[ \delta = Nh \rightarrow \mathcal{R}_{[0,\delta]}^{A_c}(\mathcal{D}) = \bigcup_{i=0}^{N-1} \mathcal{R}_{[ih,(i+1)h]}^{A_c}(\mathcal{D}) \]
Reachability Analysis

\[ \bigcup_{i=0}^{N-1} \mathcal{R}^{Ac}_{ih,(i+1)h}(D) \subseteq \bigcup_{i=0}^{N-1} \overline{\mathcal{R}}^{Ac}_{ih,(i+1)h}(D) = \bigcup_{i=0}^{N-1} \Omega_i \]
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\[ \Omega_0 := CH(D, e^{hA_c} D) \oplus \epsilon_h(D), \quad \text{[Le Guernic 2009]} \]

\[ \Omega_i := e^{hA_c} \Omega_{i-1} \]
Reachability Analysis

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\bigcup_{i=0}^{N-1} R_{[ih,(i+1)h]}^A(D) \subseteq \bigcup_{i=0}^{N-1} R_{[ih,(i+1)h]}^A(D) = \bigcup_{i=0}^{N-1} \Omega_i
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\[
\Phi(S) = \Gamma_H \left( R_{[0,\delta]}^A(e^{TA_c} A_d S) \right)
\]

where \( \Gamma_H(S) \) is the smallest polytope whose facets directions are given by \( H \) (the matrix defining the direction of \( S \)'s facets) and containing \( S \).
R-1: Compute an initial set: Choose this set to be a common contracting symmetric polytopic set $\mathcal{P}$ for $L' = L + 1$ discrete-time systems given by:

$$x_{k+1} = e^{T_i A_c A_d} x_k$$

$$T_i = T + i\delta/L; \ i \in \mathbb{N}_{[0,L]}, \ L \in \mathbb{N}_{\geq 1}.$$
**R-1:** Compute an initial set: Choose this set to be a common contracting symmetric polytopic set $\mathcal{P}$ for $L' = L + 1$ discrete-time systems given by:

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$$\Delta_0 := \Delta$$

$$\Delta_{j+1} := \left( \bigcap_{i=1}^{L'} \text{Predecessor}(\Delta_j, \lambda_j e^{T_i A_c} A_d) \right) \bigcap \Delta_0.$$ 

*stop if* $\Delta_{j+1} = \Delta_j$

→ Alternative method using forward reachability [Athanasopoulos and Lazar (2014)].
Stability verification using reachability analysis

Algorithm 1

- **R-1:** Compute an initial set $\mathcal{P}$.
- **R-2:** let $\mathcal{P}_0 \coloneqq \mathcal{P}$ and $H$ is given by the facets of $\mathcal{P}$.

1: **for** $k = 1$ to $k_{max}$ **do**
2: \[ \mathcal{P}_k \coloneqq \overline{\Phi}(\mathcal{P}_{k-1}) \coloneqq \Gamma_H \left( \mathcal{R}_{[0,\delta]}^{A_c} (e^{T A_c A_d \mathcal{P}_{k-1}}) \right) \] \hfill \text{set propagation}
3: \textbf{if} $\mathcal{P}_k \subseteq \text{int} \left( \bigcup_{j=0}^{i-1} (\mathcal{P}_j) \right)$, \textbf{then} \hfill \text{stability check}
4: \hspace{1cm} return true;
5: \hspace{1cm} \textbf{end if}
6: \textbf{end for}
7: return unknown;
Sketch of the Algorithm

\[ P_0 \]
Sketch of the Algorithm

\[ \mathcal{R}_{[0,\delta]}^{A_c}(e^{T_{A_c}A_dP_{k-1}}) \]
Sketch of the Algorithm

\[ P_1 := \Phi(P_0) := \Gamma_H \left( \overline{R}_{[0, \delta]}^{A_c} (e^{T_A c} A_d P_0) \right) \not\subset P_0 \]
Sketch of the Algorithm

\( \mathcal{P}_1 \cup \mathcal{P}_0 \)
\[ P_2 := \Phi(P_1) \subset (P_0 \cup P_1) \]
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1. Problem Formulation
2. Stability Conditions for NPILS
3. Stability Verification using Reachability Analysis
4. Examples
5. Extension to time contract design
6. Ongoing and future work
Consider a $NPILS$ with

$$A_c = \begin{pmatrix} 0 & -3 & 1 \\ 1.4 & -2.6 & 0.6 \\ 8.4 & -18.6 & 4.6 \end{pmatrix}, \quad A_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
Examples

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- \(A_d e^{TA_c}\) is Schur for \(T \in [0, 0.58]\).
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\( A_d e^{TA_c} \) is Schur for \( T \in [0, 0.58] \).

