Robust Nonlinear Model Predictive Control for Regulation of Microalgae Culture in a Continuous Photobioreactor

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GT CPNL, Université Pierre et Marie CURIE, Paris, June 11, 2015
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**Context**

biological & chemical characteristics → growing demand of organic products

**Applications**
- food;
- pharmacology, chemistry;
- sustainable environment.

↓

Industrial success

![](image)
Industrial regulation

- partial pressure of dissolved oxygen and dissolved carbon dioxide;
- pH and temperature.

Nonlinear control strategies

- optimization-based approaches [Abdollahi and Dubljevic, 2012];
- adaptive approaches [Mailleret et al., 2004];
- sliding mode control [Selisteanu et al., 2007];
- Generic Model Control (nonlinear PI) [Jenzsch et al., 2006];
- exact linearization approach [Ifrim et al., 2013, Tebbani et al., 2015];
- backstepping approach [Dochain and Perrier, 2004].

Choice (advanced & optimal)

- Nonlinear Model Predictive Control [Camacho and Bordons, 2004, Kerrigan and Maciejowski, 2004]
Assumptions

- Photobioreactor in continuous mode \( (F_{in} = F_{out}) \);
- without any additional biomass in the feed;
- neglecting the effect of gas exchanges.

Mass balances [Masci et al., 2010]

\[
\begin{align*}
\dot{X}(t) &= \mu(Q,I) X(t) - D(t)X(t) \\
\dot{Q}(t) &= \rho(S) - \mu(Q,I) Q(t) \\
\dot{S}(t) &= (S_{in} - S(t))D(t) - \rho(S)X(t)
\end{align*}
\]

with:

\[
\begin{align*}
\rho(S) &= \rho_m \frac{S}{S + K_s} \quad \text{(Monod)} \\
\mu(Q,I) &= \bar{\mu}(1 - \frac{K_Q}{Q}) \mu_I \quad \text{(Droop)} \\
\mu_I &= \frac{I}{I + K_{sl} + I^2 K_{il}} \quad \text{(Haldane)}
\end{align*}
\]

Model parameters [Goffaux and Vande Wouwer, 2008], [Munoz-Tamayo et al., 2014].
**Dynamical model**

\[
\begin{cases}
\dot{x}(t) = f(x(t), u(t), \theta) \\
y(t) = X(t) \\
x(t_0 = 0) = x_0
\end{cases}
\]

with:

\[
x = [X \quad Q \quad S]^T, \quad u = D, \\
\begin{bmatrix}
\mu(Q, I)X - DX \\
\rho(S) - \mu(Q, I)Q \\
(S_{in} - S)D - \rho(S)X
\end{bmatrix}, \\
\theta = \begin{bmatrix}
\rho_m & K_s & \bar{\mu} & K_Q & K_{sl} & K_{il}
\end{bmatrix}^T
\]

**Steady state characterization**

given \(X \implies\) characterize the corresponding \(Q, S \& D\)

- At the equilibrium: \(f(x(t), u(t), \theta) = 0\)

\[
\begin{cases}
D = \pi_1(Q, I) \\
S = \pi_2(Q, I) \\
X = \pi_3(Q, S, I)
\end{cases}
\]

- After developments (taking \(Q\) as an unknown variable):

\[
Q_{1,2} = \frac{\bar{\mu}\mu(I)(S_{in} + K_s) + (\rho_m + \bar{\mu}\mu Q)X \pm \sqrt{\Delta}}{2\bar{\mu}\mu X}
\]

with

\[
\Delta = \left((\rho_m + \bar{\mu}\mu Q)X - \bar{\mu}\mu(S_{in} + K_s)\right)^2 + 4\bar{\mu}\mu K_s \rho_m X > 0
\]

- \(Q_1 > \frac{\rho_m}{\bar{\mu}\mu I} + K_Q \implies Q_1 \not\in\) admissible solution.

