Lyapunov techniques for stabilization of switched linear systems of conservation laws

Pierre-Olivier Lamare\textsuperscript{a}, Antoine Girard\textsuperscript{a}, Christophe Prieur\textsuperscript{b}

\textsuperscript{a}Laboratoire Jean Kuntzmann
\textsuperscript{b}GIPSA-lab

October 17, 2013
GT SDH, Paris
Outline

Motivation

Stability and Stabilizability

Results on the Saint-Venant equation

Conclusion
Motivation – Control of a network of open channels

The non-linear case

\[
\frac{\partial}{\partial t} \begin{pmatrix} H_i \\ V_i \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} H_i V_i \\ \frac{V_i^2}{2} + gH_i \end{pmatrix} = 0, \quad i = 1, \ldots, M. \quad (1)
\]
Motivation – Control of a network of open channels

The non-linear case

\[ \begin{aligned}
\frac{\partial}{\partial t} \begin{pmatrix} H_i \\ V_i \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} H_i V_i \\ \frac{V_i^2}{2} + gH_i \end{pmatrix} &= 0, \quad i = 1, \ldots, M .
\end{aligned} \] (1)
Linearization

Linearization around the steady state in each pool: \((H_i^*, V_i^*)\)
Linearization

Linearization around the steady state in each pool: \((H_i^*, V_i^*)\)

Defining the deviations of the state \(H_i(x, t)\), \(V_i(x, t)\) w.r.t. the steady state as

\[
h_i(x, t) = H_i(x, t) - H_i^*,
\]

\[
v_i(x, t) = V_i(x, t) - V_i^*,
\]
Linearization

Linearization around the steady state in each pool: \((H_i^*, V_i^*)\)
Defining the deviations of the state \(H_i(x, t), V_i(x, t)\) w.r.t. the steady state as

\[
h_i(x, t) = H_i(x, t) - H_i^*, \quad (2)
\]

\[
v_i(x, t) = V_i(x, t) - V_i^*, \quad (3)
\]

it gives

\[
\frac{\partial}{\partial t} \begin{pmatrix} h_i \\ v_i \end{pmatrix} + \begin{pmatrix} V_i^* & H_i^* \\ g & V_i^* \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} h_i \\ v_i \end{pmatrix} = 0. \quad (4)
\]
Change of variables

Making the diagonalization of the system (4) we define the Riemann coordinates $y_i$, $i = 1, \ldots, 2M$:

$$y_i = v_i + \sqrt{\frac{g}{H_i^*}} h_i, \quad i = 1, \ldots, M,$$

$$y_{i+M} = v_i - \sqrt{\frac{g}{H_i^*}} h_i, \quad i = M + 1, \ldots, 2M. \quad (5)$$

To be sure that the matrix of the system in (4) is diagonalizable, the following assumption holds:

$$gH_i^* - V_i^* 2i > 0, \quad i = 1, \ldots, M. \quad (7)$$

This assumption is called fluviality assumption.

Thus the system is rewritten:

$$\partial_t y + \Lambda \partial_x y = 0,$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{2M})$ and $y = (y_1, \ldots, y_{2M})^\top$. 

P.-O. Lamare, LJK Switched Hyperbolic Systems
Change of variables

Making the diagonalization of the system (4) we define the Riemann coordinates $y_i, i = 1, \ldots, 2M$:

$$y_i = v_i + \sqrt{\frac{g}{H_i^*}} h_i, \quad i = 1, \ldots, M,$$

(5)

$$y_{i+M} = v_i - \sqrt{\frac{g}{H_i^*}} h_i, \quad i = M + 1, \ldots, 2M.$$  

(6)

To be sure that the matrix of the system in (4) is diagonalizable, the following assumption holds:

$$gH_i^* - V_i^{*2} > 0, \quad i = 1, \ldots, M.$$  

(7)

This assumption is called fluviality assumption.
**Change of variables**

Making the diagonalization of the system (4) we define the Riemann coordinates $y_i$, $i = 1, \ldots, 2M$:

$$ y_i = v_i + \sqrt{\frac{g}{H_i^*}} h_i , \quad i = 1, \ldots, M , $$

$$ y_{i+M} = v_i - \sqrt{\frac{g}{H_i^*}} h_i , \quad i = M + 1, \ldots, 2M . $$

To be sure that the matrix of the system in (4) is diagonalizable, the following assumption holds:

$$ gH_i^* - V_i^*^2 > 0 , \quad i = 1, \ldots, M . $$

This assumption is called *fluviality assumption*. Thus the system is rewritten:

$$ \partial_t y + \Lambda \partial_x y = 0 . $$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{2M})$ and $y = (y_1, \ldots, y_{2M})^\top$. 