\( \prod_{i \in \mathbb{N}[1,5]} A_d e^{T_iA_c} \) has eigen values outside the unit circle for \( T_1 = 0.515, T_i = 0.1, \forall i \in \mathbb{N}[2,5] \).
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- \(\bar{\delta} = 0.415\) as an upper bound for \(\delta\), if \(T = 0.1\).
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- \(T = 0.1\), LMI approach \(\rightarrow \delta = 0.2\) (Hetel et.al.)
Examples

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\]

- \(A_d e^{T A_c}\) is Schur for \(T \in [0, 0.58]\).
- \(\prod_{i \in \mathbb{N}_{[1,5]}} A_d e^{T_i A_c}\) has eigen values outside the unit circle for \(T_1 = 0.515, T_i = 0.1, \forall i \in \mathbb{N}_{[2,5]}\);
- \(\bar{\delta} = 0.415\) as an upper bound for \(\delta\), if \(T = 0.1\).
- \(T = 0.1\), LMI approach \(\rightarrow \delta = 0.2\) (Hetel et.al.)
- Set theory approach \(\rightarrow \delta = 0.375\) (Fiacchini and Moraescu)
Consider a NPILS with

\[
A_c = \begin{pmatrix}
0 & -3 & 1 \\
1.4 & -2.6 & 0.6 \\
8.4 & -18.6 & 4.6
\end{pmatrix}
\quad \text{and} \quad
A_d = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\(A_d e^{TA_c}\) is Schur for \(T \in [0, 0.58]\).

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\prod_{i \in \mathbb{N}_{[1,5]}} A_d e^{T_i A_c}\] has eigen values outside the unit circle for

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\(T = 0.1\), LMI approach \(\rightarrow \delta = 0.2\) (Hetel et.al.)

Set theory approach \(\rightarrow \delta = 0.375\) (Fiacchini and Morarescu)

\textbf{Algorithm 1} (L' = 2, N = 100) \(\rightarrow \delta = 0.414\).
Examples (cont’d)

- sampled-data systems:

\[
\dot{z}(t) = Az(t) + Bu(t_k), \quad \forall t \in (t_k, t_{k+1}), \quad k \in \mathbb{N}
\]

\[
u(t_k) = Kz(t_k^-)
\]

\[
t_{k+1} - t_k \in [T, T + \delta],
\]
Examples (cont’d)

- **sampled-data systems:**

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\dot{z}(t) = Az(t) + Bu(t_k), \ \forall t \in (t_k, t_{k+1}), \ k \in \mathbb{N}
\]

\[
u(t_k) = Kz(t_k^-)
\]

\[
t_{k+1} - t_k \in [T, T + \delta],
\]

- **NPILS →**

\[
A_c = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad A_d = \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix}, \quad x(t) = \begin{pmatrix} z(t) \\ u(t_k) \end{pmatrix}
\]

\[
t_{k+1} - t_k \in [T, T + \delta].
\]
Examples (cont’d)

\[
A = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, \quad K = \begin{pmatrix} -3.75 & -11.5 \end{pmatrix} \tag{2}
\]

\[
A = \begin{pmatrix} 0 & 1 \\ -2 & 0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{3}
\]
### Examples (cont’d)

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<tr>
<th></th>
<th>System (2)</th>
<th>System (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\delta_{\text{max}}$</td>
<td>$T$</td>
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<tr>
<td>Briat (2013)</td>
<td>1.7279</td>
<td>0.4</td>
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<td>Fridman et.al.(2004)</td>
<td>0.869</td>
<td>-</td>
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<td>Naghshtabrizi et.al. (2008)</td>
<td>1.113</td>
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<td>Fridman (2010)</td>
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A time contract defines the time instants at which certain functions are computed in a model.
Extension to time contract design

- A time contract defines the time instants at which certain functions are computed in a model.
- In NPILS: $g_k = t_{k+1} - t_k = T + \tau_k; \ g_0 = 0; \ \tau_k \in [0, \delta]$. 
Extension to time contract design

- A time contract defines the time instants at which certain functions are computed in a model.
- In *NPILS*: \( g_k = t_{k+1} - t_k = T + \tau_k; \ g_0 = 0; \ \tau_k \in [0, \delta] \).
- Solve *Problem 2*: [Contract synthesis] Synthesize a set \( \Pi \) such that for all \( (T, \delta) \in \Pi \), *NPILS* is *GUES*.
A time contract defines the time instants at which certain functions are computed in a model.

In **NPILS**: \( g_k = t_{k+1} - t_k = T + \tau_k \); \( g_0 = 0 \); \( \tau_k \in [0, \delta] \).

Solve **Problem 2**: [Contract synthesis] Synthesize a set \( \Pi \) such that for all \((T, \delta) \in \Pi\), **NPILS** is **GUES**.

\( g_k \in [T_m, T_M] \), where \( T_m = T \), and \( T_M = T + \delta \).

Proceed and find a set of efficient solutions for \([T_m, T_M]\) such that the system is **GUES**.
Extension to time contract design

- A time contract defines the time instants at which certain functions are computed in a model.

- In \textit{NPILS}: \( g_k = t_{k+1} - t_k = T + \tau_k; \ g_0 = 0; \ \tau_k \in [0, \delta] \).

