Stability of piecewise affine systems with state-dependent delay, and application to congestion control

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Some motivations

Systems with state-dependent delays:

- Several applications: biological systems, metal rolling systems, communication networks (fluid flow models), ...;
- New challenges: nonlinearities due to the state-dependency of the delay.

⇒ Classic stability tools for time-delay systems may not be applied.
Some motivations

**Literature:**

Stability and/or stabilization of systems with state-dependent delay:
...

None of these works concerns the congestion control problem.

Congestion control using fluid flow models:
[Liu et al., INFOCOM 2005], [Michiels et al., Int. J. Control 2006],
[Briat et al., IEEE Trans. Autom. Control 2010], ...

Most of these works ignore the state-dependency of the delay
⇒ This may lead to false stability results.
Some motivations

We consider the fluid flow model of communication network from [Avrachenkov and Paszke, ICICAR 2004]

\[
\begin{align*}
\dot{y}(t) &= \begin{cases} 
  z(t - y(t)/\mu - d) - \mu, & y(t) > 0, \\
  \max\{0, z(t - y(t)/\mu - d) - \mu\}, & y(t) = 0,
\end{cases} \\
\dot{z}(t) &= u(t),
\end{align*}
\]

where

\[
\begin{align*}
  y(t) &= \text{amount of data in the buffer} \\
  z(t) &= \text{sending rate of the data source} \\
  \mu &= \text{(constant) service rate of the router} \\
  d &= \text{propagation delay}
\end{align*}
\]
Some motivations

Denoting $x_1(t) = y(t) - y_d$ and $x_2(t) = z(t) - \mu$, with $y_d$ the desired amount of data in the router’s buffer, this model can be embedded ([Balakrishnan et al., IEEE CL 2007]) into the following switched system:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t - (x_1(t) + y_d)/\mu - d), \\
\dot{x}_2(t) &= u(t), \\
\text{for } x_1(t) > -y_d \text{ or } x_2(t - (x_1(t) + y_d)/\mu - d) &\geq 0, \\
\dot{x}_1(t) &= -x_1(t) - y_d, \\
\dot{x}_2(t) &= u(t), \\
\text{for } x_1(t) \leq -y_d \text{ and } x_2(t - (x_1(t) + y_d)/\mu - d) &\leq 0.
\end{align*}
\]

This motivates us to analyse the stability of congestion control with tools adapted from works about piecewise affine systems.
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System description

We consider the piecewise affine system with a time- and state-dependent delay and delayed-state-dependent switching

\[ \dot{x}(t) = A_i x(t) + A_d_i x(t - \tau(t, x(t))) + b_i, \text{ for } G \xi(t) \in X_i, \]

with \( \xi(t) = \begin{bmatrix} x(t) \\ x(t - \tau(t, x(t))) \end{bmatrix}, G \in \mathbb{R}^{n_G \times 2n}, \) and a covering \( \{X_i\}_{i \in \mathcal{I}} \) of the space \( \mathbb{R}^{n_G} \) into a finite number of (possibly unbounded) polyhedral cells with pairwise disjoint interiors.
Some assumptions

**Assumption 1:**
The system is linear, or piecewise linear around the origin:

\[ b_i = 0_{n \times 1} \text{ if } 0_{n \times 1} \in X_i. \]
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The delay is lower-bounded and positive:

\[ \tau(t, x) \geq h_0 \geq 0. \]
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The delay is lower-bounded and positive :

\[ \tau(t, x) \geq h_0 \geq 0. \]

**Assumption 3 :**
The variations of the delay are norm-state-bounded in a neighbourhood of the origin :

\[ \|x\|_L \leq 1 \implies |\tau(t, x) - \tau_0| \leq c + \|x\|_{\Psi}, \]

where \( \|x\|_M \) denotes \( \sqrt{x^T M x} \).
Objectives:

1. Find constructive conditions that guarantee the local exponential stability of the system for a given decay-rate;
2. Provide an under-approximation of the domain of attraction.
Challenges and contributions:

1. New constructive stability tools for piecewise affine systems with time- and state-dependent delay and delayed-state dependent switching law;

2. Stability tools for the congestion control problem which take into account the state-dependency of the delay.
The stability analysis is performed in 2 steps.

1. Stability analysis in the case of time-varying (non state-dependent) delay.


*Ideas*: Consider the time- and state-dependent delay as a saturated input, and design the domain of attraction of the system.
Lyapunov-Krasovskii functional design

Following [Johansson and Rantzer, IEEE TAC, 1998], we design a Lyapunov function with a piecewise quadratic part.

