Event-triggered control of nonlinear singularly perturbed systems based only on the slow dynamics

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Overview

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Main result

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Event-triggered control

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Main result
Semiglobal practical stabilization
Global asymptotic stabilization

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Conclusion

How to efficiently use the network?

- **Time-triggering**: \( t_{i+1} - t_i = T, \quad T > 0 \)
  
  \( \Rightarrow \) Transmit at every \( T \) seconds regardless the system’s need

- **Event-triggering**: \( t_{i+1} - t_i = T(x(t_i), x(t)) \)
  
  \( \Rightarrow \) Adapts transmissions to the state
Event-triggered control

- Implementable event-triggered controllers
  - guarantee stability
  - prevent Zeno

- Many results in the literature on nonlinear systems
  - state feedback: [Årzén, IFAC’99], [Tabuada, IEEE TAC’07]
    [Postoyan et al., IEEE TAC’15], . . .
  - output feedback: [Forni et al., Automatica’14]
    [Tallapragada & Chopra, IEEE CDC’12], . . .

- No adapted result for two-time scale systems
Two-time scale systems

Event-triggered singularly perturbed systems
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▶ Plant model

\[
\dot{x} = f(x, z, u) \quad \text{(slow)} \\
\varepsilon \dot{z} = g(x, z, u) \quad \text{(fast)}
\]

▶ \( \varepsilon > 0 \quad (\varepsilon \to 0 \Rightarrow \dot{z} \gg \dot{x}) \)

▶ **Applications:** Control, power systems, biology, economics, …
Context of control systems

\[ \dot{x} = f(x, z, u), \quad \varepsilon \dot{z} = g(x, z, u) \]

- Two-time scale models may naturally appear in
  - Robust stabilization w.r.t. unmodelled dynamics
  - High gain in the feedback
  - Electro-mechanical systems

- Issues
  - ill-conditioned controllers
  - instability

\[ \Rightarrow \quad \text{Singular perturbation framework} \]
Singular perturbation (continuous-time)

Original model

\[ \dot{x} = f(x, z, u), \quad \varepsilon \dot{z} = g(x, z, u) \]  

1. Approximate models

   ▶ Set \( \varepsilon = 0 \Rightarrow z = h(x, u_s) \)

   \[ \dot{x} = f(x, h(x, u_s), u_s) \]  

   ▶ Change coordinates \( y := z - h(x, u_s) \), \( \tau = \frac{t - t_0}{\varepsilon} \)

   \[ \frac{dy}{d\tau} = g(x, y + h(x, u_f), u_f) \]  

2. Stabilize (2), (3) separately

3. \( u = u_s + u_f \) stabilizes (1) (under some conditions)
Approximate models

\[
\begin{align*}
\dot{x} &= f(x, h(x, u_s), u_s) \quad \text{(slow)} \\
\frac{dy}{d\tau} &= g(x, y + h(x, u_f), u_f) \quad \text{(fast)}
\end{align*}
\]

1. Ignore the fast (stable) dynamics
2. Stabilize only slow dynamics \( u_s = k(x) \) \under{\text{under conditions}}
3. \( u = u_s \) stabilize original system

▶ **Intuition:** \( u_s \) is robust w.r.t. unmodelled dynamics (\( \varepsilon \) small)

▶ **Pros:** simplify the control design problem
Objective

- Event-triggered control
  - nonlinear singularly perturbed systems
  - based only on approximate slow dynamics
- The problem not addressed in the literature

Challenges

- Zeno phenomenon $\Rightarrow$ due to neglected fast dynamics
- Guarantee stability of the original system

stability of approximate models $\not\Rightarrow$ stability of original system
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Counter example [Christofides & Teel, IEEE TAC’96]

Original model

\[
\begin{align*}
\dot{x} &= -x^3 + \varepsilon x \\
\varepsilon \dot{z} &= -z
\end{align*}
\]

Approximate slow

Set \( \varepsilon = 0 \) \( \Rightarrow \) \( z = h(x) = 0 \)
\[
\dot{x} = -x^3 \quad \text{(GAS)}
\]

Approximate fast

\[
\begin{align*}
h(x) &= 0 \Rightarrow y = z - h(x) = z \\
\frac{dy}{d\tau} &= -y \quad \text{(GES)}
\end{align*}
\]

Check original system

Jacobian matrix

\[
\begin{bmatrix}
-3x^2 + \varepsilon & 0 \\
0 & -\frac{1}{\varepsilon}
\end{bmatrix}
\]

\( x = 0, z = 0 \)

\[
\begin{bmatrix}
\varepsilon & 0 \\
0 & -\frac{1}{\varepsilon}
\end{bmatrix}
\]

\( \Rightarrow \) Unstable
Introduction

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Design approach

We design event-triggered controllers by emulation:

