Analysis and control of large scale-networks

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Network structure

- Networks are obtained interconnecting several smaller networks/clusters.
- For analysis purposes we can partition large networks into smaller ones.
- Communication or energy saving strategies may lead to partitions.
Motivation

- Networks of mobile agents with communication/energy saving constraints
- Social networks
Framework

• The network is partitioned in several clusters.

• The state of the agents evolves continuously by taking into account neighbors belonging to the same cluster.

• Some agents can update their state at some discrete time instants by taking into account the state of neighbors belonging to different clusters.

• The activation of inter-clusters connections
  ○ synchronous
  ○ asynchronous
Consensus in networks with clustered pattern

Synchronized resets

- Characterization of consensus value
- Control of consensus value and convergence speed
Consensus in networks with clustered pattern

Asynchronous resets

- **Time-triggered reset instants**
  - quasi-periodic resets

- **Event-triggered reset instants**

\[
t_{\phi_i}(k) = \min_{t \geq t_{\phi_i}(k-1)} \left\{ d_i(t) \leq \frac{d_i(t_{\phi_i}(k-1))}{a_i} \right\},
\]

- **Stochastic reset instants**
  - Poisson renewal process
Let $G = (\mathcal{V}, \mathcal{E})$ be a digraph
$\mathcal{V}$ partitioned in $m$ clusters $\mathcal{C}_1, \ldots, \mathcal{C}_m$
x$_i$ the state of agent $i$, $x = (x_1, \ldots, x_n) \top$
x$_{C_i}$ the vector collecting the states of agents in $\mathcal{C}_i$

Dynamics:

\[
\begin{cases}
\dot{x}(t) = -Lx(t), & \forall t \in \mathbb{R}_+ \setminus \mathcal{T} \\
x(t_k) = P(t_k)x(t_k^-) & \forall t_k \in \mathcal{T} \\
x(0) = x_0
\end{cases}
\] (1)

where $L = \text{diag}(L_1, L_2, \ldots, L_m)$

Network structure:

Each cluster as well as the network between clusters is weakly connected
Interactions

• ∃α > 0 such that, for all t_k, if P_{i,j}(t_k) ≠ 0, i, j ∈ {1, . . . , n} then P_{i,j}(t_k) ≥ α.

Ensures minimal influence between clusters

• ∃δ_{max} > 0 such that : if (i, j) ∈ E(t_k) an inter-cluster link there exists T ∈ [t_k, t_k + δ_{max}] such that (i, j) ∈ E(T).

Imposes recurrent activation of inter-cluster links
Interactions

- Consider the extraction function

$$\phi_i(h + 1) = \min\{k > \phi_i(h) | \exists u \in C_i, v \in V \setminus C_i, P_{u,v}(t_k) > 0\}.$$ 

$$\exists \delta \in (0, \delta_{\text{max}}] \text{ such that } t_{\phi_i(k+1)} - t_{\phi_i(k)} \geq \delta, \ \forall i \in \{1, \ldots, m\}.$$ 

Guarantees minimal influence within clusters
Remarks

- Dynamics (1) can be rewritten as

\[
\begin{align*}
    x(t) &= e^{-L(t-t_k)}P(t_k)x(t_k^-), \quad \forall k \in \mathbb{N}, \forall t \in [t_k, t_{k+1}) \\
    x(0) &= x_0
\end{align*}
\]  

(2)

- No global minimal dwell time is imposed i.e. \(t_{k+1} - t_k\) is not lower-bounded.

- \[
\begin{align*}
    x_{C_i}(t) &= e^{-L_i(t-t_{\phi_i(k)})}P_{C_i}(t_{\phi_i(k)})x(t_{\phi_i(k)}^-), \quad \forall k \in \mathbb{N}, \forall t \in [t_{\phi_i(k)}, t_{\phi_i(k+1)}) \\
    x(0) &= x_0
\end{align*}
\]  

(3)
Instrumental result

Proposition

Let $\bar{G}$ be a directed graph with $n$ vertices containing a spanning tree with the root $\ell$. Let $\bar{L}$ be its Laplacian matrix. The matrix $e^{-\bar{L}t}$ is stochastic for all $t \geq 0$ and

$$\forall \delta > 0 \ \exists \bar{\alpha} > 0, \delta_M > \delta \text{ such that}$$

$$\begin{cases} (e^{-\bar{L}t})_{i,\ell} > \bar{\alpha}, \\ (e^{-\bar{L}t})_{i,i} > \bar{\alpha} \end{cases}, \forall i \in \{1, \ldots, n\}$$

for all $t \in [\delta, \delta_M]$. 