**Natural design:**

\[
\bar{V}_0(t) = \begin{bmatrix} \xi(t) \\ 1 \end{bmatrix}^T \bar{P}_i \begin{bmatrix} \xi(t) \\ 1 \end{bmatrix}, \text{ for all } G\xi(t) \in X_i, \ i \in I.
\]
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The derivative of the delay \( \frac{d}{dt}(\tau(t, x(t))) \) appears in the derivative of \( \bar{V}_0 \)!
Lyapunov-Krasovskii functional design

Following [Johansson and Rantzer, IEEE TAC, 1998], we design a Lyapunov function with a piecewise quadratic part.

**Natural design:**

\[
\tilde{V}_0(t) = \begin{bmatrix} \xi(t) \\ 1 \end{bmatrix}^T \bar{P}_i \begin{bmatrix} \xi(t) \\ 1 \end{bmatrix}, \text{ for all } G\xi(t) \in X_i, \ i \in \mathcal{I}.
\]

**Better design:**

\[
\tilde{V}_0(t) = \begin{bmatrix} x(t) \\ 1 \end{bmatrix}^T \bar{P}_i \begin{bmatrix} x(t) \\ 1 \end{bmatrix}, \text{ for all } G'x(t) \in X'_i, \ i \in \mathcal{I},
\]

with a matrix \( G' \in \mathbb{R}^{ng' \times n} \) and a covering \( \{X'_i\}_{i \in \mathcal{I}} \) of the space \( \mathbb{R}^{ng'} \) designed to satisfy

\[
G \begin{bmatrix} x \\ y \end{bmatrix} \in X_i \Rightarrow G'x \in X'_i.
\]
Lyapunov-Krasovskii functional design

Example:

System’s switching law defined with

\[ G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]
Lyapunov-Krasovskii functional design

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System’s switching law defined with

\[ G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

Lyapunov function’s switching law defined with

\[ G' = [1 0] \]
Lyapunov-Krasovskii functional design

Complete Lyapunov-Krasovskii functional:

\[ \mathcal{V}(t) \triangleq V(t, x_t, \dot{x}_t) = \mathcal{V}_0(t) + \mathcal{V}_1(t), \]

with

\[ \mathcal{V}_0(t) \triangleq V_0(x(t)) = \begin{bmatrix} x(t) \\ 1 \end{bmatrix}^T \bar{P}_i \begin{bmatrix} x(t) \\ 1 \end{bmatrix}, \text{ for } G'x(t) \in X'_i, \ i \in \mathcal{I}, \]

\[ \mathcal{V}_1(t) \triangleq V_1(t, x_t, \dot{x}_t) = \sum_{k=1}^{2} \int_{t-h_{k-1}}^{t} e^{2\alpha(s-t)} x^T(s) S_k x(s) ds \]

\[ + \sum_{k=1}^{2} (\bar{h}_k - \bar{h}_{k-1}) \int_{-h_k}^{-\bar{h}_{k-1}} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{x}^T(s) R_k \dot{x}(s) ds d\theta. \]
Main result

Using some classic tools from stability analysis with Lyapunov-Krasovskii functionals, we obtain the following:

**Theorem:**
Given scalars $h_0 \leq h_1 \leq h_2$, if there exist parameters such that some LMI (positivity and exponential decrease of $\tilde{V}$) and equality (continuity of $\tilde{V}$) constraints are satisfied, then the PWA system is globally exponentially stable for any time-varying delay $\tau(t) \in [\bar{h}_1, \bar{h}_2]$. 
Saturated delay

We treat the delay as a saturated input. We recall the assumptions
\[\tau(t, x) \geq h_0 \geq 0, \quad \|x\|_L \leq 1 \implies |\tau(t, x) - \tau_0| \leq c + \|x\|_\Psi.\]

We consider bounds on the delay \(\tau(t, x) \in [\bar{h}_1, \bar{h}_2]\) defined as
\[\bar{h}_1 = \max(h_0, \tau_0 - \eta), \quad \bar{h}_2 = \tau_0 + \eta,\]
for some scalar \(\eta \geq c\), and a neighbourhood of the origin defined by
\[\mathcal{X}_\beta(P) = \{x \in \mathbb{R}^n \mid x^T P x \leq \beta\},\]

**Lemma :**
\[\forall x \in \mathcal{X}_\beta(P), \quad \tau(t, x) \in [\bar{h}_1, \bar{h}_2]\] if the following LMIIs are satisfied:
\[
\gamma_1 \begin{bmatrix}
-L & 0_{n \times 1} \\
* & 1
\end{bmatrix}
- \begin{bmatrix}
-P & 0_{n \times 1} \\
* & \beta
\end{bmatrix} \succeq 0_{(n+1) \times (n+1)},
\]
\[
\gamma_2 \begin{bmatrix}
-\Psi & 0_{n \times 1} \\
* & (\eta - c)^2_r
\end{bmatrix}
- \begin{bmatrix}
-P & 0_{n \times 1} \\
* & \beta
\end{bmatrix} \succeq 0_{(n+1) \times (n+1)}. 
\]
Main result