1. stabilize the plant in continuous-time (neglect sampling)
2. consider sampling
3. design the triggering condition to maintain stability

Pros: versatile tools to handle nonlinear systems
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Emulation approach

Plant model

\begin{align*}
\dot{x} &= f(x, z, u) \\
\varepsilon \dot{z} &= g(x, z, u)
\end{align*}

Approximate models

\begin{align*}
\dot{x} &= f(x, h(x, u_s), u_s) \quad \text{(slow)} \\
\frac{dy}{d\tau} &= g(x, y + h(x, u_f), u_f) \quad \text{(fast, stable)}
\end{align*}

1. Stabilize approximate slow (no sampling)

\[ u_s = K(x(t)) \]

2. Consider the network \( u_s = K(x(t_i)) \)

\begin{align*}
e_x(t) &= x(t_i) - x(t) \quad \forall t \in [t_i, t_{i+1}) \\
e_x(t_{i+}^+) &= 0
\end{align*}

3. Event-triggering condition

4. Verify stability of original system \( u = u_s \)
Hybrid model

- The closed-loop system exhibits interaction
  - continuous-time evolutions \((\dot{x}, \dot{y}, \dot{e}_x) \Rightarrow \text{ODE}\)
  - discrete transitions \((e^+_x, y^+) \Rightarrow \text{difference equations}\)

- Hybrid model truly describe system behaviour
Hybrid model [Goebel et al., Hybrid Dynamical Systems]

\[ q = (x, y, e_x) \]

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{e}_x
\end{pmatrix}
= \begin{pmatrix}
f_x(x, y, e_x) \\
\frac{1}{\epsilon} f_y(x, y, e_x) \\
-f_x(x, y, e_x)
\end{pmatrix}
q \in C,
\begin{pmatrix}
x^+ \\
y^+ \\
e_{x}^+
\end{pmatrix}
= \begin{pmatrix}
x \\
h_y(x, y, e_x) \\
0
\end{pmatrix}
q \in D
\]

Problem statement

Design approach

Based only on \( x \)

Stability of overall system

\[ t_{i+1} - t_i \geq \Delta \text{ (dwell-time)} \]

Challenges

- \( y \) changes at jumps
- Zeno
Main result

- Semiglobal practical stabilization
- Global asymptotic stabilization
## Semiglobal practical stabilization

### Assumptions

<table>
<thead>
<tr>
<th>Plant model</th>
<th>Approximate models</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{x} = f(x, z, e_x) )</td>
<td>( \dot{x} = f_s(x, e_x) ) (slow)</td>
</tr>
<tr>
<td>( \varepsilon \dot{z} = g(x, z, e_x) )</td>
<td>( \frac{dy}{d\tau} = g_f(x, y, e_x) ) (fast)</td>
</tr>
</tbody>
</table>

1. \( g(x, z, u) = 0 \) has \( n \geq 1 \) isolated real roots

2. The origin of the approximate fast dynamic is GAS

3. The approximate slow model is ISS w.r.t. \( e_x \)

4. The interconnection between the slow and fast dynamics satisfy some conditions

5. The dynamics of \( V_y \) along jumps of the states \( x, y \) satisfy some conditions
Semiglobal practical stabilization
Triggering condition

**Approximate slow model ISS w.r.t. \( e_x \)**

\[ \exists V: \text{continuously differentiable, } \alpha_1 > 0, \ \alpha_x, \overline{\alpha}_x, \gamma \in \mathcal{K}_\infty \text{ s.t.} \]

\[
\begin{align*}
\alpha_x(|x|) & \leq V(x) \leq \overline{\alpha}_x(|x|), & \forall x \in \mathbb{R}^{n_x} \\
\frac{\partial V_x}{\partial x} f(x, e) & \leq -\alpha_1 V_x(x) + \gamma(|e_x|), & \forall (x, e) \in \mathbb{R}^{2n_x}
\end{align*}
\]

Tabuada (TAC’07)

\[
\gamma(|e_x|) \leq \sigma \alpha_1 V_x(x), \quad \sigma \in (0, 1) \quad (1)
\]

- **Issue:** \((x, e_x) = (0, 0) \Rightarrow (1) \text{ always violated} \Rightarrow \text{Zeno}\)
Semiglobal practical stabilization

Triggering condition

Approximate slow model ISS w.r.t. $e_x$

\[ \exists V: \text{continuously differentiable, } \alpha_1 > 0, \alpha_x, \alpha_x, \gamma \in \mathcal{K}_\infty \text{ s.t.} \]
\[ \alpha_x(|x|) \leq V(x) \leq \alpha_x(|x|), \quad \forall x \in \mathbb{R}^{n_x} \]
\[ \frac{\partial V}{\partial x} f(x, e) \leq -\alpha_1 V_x(x) + \gamma(|e_x|), \quad \forall (x, e) \in \mathbb{R}^{2n_x} \]