**Constraints [Bernard and Gouzé, 1995]**

\[
\begin{cases}
X \geq 0, \\
K_Q \leq Q \leq \frac{\rho_m}{\bar{\mu}\mu I} + K_Q, \\
0 \leq S \leq S_{in}, \\
D \geq 0
\end{cases}
\]
Find a stabilizing control strategy that
- minimizes the objective functional \((X \rightarrow X' \& D \rightarrow D')\);
- satisfies constraints;
- is robust towards uncertainties.

**Algorithm**

1. initialisation.
2. get the new state \(x\);
3. solving the optimization problem over a finite moving horizon \(\downarrow\) (trust-region-reflective algorithm)
4. optimal input \(u\);
5. apply only the first value of the optimal sequence \(u\);
6. \(t \leftarrow t + 1\). Go to 2.

**Figure:** Overall NMPC strategy.
Problem formulation

Discrete prediction model (Runge-Kutta scheme)

\[
\begin{align*}
    x_{k+1} &= x_k + \int_{t_k}^{t_{k+1}} f(x(\tau), u_k, \theta) d\tau \\
    y_k &= Hx_k
\end{align*}
\] (9)

where

- \( x_k \): the discrete state vector at time \( k \);
- \( y_k \): the sampled measurement at time \( k \);
- \( u(k) \): control action (parametrized using a piecewise-constant approximation);
  \( u(\tau) = u(k), \ \tau \in [kT_s, (k+1)T_s] \)
- \( H \): measurement matrix.

Discrete state trajectory

\[
x_{k+1} = g(t_0, t_{k+1}, x_0, u_{t_0}^{t_k}, \theta)
\] (10)

with \( u_{t_0}^{t_k} \) the control sequence from the initial time instant \( t_0 \) to the time instant \( t_k \).
Mathematical formulation of NMPC

\[ \tilde{u}_{k}^{k+N_p-1} = \arg \min_{u_{k}^{k+N_p-1}} \Pi(u_{k}^{k+N_p-1}, \theta) \]  

(11)

where

\[ \Pi(u_{k}^{k+N_p-1}, \theta) = \|\hat{y}_{k+1}^{k+N_p} - \bar{y}_{k+1}^{k+N_p}\|^2_P + \|u_{k}^{k+N_p-1} - \bar{u}_{k}^{k+N_p-1}\|^2_R \]  

(12)

with

\[ \hat{y}_{k+1}^{k+N_p} = \begin{bmatrix} Hg(t_k, t_{k+1}, x_k, u_k, \theta) \\ Hg(t_k, t_{k+2}, x_k, u_k^{k+1}, \theta) \\ \vdots \\ Hg(t_k, t_{k+N_p}, x_k, u_k^{k+N_p-1}, \theta) \end{bmatrix} : \text{the predicted output,} \]

\[ \bar{y}_{k+1}^{k+N_p} = [y_{k+1}^{r}, \ldots, y_{k+N_p}^{r}]^\top : \text{the setpoint values,} \]

\[ u_{k}^{k+N_p-1} = [u_k, \ldots, u_{k+N_p-1}]^\top : \text{the optimization variable,} \]

\[ \bar{u}_{k}^{k+N_p-1} = [u_{k}^{r}, \ldots, u_{k+N_p-1}^{r}]^\top : \text{the reference control sequence.} \]

and the tuning diagonal matrices \( P \geq 0, R > 0 \).
In the case of uncertain parameters \((\Theta = [\theta^-, \theta^+])\): 

\[
\tilde{u}^{k+N_p-1}_k = \arg \min_{u^{k+N_p-1}_k} \max_{\hat{\theta} \in \Theta} \Pi(u^{k+N_p-1}_k, \hat{\theta})
\]  

(13)

with

\[
\Pi(u^{k+N_p-1}_k, \hat{\theta}) = ||\hat{y}^{k+N_p}_{k+1} - \bar{y}^{k+N_p}_{k+1}||^2_P + ||u^{k+N_p-1}_k - \bar{u}^{k+N_p-1}_k||^2_R
\]  