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- \( g_k \in [T_m, T_M] \), where \( T_m = T \), and \( T_M = T + \delta \).

- Proceed and find a set of efficient solutions for \( [T_m, T_M] \) such that the system is \textit{GUES}.

\textbf{Proposition}

\begin{quote}
If the \textit{NPILS} is \textit{GUES} for parameters \( T_m, T_M \) with \( T_m \leq T_M \) then it is \textit{GUES} for all parameters \( T'_m, T'_M \) with \( T_m \leq T'_m \leq T'_M \leq T_M \).
\end{quote}
Reconsider the sampled data system with

\[ A = \begin{pmatrix} 0 & 1 \\ -2 & 0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \end{pmatrix}. \]

Valid contract parameters in the (left): \((T_m, T_M)\) domain; (right): \((T, \delta)\) domain (A solution to Problem 2).
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The approach has been extended to the stochastic case.

\[ t_{k+1} - t_k \sim \mathcal{U}([T, T + \delta]). \]

Consider classes of impulsive systems with different time contract:

\[
\begin{align*}
\dot{x}(t) &= A_c x(t), \quad \forall t \neq t^s_k, t \neq t^a_k, k \in \mathbb{N} \\
x(t^s_{k+1}) &= A_s x(t^s_k), \\
x(t^a_{k+1}) &= A_a x(t^a_k), \\
x(0) &= x_0 \in \mathcal{X}_0.
\end{align*}
\]
Consider classes of impulsive systems with different time contract:

\[
\dot{x}(t) = A_c x(t), \quad \forall t \neq t_k^s, \; t \neq t_k^a, \; k \in \mathbb{N}
\]

\[
x(t_k^s) = A_s x(t_k^s),
\]

\[
x(t_k^a) = A_a x(t_k^a),
\]

\[
x(0) = x_0 \in \mathcal{X}_0.
\]

The time contract has the form:

\[
h_k = t_k^s - t_{k-1}^s \in [\max (\tau_{k-1}, h), \bar{h}]; \quad t_0^s = 0
\]

\[
\tau_{k-1} = t_{k-1}^a - t_{k-1}^s \in [\underline{\tau}, \overline{\tau}].
\]
Ongoing and future work

- Consider classes of impulsive systems with different time contract:

  \[
  \dot{x}(t) = A_c x(t), \forall t \neq t^s_k, t \neq t^a_k, k \in \mathbb{N}
  \]

  \[
  x(t^s_k) = A_s x(t^s_k),
  \]

  \[
  x(t^a_k) = A_a x(t^a_k),
  \]

  \[
  x(0) = x_0 \in \mathcal{X}_0.
  \]

  The time contract has the form:

  \[
  h_k = t^s_k - t^s_{k-1} \in [\max(\tau_{k-1}, h), \overline{h}]; \quad t^s_0 = 0
  \]

  \[
  \tau_{k-1} = t^a_{k-1} - t^s_{k-1} \in [\underline{\tau}, \overline{\tau}].
  \]

  **Stability verification problem:** Verify GUES of the system.
Ongoing and future work

Consider classes of impulsive systems with different time contract:

\[
\dot{x}(t) = A_c x(t), \forall t \neq t_k^s, t \neq t_k^a, k \in \mathbb{N}
\]

\[
x(t_{k}^{s+}) = A_s x(t_k^s),
\]

\[
x(t_{k}^{a+}) = A_a x(t_k^a),
\]

\[
x(0) = x_0 \in \mathcal{X}_0.
\]

The time contract has the form:

\[
h_k = t_k^s - t_{k-1}^s \in \left[\max (\tau_{k-1}, h), h\right]; \quad t_0^s = 0
\]

\[
\tau_{k-1} = t_{k-1}^a - t_{k-1}^s \in [\tau, \overline{\tau}].
\]

**Stability verification problem**: Verify GUES of the system.

**Contract design problem**: Synthesize a set \(\Pi\) such that for all \((\tau, \overline{\tau}, h, h) \in \Pi\), the system is **GUES**.
Ongoing and future work

- **Control design problem**: Design the control input given the timing contract.
Ongoing and future work

- **Control design problem**: Design the control input given the timing contract.
- **Codesign problem**: Design the control input and the timing contract for a given set of specification.
Ongoing and future work

The approach has been extended to the stochastic case.

\[ t_{k+1} - t_k \sim U(T, T + \delta) \].

Consider classes of impulsive systems with different time contracts:

\[ \dot{x}(t) = A_c x(t), \quad \forall t \neq t_{sk}, \ t \neq t_{ak}, \ k \in \mathbb{N} \]

\[ x(t_{sk} + k) = A_s x(t_{sk}), \]

\[ x(t_{ak} + k) = A_a x(t_{ak}) \],

\[ x(0) = x_0 \in X_0 \].

Control design problem: Design the control input given the timing contract.

Codesign problem: Design the control input and the timing contract for a given set of specifications.

Thank you!!!! Questions???