Tabuada (TAC’07)

\[ \gamma(|e_x|) \leq \sigma \alpha_1 V_x(x), \quad \sigma \in (0, 1) \quad (1) \]

- **Issue**: $(x, e_x) = (0, 0) \Rightarrow (1)$ always violated $\Rightarrow$ Zeno

- **Idea**: add a dead zone to the triggering threshold

\[ \gamma(|e_x|) \leq \max\{\sigma \alpha_1 V_x(x), \rho\} \]
Semiglobal practical stabilization

Result

$q = (x, y, e_x)$

\[ \dot{q} = F(q) \quad q \in C, \quad q^+ = G(q) \quad q \in D \]

\[ C = \left\{ q : \gamma(|e_x|) \leq \max\{\sigma \alpha_1 V_x(x), \rho\} \right\} \]

\[ D = \left\{ q : \gamma(|e_x|) \geq \max\{\sigma \alpha_1 V_x(x), \rho\} \right\} \]

Theorem

For any $\Delta, \rho > 0$, there exist $\beta_\Delta \in \mathcal{K}\mathcal{L}$, $\gamma_\Delta \in \mathcal{K}$ and $\epsilon^*(\Delta) > 0$ s.t. for any $\epsilon(\Delta) \in (0, \epsilon^*(\Delta))$, any solution $(x, y, e_x)$ with $|(x(0, 0), y(0, 0), e_x(0, 0))| \leq \Delta$ satisfies, $\forall (t, j) \in \text{dom} (x, y, e_x)$

\[ |(x(t, j), y(t, j))| \leq \beta_\Delta(|(x(0, 0), y(0, 0), e_x(0, 0))|, t + j) + \gamma_\Delta(\rho) \]

Moreover, if $(x, y, e_x)$ is maximal, then it is complete.

⇒ Tradeoff between $\tau_{\min}$ and the convergence region
Main result

- Semiglobal practical stabilization

- Global asymptotic stabilization
Global asymptotic stabilization

Model

We introduce a timer

\[
\dot{\tau} = 1 \quad \forall t \in [t_i, t_{i+1}), \quad \tau^+ = 0 \quad t = t_i^+
\]

\[q = (x, y, e_x, \tau)\]

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{e}_x \\
\dot{\tau}
\end{pmatrix}
= \begin{pmatrix}
f_x(x, y, e_x) \\
\frac{1}{\varepsilon} f_y(x, y, e_x) \\
-f_x(x, y, e_x) \\
1
\end{pmatrix}
q \in C,
\]

\[
\begin{pmatrix}
x^+ \\
y^+ \\
e_{x}^+ \\
\tau^+
\end{pmatrix}
= \begin{pmatrix}
x \\
h_y(x, y, e_x) \\
0 \\
0
\end{pmatrix}
\]

Stronger stability property \(\Rightarrow\) additional condition
Global asymptotic stabilization
Assumptions

Same conditions as before

1. \( g(x, z, u) = 0 \) has \( n \geq 1 \) isolated real roots

2. The origin of the approximate fast dynamic is GAS

3. The approximate slow model is ISS w.r.t. \( e_x \)

4. The interconnection between the slow and fast dynamics satisfy some conditions

5. The dynamics of \( V_y \) along jumps of the states \( x, y \) satisfy some conditions

Additional condition

\[ \exists M, N \geq 0 \text{ such that, } \forall (x, y) \text{ and for almost all } e_x \]

\[ \langle \nabla |e_x|, -f_x(x, y, e_x) \rangle \leq M|e_x| + N(\sqrt{V_x(x)} + \sqrt{V_y(x, y)}) \]
Global asymptotic stabilization
Event-triggering mechanism

\[
\frac{\partial V_x}{\partial x} f(x, e) \leq -\alpha_1 V_x(x) + \gamma^2 |e_x|^2
\]
\[
\langle \nabla |e_x|, -f_x(x, y, e_x) \rangle \leq M |e_x| + N (\sqrt{V_x(x)} + \sqrt{V_y(x, y)})
\]

Time-triggered [Nešić et al., IEEE TAC’09]

\[ C = \{ q : \tau \in [0, T] \} \]
\[ D = \{ q : \tau = T \} \]
\[ \Rightarrow \quad T < T(\gamma, M, N) \quad \text{(MATI)} \]
Global asymptotic stabilization
Event-triggering mechanism

\[
\frac{\partial V_x}{\partial x} f(x, e) \leq -\alpha_1 V_x(x) + \gamma^2 |e_x|^2 \\
\langle \nabla |e_x|, -f_x(x, y, e_x) \rangle \leq M|e_x| + N(\sqrt{V_x(x)} + \sqrt{V_y(x, y)})
\]