(14)

and

\[
\hat{y}^{k+N_p}_{k+1} = 
\begin{bmatrix}
Hg(t_k, t_{k+1}, x_k, u_k, \hat{\theta}) \\
Hg(t_k, t_{k+2}, x_k, u^{k+1}_k, \hat{\theta}) \\
\vdots \\
Hg(t_k, t_{k+N_p}, x_k, u^{k+N_p-1}_k, \hat{\theta})
\end{bmatrix}
\]  

(15)

The optimal control sequence \(u^{k+N_p-1}_k\) is determined so that the maximum deviation for all trajectories over all possible data scenarii is minimized \([\text{Yu}, 1998], [\text{Kerrigan and Maciejowski}, 2004]\).
Taylor series expansion (limited to the first-order)

\( g \) is linearized around \( \bar{u}_k^{k+N_p-1} \) & \( \theta_{nom} = \frac{\theta^+ + \theta^-}{2} \) for \( j = 1, N_p \):

\[
\begin{align*}
    g(t_k, t_{k+j}, x_k, u_k^{k+j-1}, \hat{\theta}) & \approx
    g(t_k, t_{k+j}, x_k, \bar{u}_k^{k+j-1}, \theta_{nom}) \\
    & + \nabla g_u(t_{k+j-1})(u_k^{k+j-1} - \bar{u}_k^{k+j-1}) \\
    & + \nabla g_\theta(t_{k+j})(\hat{\theta} - \theta_{nom})
\end{align*}
\]

(16)

**Sensitivity functions**

\[
\nabla g_u(t_{k+j-1}) = \frac{\partial g(t_k, t_{k+j}, x_k, u_k^{k+j-1}, \theta)}{\partial u_k^{k+j-1}} \\
\bigg|_{u_k^{k+j-1} = \bar{u}_k^{k+j-1}} \bigg|_{\theta = \theta_{nom}}
\]

(17)

\[
\nabla g_\theta(t_{k+j}) = \frac{\partial g(t_k, t_{k+j}, x_k, u_k^{k+j-1}, \theta)}{\partial \theta} \\
\bigg|_{u_k^{k+j-1} = \bar{u}_k^{k+j-1}} \bigg|_{\theta = \theta_{nom}}
\]

(18)
Dynamics of sensitivity functions with respect to $\theta$ and $u$

- **Analytical derivation** [Dochain, 2008]

\[
\frac{d}{dt}(\nabla g_\theta) = \frac{\partial f(x, u, \theta_{\text{nom}})}{\partial x} \nabla g_\theta + \frac{\partial f(x, u, \theta)}{\partial \theta} \bigg|_{\theta=\theta_{\text{nom}}} 
\]  \hspace{0.5cm} (19)

**Initial conditions:** \( \nabla g_\theta(t_k) = 0_{3 \times 6} \)

- The computing of $\nabla g_u$ is done numerically by finite differences.

\[
\hat{y}_{k+1} \approx \bar{G}_{\theta, k+1}^{k+N_p} (\hat{\theta} - \theta_{\text{nom}})
\]

\[
\begin{align*}
\hat{y}_{k+1}^{k+N_p} & \approx \bar{G}_{\text{nom}, k+1}^{k+N_p} + \bar{G}_{u, k}^{k+N_p-1} (u_{k}^{k+N_p-1} - \bar{u}_{k}^{k+N_p-1}) + \bar{G}_{\theta, k+1}^{k+N_p} (\hat{\theta} - \theta_{\text{nom}}) \\
\end{align*}
\]  \hspace{0.5cm} (20)

with

- $\bar{G}_{\text{nom}, k+1}^{k+N_p}$: predicted output for the nominal case.
- $\bar{G}_{u, k+1}^{k+N_p}$: Jacobian matrices related to the control sequence.
- $\bar{G}_{\theta, k+1}^{k+N_p}$: Jacobian matrices related to the parameters.
New cost function