- For a given $\delta$, we can consider a unique value $\alpha = \min \alpha_i$ satisfying the Minimal influence assumption.
Main result

For all time $t \in \mathbb{R}_+$, we define the global diameter of the group as

$$\Delta(t) = \bar{x}(t) - \underline{x}(t)$$

with

$$\bar{x}(t) = \max_{i \in \{1,...,n\}} x_i(t) \text{ and } \underline{x}(t) = \min_{i \in \{1,...,n\}} x_i(t).$$

**Theorem**

There exists some positive decay rate $\beta \in [0, 1)$ such that for all $t \in \mathbb{R}_+$,

$$\Delta(2m\delta_{\text{max}} + t) \leq \beta \Delta(t)$$
proof ideas

Reset dynamics:

Continuous dynamics:

Combining the previous results repetitively we show that, after some time, no state can be arbitrarily close to $x(t)$. 
Example 1

$$L = \begin{pmatrix} 3 & 0 & -3 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0.7 & 0 & 0 & 0.3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.25 & 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

$$\delta = 4 \text{ and } \delta_{\text{max}} = 8$$

**Figure**: Left: Consensus of the five agents grouped in 2 clusters. Right: Zoom in pointing out that the resets are not synchronized.
Example 2

We consider a graph with 30 vertices partitioned in three clusters and the inter-clusters network strongly connected. \( \delta = 10 \) and \( \delta_{\text{max}} = 20 \).

\[ \delta = 10 \quad \text{and} \quad \delta_{\text{max}} = 20. \]

**Figure**: Cycle inter-cluster structure: the top (blue) cluster influences the middle (green) clusters which influences the bottom (red) cluster which influences the top (blue) cluster.
Event-triggered reset strategies

- cluster diameter monitoring strategy:
  \[ d_i(t) = \max_{j \in C_i} x_j(t) - \min_{j \in C_i} x_j(t) \]

- distance to local agreement strategy:
  \[ d_i(t) = x_i(t) - x_i^*(t_{\phi_i(k-1)}), \quad x_i^*(t) = v_i^\top x_{C_i}(t_{\phi_i(k-1)}) \]

- distance to neighbors strategy
  \[ d_i(t) = \max_{j \in N_i} (x_i(t) - x_j(t)) \]
Assumptions

Network structure

• For each cluster $C_i$, the induced graph $(C_i, \mathcal{E} \cap (C_i \times C_i))$ contains a spanning tree with the root $r_i$,

• eventually re-ordering the clusters, the following holds : for all $i \geq 2$ there exist $j < i$ and $l_i \in C_j$ such that $(l_i, r_i) \in \mathcal{E}$.

Minimal influence $\exists \alpha > 0$ such that, for all $t_k$, if $P_{i,j}(t_k) \neq 0$, $i, j \in \{1, \ldots, n\}$ then $P_{i,j}(t_k) \geq \alpha$

Maximal influence

The components of the Laplacian matrix $L$ are uniformly bounded i.e. there exists $\bar{\alpha} > 0$ finite real number such that $|L_{i,j}| \leq \bar{\alpha}$, $\forall i, j \in \{1, \ldots, n\}$. 
Diameter monitoring strategy

Reset sequence:

**Definition**

Let us recall that $d_i(t) = \max_{j \in C_i} x_j(t) - \min_{j \in C_i} x_j(t)$. The reset sequence $(t_k)_{k \in \mathbb{N}}$ associated with the dynamics (1) is defined as follows: for all $i \in \{1, \ldots, m\}$ and for all $k \geq 0$,