Time-triggered [Nešić et al., IEEE TAC’09]

\[C = \{ q : \tau \in [0, T] \} \quad \Rightarrow \quad T < T(\gamma, M, N) \quad (MATI)\]

\[D = \{ q : \tau = T \}\]

Event-triggered [Tabuada, IEEE TAC’07]

\[C = \{ q : \gamma^2 |e_x|^2 \leq \sigma \alpha_1 V_x(x) \} \]

\[D = \{ q : \gamma^2 |e_x|^2 \geq \sigma \alpha_1 V_x(x) \} \]

**Idea:** combine event-triggering and time-triggering
Global asymptotic stabilization
The proposed mechanism

Time-triggered [Nešić et al., IEEE TAC’09]

\[ C = \left\{ q : \tau \in [0, T] \right\} \]
\[ D = \left\{ q : \tau = T \right\} \]

Event-triggered [Tabuada, IEEE TAC’07]

\[ C = \left\{ q : \gamma^2 |e_x|^2 \leq \sigma \alpha V_x(x) \right\} \]
\[ D = \left\{ q : \gamma^2 |e_x|^2 \geq \sigma \alpha V_x(x) \right\} \]

\[ C = \left\{ q : \gamma^2 |e_x|^2 \leq \sigma \alpha V_x(x) \text{ or } \tau \in [0, T] \right\} \]
\[ D = \left\{ q : \gamma^2 |e_x|^2 \geq \sigma \alpha V_x(x) \text{ and } \tau \geq T \right\} \]

- \( t_{i+1} - t_i \geq T \Rightarrow \text{No Zeno} \)
- \( \gamma^2 |e_x|^2 \leq \sigma \alpha V_x(x) \) allow to reduce transmissions
- Outperform (or at least) periodic setups of [Nešić et al.]
- Ensure GAS property
Global asymptotic stabilization
Result

**Additional condition**

There exist $M, N \geq 0$ such that, $\forall (x, y)$ and for a.a. $e_x$

$$\langle \nabla |e_x|, -f_x(x, y, e_x) \rangle \leq M|e_x| + N(\sqrt{V_x(x)} + \sqrt{V_y(x, y)})$$

**Theorem**

There exist $\beta \in KL$ and $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$, any solution $(x, y, e_x, \tau)$ satisfies, $\forall (t, j) \in \text{dom } (x, y, e_x, \tau)$

$$|x(t, j), y(t, j)| \leq \beta(|(x(0, 0), y(0, 0), e_x(0, 0), \tau(0, 0))|, t + j)$$

Moreover, if $(x, y, e_x, \tau)$ is maximal, then it is complete.
Case study

We show that all conditions are satisfied by

- Globally Lipschitz systems
- Linear systems
Example: control of F-8 aircraft

**Plant model** [Kokotović et al., Academic Press’86]

\[
\begin{align*}
\dot{x} &= A_{11} x + A_{12} z + B_1 u \\
\varepsilon \dot{z} &= A_{21} x + A_{22} z + B_2 u
\end{align*}
\]

\(x \in \mathbb{R}^2 \quad z \in \mathbb{R}^2 \quad \varepsilon = 0.0336\)

- **Approximate models**

\[
\begin{align*}
\dot{x} &= A_0 x + B_0 u & \text{(slow)} \\
\frac{dy}{d\tau} &= A_{22} y & \text{(fast)}
\end{align*}
\]

- \(V_x(x) = x^T P_1 x, \quad V_y(x, y) = y^T P_2 y \quad \Rightarrow \quad \text{all conditions hold}\)
Example: control of F-8 aircraft

Semiglobal practical stabilization

\[ \gamma_1 |e(x)|^2 \leq \max \{ \sigma \alpha_1 V_x(x), \rho \} \]
Example: control of F-8 aircraft
Global asymptotic stabilization

\[ \gamma_1 |e_x|^2 \leq \sigma \alpha_1 V_x(x) \text{ or } \tau \in [0, T^*] \]
Conclusion

- Event-triggered of nonlinear singularly perturbed systems
- Triggering condition based only on slow dynamics
- Zeno phenomenon is prevented
- Semiglobal practical stabilization and GAS (additional condition)
Conclusion

- Event-triggered of nonlinear singularly perturbed systems
- Triggering condition based only on slow dynamics
- Zeno phenomenon is prevented
- Semiglobal practical stabilization and GAS (additional condition)

Future research

- Extension to the general case (the fast dynamic not stable)
- Investigate the strategy for other classes of nonlinear systems
- Output feedback event-triggered control
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Event-triggered control of nonlinear singularly perturbed systems based only on the slow dynamics.

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Output feedback event-triggered control.

Thank You!