\[
\Pi(u^{k+N_p-1}_k, \hat{\theta}) = \left\| u^{k+N_p-1}_k - \bar{u}^{k+N_p-1}_k \right\|^2_R + \\
\left\| \bar{G}^{k+N_p-1}_{u,k} (u^{k+N_p-1}_k - \bar{u}^{k+N_p-1}_k) - \\
(\bar{y}^{k+1}_k - \bar{G}^{k+N_p}_{\text{nom},k+1} - \bar{G}^{k+N_p}_{\theta,k+1} (\hat{\theta} - \theta_{\text{nom}})) \right\|^2_P
\]

(21)

Bounded parametric error (uncertain parameters are uncorrelated)

\[
\hat{\theta} - \theta_{\text{nom}} = \gamma \delta \theta_{\text{max}}
\]

with \(\delta \theta_{\text{max}} = \frac{\theta^+ - \theta^-}{2}\) and \(\|\gamma\| \leq 1\)

Approach

move from Min-Max optimization of \(N_p \times N_\theta\) dimension to a scalar robust regularized optimization [Benattia et al., 2015].

Robust Regularized Least Squares Problem

\[
\tilde{z} = \arg \min_z \max_{E_b} \left\| z \right\|^2_R + \left\| Az - (b + C \gamma E_b) \right\|^2_P
\]

(22)

with

\[
\left\{ \\
\begin{align*}
z &= u^{k+N_p-1}_k - \bar{u}^{k+N_p-1}_k, \\
A &= \bar{G}^{k+N_p}_{u,k} \\
b &= \bar{y}^{k+1}_k - \bar{G}^{k+N_p}_{\text{nom},k+1}, \\
C &= \bar{G}^{k+N_p}_{\theta,k+1}, \;
E_b = -\delta \theta_{\text{max}}
\end{align*}
\right.
\]

(23)
Regularized Robust Design Criterion for Uncertain Data [Sayed et al., 2002]

**Unique global minimum** \( z(\lambda^o) \)

\[
z(\lambda^o) = [R + A^T P(\lambda^o)A]^{-1} A^T P(\lambda^o)b
\]  
(24)

**Modified weighting matrix** \( p(\lambda^o) \)

\[
p(\lambda^o) = P + PC(\lambda^o I - C^T PC)^\dagger C^T P
\]  
(25)

**Nonnegative scalar parameter** \( \lambda^o (\lambda \in \mathbb{R}^+) \)

\[
\lambda^o = \arg \min_{\lambda \geq \|C^T PC\|} \|z(\lambda)\|^2_R + \lambda \| - E_b \|^2 + \|Az(\lambda) - b\|_P^2
\]  
(26)

with \[
\left\{
\begin{array}{l}
z(\lambda) = [R + A^T P(\lambda)A]^{-1} A^T P(\lambda)b \\
P(\lambda) = P + PC(\lambda I - C^T PC)^\dagger C^T P
\end{array}
\right.
\]

**Linearized Robust MPC (LRMPC)**

\[
\tilde{u}_{k+Np-1} = \tilde{u}_{k+Np-1} + [R + \tilde{G}_{u,k}^{k+Np-1} P(\lambda^o)\tilde{G}_{u,k}^{k+Np-1}]^{-1}[\tilde{G}_{u,k}^{k+Np-1} P(\lambda^o)(\tilde{y}_{k+1} - \tilde{G}_{nom,k+1})
\]

\[
\tilde{y}_{k+1} = \tilde{y}_{k+1} + [R + \tilde{G}_{y,k}^{k+Np-1} P(\lambda^o)\tilde{G}_{y,k}^{k+Np-1}]^{-1}[\tilde{G}_{y,k}^{k+Np-1} P(\lambda^o)(\tilde{u}_{k+Np} - \tilde{G}_{nom,k+1})
\]
Influence of the sampling time

Parameters uncertainties ±20% [Goffaux and Vande Wouwer, 2008]