- if $d_i(t_{\phi_i(k-1)}) = 0$, $t_{\phi_i(k)} = t_{\phi_i(k-1)} + \delta$ with 
  \[ \delta = \min_{i \in \{1, \ldots, m\}} \frac{1}{2n_i \alpha} \ln(a_i), \]
- otherwise $t_{\phi_i(k)} = \min_{t \geq t_{\phi_i(k-1)}} \left\{ d_i(t) \leq \frac{d_i(t_{\phi_i(k-1)})}{a_i} \right\}$,

where the $a_i > 1$ are design parameters fixed a priori. (for consistency, we denote $t_{\phi_i(-1)} = 0$).
Distance to local agreement strategy

**Definition**

Let $x_i^*$ the local agreement of the cluster $C_i$ at time $t$. We recall that $d_i(t) = |x_{r_i}(t) - x_i^*(t)|$. Considering $\epsilon > 0$ a fixed scalar, the reset sequence $(t_k)_{k \in \mathbb{N}}$ associated with the dynamics (1) is defined as follows: for all $i \in \{1, \ldots, m\}$ and for all $k \geq 0$,

- if $d_i(t_{\phi_i(k-1)}) > \epsilon$ we define
  
  $$t_{\phi_i(k)} = \min_{t \geq t_{\phi_i(k-1)}} \left\{ d_i(t) \leq \frac{d_i(t_{\phi_i(k-1)})}{a_i} \right\},$$

- otherwise, $t_{\phi_i(k)} = t_{\phi_i(k-1)} + \delta$ with
  
  $$\delta = \min_{i \in \{1, \ldots, m\}} \frac{a_i - 1}{a_i} \frac{\epsilon}{2n_i \bar{\alpha} \Delta_i(0)},$$

where the $a_i > 1$ are design parameters fixed a priori.
Distance to neighbors strategy - fully decentralized

Definition

\[ d_i(t) = \max_{j \in N_{r_i}} | x_{r_i}(t) - x_j(t) | . \]

Let \( \epsilon > 0 \) a fixed scalar for all \( i \in \{1, \ldots, m\} \) and for all \( k \geq 0 \),

- if \( d_i(t_{\phi_i(k-1)}) > \epsilon \) we define

\[ t_{\phi_i(k)} = \min_{t \geq t_{\phi_i(k-1)}} \left\{ d_i(t) \leq \frac{d_i(t_{\phi_i(k-1)})}{a_i} \right\} , \]

- otherwise, \( t_{\phi_i(k)} = t_{\phi_i(k-1)} + \delta \) with

\[ \delta = \min_{i \in \{1, \ldots, m\}} \frac{a_i - 1}{a_i} \frac{\epsilon}{n_i \bar{\alpha} \Delta_i(0)} , \]

where the \( a_i > 1 \) are design parameters fixed a priori.
Main results

**Theorem**

Let us consider the dynamics (1) under assumptions Network structure, Minimal influence and Maximal influence. The associated reset sequence introduced in each of the previous definitions satisfies the assumptions Recurrent activation of inter-cluster links and Dwell time.

**Corollary**

Let us consider the dynamics (1) with the reset rule introduced in Definition 2. If assumptions Network structure, Minimal influence and Maximal influence hold, there exists some positive decay rate $\beta \in [0, 1)$ such that for all $t \in \mathbb{R}_+$,

$$\Delta(2m\delta_{\text{max}} + t) \leq \beta \Delta(t).$$
Numerical examples

\[
L = \begin{pmatrix}
4 & -2 & -2 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
0 & -2 & 2 & 0 & 0 \\
0 & 0 & 0 & 3 & -3 \\
0 & 0 & 0 & -1 & 1 \\
\end{pmatrix}
\] (5)

\[
\begin{cases}
x_1(t_{\phi_1(k)}) = 0.45 x_1(t_{\phi_1(k)}^-) + 0.55 x_5(t_{\phi_1(k)}^-) \\
x_4(t_{\phi_2(k)}) = 0.25 x_3(t_{\phi_2(k)}^-) + 0.75 x_4(t_{\phi_2(k)}^-)
\end{cases}
\]
Role of $a_i$ - second rule