**Theorem:**
If there exist parameters such that

1. the conditions from the previous Lemma are satisfied
2. the conditions from the Theorem in the time-varying delay case are satisfied

Then, the PWA system is locally exponentially stable, and an under-approximation of the domain of attraction is obtained as a union of ellipsoids:

\[
D = \bigcup_{i \in \mathcal{I}} \left\{ x_0 \in \mathbb{R}^n \mid G' x_0 \in X'_i \text{ and } \begin{bmatrix} x_0 \\ 1 \end{bmatrix}^T B_i \begin{bmatrix} x_0 \\ 1 \end{bmatrix} \leq \beta \right\}.
\]
Example 1 - Application to congestion control

We consider the fluid flow model of communication network from [Avrachenkov and Paszke, ICICAR 2004]

\[
\begin{align*}
\dot{y}(t) &= \begin{cases} 
z(t - y(t)/\mu - d) - \mu, & y(t) > 0, \\ 
\max\{0, z(t - y(t)/\mu - d) - \mu\}, & y(t) = 0,
\end{cases} \\
\dot{z}(t) &= u(t),
\end{align*}
\]

with a linear controller

\[
u(t) = -k_1(y(t) - y_d) - k_2(z(t) - \mu).
\]
Example 1 - Application to congestion control

Denoting $x_1(t) = y(t) - y_d$ and $x_2(t) = z(t) - \mu$, we embed the model into the switched system

$$\dot{x}(t) = A_i x(t) + A_{di} x(t - \tau(x(t))) + b_i, \text{ for } G\xi(t) \in X_i,$$

with

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tau(x(t)) = \max \left( \frac{x_1(t) + y_d}{\mu} + d, d \right),$$

$$A_1 = A_2 = \begin{bmatrix} -1 & 0 \\ -k_1 & -k_2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ -k_1 & -k_2 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{d2} = A_{d3} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$b_1 = b_2 = \begin{bmatrix} -y_d \\ 0 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
Example 1 - Application to congestion control

\[ X_1 = \{ G\xi(t) \mid x_1(t) \leq -y_d \text{ and } x_2(t - \tau(t, x(t))) \leq 0 \}, \]
\[ X_2 = \{ G\xi(t) \mid x_1(t) \leq -y_d \text{ and } x_2(t - \tau(t, x(t))) \geq 0 \}, \]
\[ X_3 = \{ G\xi(t) \mid x_1(t) > -y_d \}, \]
Example 1 - Application to congestion control

Results obtained for a decay rate $\alpha = 0.01$, and system parameters $d = 0.8$, $\mu = 4$, $y_d = 0.2$, and $k_1 = k_2 = 1$.

$D$ : Estimation of the domain of attraction
$X_\beta(P) :$ Positive invariant ellipsoid
Example 2 - Academic example

We consider the system from [Kulkarni et al., ACC 2004]:

\[ \dot{x}(t) = A_i x(t) + A_{d_i} x(t - \tau(t)), \text{ for } x(t) \in X_i \]

with

\[ X_i = \{ x \in \mathbb{R}^2, E_i x \geq 0 \}, \]

\[ E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \]

\[ A_1 = A_2 = A_3 = A_4 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \]

\[ A_{d_1} = A_{d_3} = \begin{bmatrix} 0 & 5 \\ -1 & 0 \end{bmatrix}, \quad A_{d_2} = A_{d_4} = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}. \]
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Obtained upper-bound for time-varying delay: \( \tau^* = 0.0149 \)
Example 2 - Academic example

We consider a state-dependent delay $\tau(x(t)) = 0.005 + \|x(t)\|^2$. 
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$D$ : Estimation of the domain of attraction
$X_\beta(P)$ : Positive invariant ellipsoid
$D_{\text{simulations}}$ : Stability domain obtained through simulations.
Conclusion and perspectives

Conclusion:
Constructive tools for the stability analysis of piecewise affine systems with time- and state-dependent delays and delayed-state-dependent switching. Two cases are treated:

- System with (possibly fast) bounded time-varying delay (global exponential stability);
- System with time- and state-dependent delay (local exponential stability with estimation of the domain of attraction as the union of ellipsoids).

Application to congestion control

Perspectives:

- Control design
- Application to other congestion control fluid flow models