- exploring the parameter subspace border \( \{\theta^-, \theta^+\} \rightarrow 2^{\dim(\theta)} \) tests;
- worst-case model mismatch:
  \[ \theta_{\text{real}} = [\rho_m^+ K_s^- \bar{\mu}^+ K_Q^- K_{sl}^- K_{il}^+] \]

- \( N_p = 5; \)
- \( P = I_{N_p} \) and \( R = I_{N_p}; \)

Compromise

- \( T_s \downarrow \Rightarrow \) first order Taylor series expansion is accurate as much as possible;
- \( T_s \uparrow \Rightarrow \) computation burden due to the state estimator and/or determination online of the optimal trajectory.

**Figure:** Biomass concentration tracking error and dilution rate evolution with time for LRMPC strategy.

**Choice**

\( T_s = 10 \text{ min} \Rightarrow \) linearization accuracy/computational burden \( \checkmark. \)
Influence of the prediction horizon

- $T_s = 10 \text{ min}$;
- $P = I_{N_p}$ and $R = I_{N_p}$;

**Compromise**
- $N_p \downarrow$ $\implies$ computation time and a sufficient vision of the system behaviour;
- $N_p \uparrow$ $\implies$ poor prediction accuracy.

**Choice**
- $N_p = 5$ $\implies$ good prediction $\checkmark$.

*Figure:* Biomass concentration tracking error evolution with time for LRMPC strategy.
Robustness w.r.t parameters uncertainties

- $y_k = X_k + e \ & \ e \sim \mathcal{N}(0, 0.1)$;
- $T_s = 10$ min and $N_p = 5$;
- $P = I_{N_p}$ and $R = I_{N_p}$;
- anticipation behavior to a setpoint change;
- dilution rate ↘ cell concentration ↗ and vice versa (biological aspect √);
- both RNMPC and LRMPC have better performances than the classical NMPC;

### LRMPC

approximation of the model through linearization + model mismatch ↓ static error

#### Figure: Biomass concentration and dilution rate evolution with time for NMPC, RNMPC and LRMPC strategies.

#### Table: Comparison of the proposed algorithms in terms of computation time at each sampling time (worst case).

<table>
<thead>
<tr>
<th>Algo.</th>
<th>Perf. indices</th>
<th>Computation time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>min</td>
</tr>
<tr>
<td>NMPC</td>
<td></td>
<td>$&lt; 10^{-5}$</td>
</tr>
<tr>
<td>RNMPC</td>
<td></td>
<td>0.43</td>
</tr>
<tr>
<td>LRMPC</td>
<td></td>
<td>$&lt; 10^{-5}$</td>
</tr>
</tbody>
</table>
Conclusion

- regulation of the biomass concentration to a desired value in a continuous photobioreactor;
- solving, online, NMPC problem using a finite parametrization of \( u \);
- robust NMPC \( \Rightarrow \) min-max problem;
- first order Taylor series expansion \( \Rightarrow \) LRMPC law;
- comparison between the proposed strategy and the (R)NMPC laws;
- robustness of the proposed strategy w.r.t model parameters uncertainties;
- reducing the computational burden.

Perspectives

- hierarchical control strategy formed by two level controller:
  predictive + integral action; \( \rightsquigarrow \)
- determination of conditions to ensure robust stability with bounded uncertainties including inequality constraints; \( \rightsquigarrow \)
- second order expansion to improve robustness; \( \uparrow \)
- estimation algorithm to reconstruct biomass concentration; \( \uparrow \)
Thank you for your attention!
Appendix A

The equilibrium is defined as follows:

\[
\begin{align*}
D & = \bar{\mu}(1 - \frac{K_Q}{Q}) \mu I \\
S & = \frac{\mu(Q,I)QK_s}{\rho_m - \mu(Q,I)Q} \\
(S_{in} - \frac{\mu(Q,I)QK_s}{\rho_m - \mu(Q,I)Q})\mu(Q,I) - \mu(Q,I)QX & = 0
\end{align*}
\]