**Figure**: Left: $a_1 = a_2 = 2$. Right: $a_1 = 200$, $a_2 = 2$
Role of $a_i$ - third rule

**Figure**: Left: $a_1 = a_2 = 2$. Right: $a_1 = 200, a_2 = 2$
Time versus event-triggering strategies

\[
L = \begin{pmatrix}
4 & -2 & -2 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
0 & -2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0.1 & -0.1 \\
0 & 0 & 0 & -0.05 & 0.05
\end{pmatrix},
\]

\[
x_1(t_{\phi_1(k)}) = 0.45x_1(t^-_{\phi_1(k)}) + 0.55x_4(t^-_{\phi_1(k)}) \\
x_4(t_{\phi_2(k)}) = 0.25x_1(t^-_{\phi_2(k)}) + 0.75x_4(t^-_{\phi_2(k)}).
\]
Reset sequences

\[ \delta_{\text{min}} = 0.1 \text{s}, \ \delta_{\text{max}} = 1 \text{s}, \ \delta_{\text{ave}} = \frac{\delta_{\text{max}} + \delta_{\text{min}}}{2} \] and the jitter \( \delta' \in [0, 0.001] \).

\[ T_1 = (t_k)_{k \geq 1} \text{ where } t_0 = 0, \ t_{k+1} = t_k + \delta_{\text{min}} + \delta', \ \forall k \geq 0, \]

\[ T_2 = (t_k)_{k \geq 1} \text{ where } t_0 = 0, \ t_{k+1} = t_k + \delta_{\text{max}} + \delta', \ \forall k \geq 0, \]

\[ T_3 = (t_k)_{k \geq 1} \text{ where } t_0 = 0, \ t_{k+1} = t_k + \delta_{\text{ave}} + \delta', \ \forall k \geq 0, \]

\[ T_4(a_1, a_2) = (t_k)_{k \geq 1} \text{ second rule with the parameters } a_1 \text{ and } a_2, \]

\[ T_5(a_1, a_2) = (t_k)_{k \geq 1} \text{ third rule with the parameters } a_1 \text{ and } a_2. \]
Comparison results

**FIGURE**: Left: $t_{\phi_1}(k) = t_{\phi_2}(k) = t_k$ are elements of $\mathcal{T}_1$, Right: $t_{\phi_1}(k) = t_{\phi_2}(k) = t_k$
Comparison results

Figure: Left: $t_{\phi_1}(k) = t_{\phi_2}(k) = t_k$ are elements of $\mathcal{T}_3$. Right: $t_{\phi_1}(k)$ and $t_{\phi_2}(k)$ are elements of $\mathcal{T}_4(20, 40)$
Poisson renewal process

- the occurrence of the reset \( t_{\phi_i(k)} \) for \( k \geq 0 \) is described by the random variable \( T_{\phi_i(k)} \).
- the sequence \( (t_{\phi_i(k)})_k \) follows a Poisson renewal process - the reset instants occurs randomly in time

\[
N_t = \sum_{k=0}^{\infty} \chi(T_{\phi_i(k)} \leq t),
\]

Remark

The process \( N_t \) has independent and stationary increments:

- for all \( 0 \leq t_0 < t_1 < \ldots < t_n \) the random variables \( N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}} \) are independent,
- for all \( t, s \in \mathbb{R}_+ \), \( N_{t+s} - N_t \) and \( N_t \) follow the same distribution - a Poisson one.
Theorem

Consider dynamics (1). Suppose that assumptions Network structure and Minimal influence are satisfied. Also, suppose that no more than one agent in each cluster $C_i$ resets its state and this agent is the one described as $r_i$. Finally, assume that these sequences of resets follow independent Poisson renewal processes. Then, there exists some positive decay rate $\beta \in [0, 1)$ and some positive constant bound $p > 0$ such that for all $t \in \mathbb{R}_+$,

$$\mathbb{P}(\Delta(2m\delta_{\text{max}} + t) \leq \beta \Delta(t)) \geq p.$$ 

Moreover, consensus occurs with probability one.
Conclusions

- consensus problem in heterogeneous network linear and linear impulsive dynamics
- lower bound the convergence speed
- different event-triggering strategies and their interest
- stochastic reset sequences

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