(27)

Taking \( Q \) as an unknown variable, the following quadratic equation must be solved:

\[
\bar{\mu} \mu_I X Q^2 - \{(S_{in} + K_s)\bar{\mu} \mu_I + (\rho_m + \bar{\mu} \mu_I K_Q)X\}Q + (\rho_m + \bar{\mu} \mu_I K_Q)S_{in} + \bar{\mu} \mu_I K_Q K_s = 0
\]

(28)

The discriminant of the equation (28) is given by:

\[
\Delta = aX^2 - bX + c = (\sqrt{a}X - \sqrt{c})^2 + (2\sqrt{a}\sqrt{c} - b)X
\]

(29)

where

\[
\begin{align*}
a & = (\rho_m + \bar{\mu} \mu_I K_Q)^2 \\
b & = -2\bar{\mu} \mu_I \{(K_s - S_{in})(\rho_m + \bar{\mu} \mu_I K_Q) - 2\bar{\mu} \mu_I K_Q K_s\} \\
c & = \bar{\mu}^2 \mu_I^2 (S_{in} + K_s)^2
\end{align*}
\]

(30)

we have that

\[
2\sqrt{a}\sqrt{c} - b = 2(\rho_m + \bar{\mu} \mu_I K_Q)\bar{\mu} \mu_I (S_{in} + K_s) + 2\bar{\mu} \mu_I \{(K_s - S_{in})(\rho_m + \bar{\mu} \mu_I K_Q) - 2\bar{\mu} \mu_I K_Q K_s\}
\]

\[
2\sqrt{a}\sqrt{c} - b = 4\bar{\mu} \mu_I K_s \rho_m
\]

(31)
Finally
\[
\Delta = (\rho_m + \bar{\mu}_1 K_Q)X - \bar{\mu}_1(S_{in} + K_s)^2 + 4\bar{\mu}_1 K_s \rho_m X
\]  
(32)

Since the discriminant \(\Delta\) is non negative, there are two real solutions:
\[
Q_{1,2}^* = \frac{S_{in} + K_s}{2X} + \frac{1}{2} \left( \frac{\rho_m}{\bar{\mu}_1} + K_Q \right) \pm \frac{\sqrt{\Delta}}{2\bar{\mu}_1 X}
\]  
(33)

Using (29), it comes:
\[
\frac{\sqrt{\Delta}}{2\bar{\mu}_1 X} = \frac{1}{2} \sqrt{a} \sqrt{1 + \alpha \frac{1}{X} + \beta \frac{1}{X^2}}
\]  
(34)

with
\[
\left\{ \begin{array}{l}
\alpha = -\frac{b}{a} \\
\beta = \frac{c}{a}
\end{array} \right.
\]  
(35)

where
\[
\sqrt{1 + \alpha \frac{1}{X} + \beta \frac{1}{X^2}} = \frac{1}{X} \sqrt{(X + \sqrt{\beta})^2 + (\alpha - 2\sqrt{\beta})X}
\]  
(36)
we have that

\[
\alpha - 2\sqrt{\beta} = \frac{2\bar{\mu}\mu_I[(K_s - S_{in})(\rho_m + \bar{\mu}\mu_I K_Q) - 2\bar{\mu}\mu_I K_Q K_s]}{(\rho_m + \bar{\mu}\mu_I K_Q)^2} - \frac{2\bar{\mu}\mu_I(S_{in} + K_s)}{(\rho_m + \bar{\mu}\mu_I K_Q)}
\]

\[
= \frac{2\bar{\mu}\mu_I(K_s - S_{in})}{(\rho_m + \bar{\mu}\mu_I K_Q)} - \frac{4\bar{\mu}^2\mu_I^2 K_s K_Q}{(\rho_m + \bar{\mu}\mu_I K_Q)^2} - \frac{2\bar{\mu}\mu_I(S_{in} + K_s)}{(\rho_m + \bar{\mu}\mu_I K_Q)} \tag{37}
\]

\[
\alpha - 2\sqrt{\beta} = \frac{4\bar{\mu}\mu_I \rho_m K_s}{(\rho_m + \bar{\mu}\mu_I K_Q)^2}
\]

Then

\[
\sqrt{1 + \alpha \frac{1}{X} + \beta \frac{1}{X^2}} = \sqrt{\left(1 + \frac{\bar{\mu}\mu_I(S_{in} + K_s)}{X(\rho_m + \bar{\mu}\mu_I K_Q)}\right)^2 + \frac{4\bar{\mu}\mu_I \rho_m K_s}{X(\rho_m + \bar{\mu}\mu_I K_Q)^2}} > 1 \tag{38}
\]

Finally

\[
Q_2 = \frac{S_{in} + K_s}{2X} + \frac{1}{2} \left(\frac{\rho_m}{\bar{\mu}\mu_I} + K_Q\right)\{1 + \sqrt{1 + \alpha \frac{1}{X} + \beta \frac{1}{X^2}}\} \tag{39}
\]
Appendix B

- Robust regularized optimization problem

\[ z^o = \arg \min_z \max_{\delta A, \delta b} \left\| z \right\|_V^2 + \left\| (A + \delta A)z - (b + \delta b) \right\|_W^2 \]  \hspace{1cm} (40)

where \( V > 0 \) and \( W \geq 0 \) are Hermitian weighting matrices.

Perturbation modeling:

\[ \begin{align*}
\delta A &= C \Delta E_a \\
\delta b &= C \Delta E_b 
\end{align*} \hspace{1cm} C \neq 0 \]  \hspace{1cm} (41)

\( \Delta \): arbitrary contraction (\( \left\| \Delta \right\| \leq 1 \)).

- Constrained two player game problem

\[ \min_z \max_{\| \kappa \| \leq \pi(z)} J(z, \kappa) \]  \hspace{1cm} (42)

with

\[ J(z, \kappa) = z^\top V z + (A z - b + C \kappa)^\top W (A z - b + C \kappa) \]  \hspace{1cm} (43)

and

\[ \kappa = \Delta (E_a z - E_b) \]  \hspace{1cm} (44)
• On the use of Lagrangean multiplier $\lambda$: (42) $\implies$ an unconstrained problem

$$
\min_{z} \max_{\kappa, \lambda} J(z, \kappa, \lambda)
$$

(45)

with

$$
J(z, \kappa, \lambda) = z^T Vz + (Az - b + C\kappa)^T W(Az - b + C\kappa) - \lambda(||\kappa||^2 - \pi(z)^2)
$$

(46)

solution of $\kappa$

1. First derivative: \( \frac{\partial J(z,\kappa,\lambda)}{\partial \kappa} = 0 \)

\[
C^T W(Az - b) + C^T WC\kappa + (Az - b)^T WC + \kappa^T C^T WC - 2\lambda\kappa = 0
\]

\[
C^T W(Az - b) + (Az - b)^T WC + C^T WC\kappa + \kappa^T C^T WC - 2\lambda\kappa = 0
\]

Then,

$$
\kappa = (\lambda I - C^T WC)^{-1} C^T W(Az - b)
$$

(47)

2. Second derivative: \( \frac{\partial^2 J(z,\mu,\kappa,\lambda)}{\partial^2 \kappa} \geq 0 \)

\[
C^T WC + C^T WC - 2\lambda \geq 0
\]

Then,

$$
\lambda \geq ||C^T WC||
$$
Using the analytic solution (47) in the cost function (46), we have:

\[
J(z, \lambda) = z^T Vz + (Az - b)^T W (Az - b) + \kappa (Az - b)^T WC \kappa + \kappa^T C^T W (Az - b) \\
+ \kappa^T C^T WC \kappa - \lambda \kappa^T \kappa + \lambda \pi (z)^2
\]  

(48)

Then,

\[
J(z, \lambda) = z^T Vz + (Az - b)^T W(\lambda) (Az - b) + \lambda \pi (z)^2
\]  

(49)

with

\[
W(\lambda) = W + WC (\lambda I - C^T WC)^{-1} C^T W
\]  

(50)

Finally, the optimization problem (45) becomes:

\[
\min_z \quad \min_{\lambda \geq \|C^T WC\|} \quad J(z, \lambda) \iff \min_{\lambda \geq \|C^T WC\|} \min_z J(z, \lambda)
\]  

(51)

Analytic solution of \(z\)

\[
\frac{\partial J(z, \lambda)}{\partial z} = 0
\]  

(52)

\[
Vz + z^T V + A^T W(\lambda) (Az - b) + (Az - b)^T W(\lambda) A + \lambda \nabla \pi (z)^2 = 0
\]  

(53)

\[
(V + A^T W(\lambda) A)z + \frac{1}{2} \lambda \nabla \pi (z)^2 = A^T W(\lambda) b
\]  

(54)
We have that:

\[\pi(z) = ||E_a z - E_b||\]  \hfill (55)

Then,

\[\nabla \pi(z)^2 = 2E_a^T (E_a z - E_b)\]  \hfill (56)

Using (56) in the analytic solution (54), it comes:

\[(V + A^T W(\lambda)A)z + \lambda E_a^T E_a z - \lambda E_a^T E_b = A^T W(\lambda)b\]
\[(V + A^T W(\lambda)A + \lambda E_a^T E_a)z = A^T W(\lambda)b + \lambda E_a^T E_b\]

Then,

\[z(\lambda) = E(\lambda)^{-1} B(\lambda)\]  \hfill (57)

with

\[\left\{\begin{array}{l}
E(\lambda) = V(\lambda) + A^T W(\lambda)A \\
B(\lambda) = A^T W(\lambda)b + \lambda E_a^T E_b
\end{array}\right.\]  \hfill (58)

and

\[\left\{\begin{array}{l}
W(\lambda^o) = W + WC(\lambda^o I - C^T WC)^+ C^T W \\
V(\lambda^o) = V + \lambda^o E_a^T E_a
\end{array}\right.\]  \hfill (59)
Replacing the analytic solution (57) in the cost function (49):

\[ J(\lambda) = z^T Rz + (Az - b)^T W(\lambda)(Az - b) + \lambda (E_a z - E_b)^T (E_a z - E_b) \]

\[ = z^T Rz + b^T W(\lambda) b + z^T A^T W(\lambda) Az - b^T W(\lambda) Az - z^T A^T W(\lambda) b + \lambda z^T E_a^T E_a z + \lambda E_b^T E_b - \lambda z^T E_a^T E_b - \lambda E_b^T E_a z \]

\[ = \lambda E_b^T E_b + b^T W(\lambda) b + z^T (R + A^T W(\lambda) A + \lambda E_a^T E_a) z - (b^T W(\lambda) A + \lambda E_b^T E_a) z - z^T (A^T W(\lambda) b + \lambda E_a^T E_b) \]

\[ = \lambda E_b^T E_b + b^T W(\lambda) b + z^T E(\lambda) z - B(\lambda)^T z - z^T B(\lambda) \]

\[ J(\lambda) = \lambda E_b^T E_b - B(\lambda)^T E(\lambda)^{-1} B(\lambda) \]

The scalar \( \lambda \) is computed from the following minimization problem:

\[ \lambda^o = \arg \min_{\lambda \geq ||H^T WH||} J(\lambda) \tag{60} \]

with

\[ J(\lambda) = \lambda ||E_a z(\lambda) - E_b||^2 + ||z(\lambda)||^2_R + ||Az(\lambda) - b||^2_{W(\lambda)} \tag{61} \]

Finally, the unique global minimum \( z^o \) given by:

\[ z^o = E(\lambda^o)^{-1} B(\lambda^o) \tag{62} \]


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