---

P.-O. Lamare, LJK

Switched Hyperbolic Systems
Boundary conditions

The channel is provided with underflow sluice gates with gate openings $u^j_i$ ($j$ is the mode)

$$H_i(t, L)V_i(t, L) = u^j_i(t)\mu_0 l \sqrt{2g \left( H_i(t, L) - H_{i+1}(t, 0) \right)}, \quad i = 1, \ldots, M-1$$

(9)

where $\mu_0$ is a positive coefficient and $l$ the width of the canal.
Boundary conditions

The channel is provided with underflow sluice gates with gate openings $u^j_i$ ($j$ is the mode)

$$H_i(t, L)V_i(t, L) = u^j_i(t)\mu_0 l \sqrt{2g \left( H_i(t, L) - H_{i+1}(t, 0) \right)} , \ i = 1, \ldots, M-1$$

(9)

where $\mu_0$ is a positive coefficient and $l$ the width of the canal.

For the last gate we have

$$H_M(t, L)V_M(t, L) = u^j_i(t)\mu_0 l \sqrt{2g \left( H_M(t, L) - H_{\text{down}} \right)}$$

(10)

where $H_{\text{down}} > 0$ denoted the down stream level.
Boundary conditions

The channel is provided with underflow sluice gates with gate openings \( u^j_i \) (\( j \) is the mode)

\[
H_i(t, L)V_i(t, L) = u^j_i(t)\mu_0 l \sqrt{2g \left( H_i(t, L) - H_{i+1}(t, 0) \right)}, \quad i = 1, \ldots, M-1
\] (9)

where \( \mu_0 \) is a positive coefficient and \( l \) the width of the canal.

For the last gate we have

\[
H_M(t, L)V_M(t, L) = u^j_i(t)\mu_0 l \sqrt{2g \left( H_M(t, L) - H_{\text{down}} \right)}
\] (10)

where \( H_{\text{down}} > 0 \) denoted the down stream level.

The inflow rate is imposed

\[
H_1(t, 0)V_1(t, 0) = Q_0(t).
\] (11)
Boundary conditions

The channel is provided with underflow sluice gates with gate openings $u_i^j$ ($j$ is the mode)

$$H_i(t, L)V_i(t, L) = u_i^j(t)\mu_0 l \sqrt{2g (H_i(t, L) - H_{i+1}(t, 0))} , \quad i = 1, \ldots, M-1$$

where $\mu_0$ is a positive coefficient and $l$ the width of the canal. For the last gate we have

$$H_M(t, L)V_M(t, L) = u_i^j(t)\mu_0 l \sqrt{2g (H_M(t, L) - H_{down})}$$

where $H_{down} > 0$ denoted the down stream level.

The inflow rate is imposed

$$H_1(t, 0)V_1(t, 0) = Q_0(t).$$

The last boundary conditions imposed the flow conservation between the pools

$$H_i(t, L)V_i(t, L) = H_{i+1}(t, 0)V_{i+1}(t, 0), \quad i = 1, \ldots, M - 1.$$
In Riemann coordinates, boundary conditions (9) and (10) are equivalent to

\[ y_{i+M}(t, L) = -k_i^j y_i(t, L), \quad i = 1, \ldots, M, \quad (13) \]

for a suitable choice of \( u_i^j \):
In Riemann coordinates, boundary conditions (9) and (10) are equivalent to

\[ y_{i+M}(t, L) = -k_i^j y_i(t, L), \quad i = 1, \ldots, M, \]  

(13)

for a suitable choice of \( u_j^i \): 

\[ u_j^i(t) = \frac{H_i(t, L) \left( \frac{1-k_i^j}{1+k_i^j} \right) (H_i(t, L) - H_i^*) + V_i^*}{\mu_0 \sqrt{H_i(t, L) - H_{i+1}(t, 0)}}, \quad i = 1, \ldots, M - 1, \]  

(14)

and

\[ u_j^M(t) = \frac{H_M(t, L) \left( \frac{1-k_M^j}{1+k_M^j} \right) (H_i(t, L) - H_M^*) + V_M^*}{\mu_0 \sqrt{H_i(t, L) - H_{down}(t, 0)}}, \]  

(15)
Motivation for switching and relative questions

Considering discontinuous controllers may add some degree of freedom w.r.t. continuous controllers.

Improve the convergence of the system to the desired behavior.

Drawback: the question of existence of solutions in the case of switched system can be hard to resolve. See for example Tucsnak, Sigalotti or Hante.

For the rest of the presentation, we suppose there exists a solution to the closed loop system presented below.
Motivation for switching and relative questions

- Considering discontinuous controllers may add some degree of freedom w.r.t. continuous controllers.
Motivation for switching and relative questions

- Considering discontinuous controllers may add some degree of freedom w.r.t. continuous controllers.
- Improve the convergence of the system to the desired behavior.
Motivation for switching and relative questions

- Considering discontinuous controllers may add some degree of freedom w.r.t. continuous controllers.
- Improve the convergence of the system to the desired behavior

**Drawback:** the question of existence of solutions in the case of switched system can be hard to resolve. See for example Tucsnak, Sigalotti or Hante.
Motivation for switching and relative questions

- Considering discontinuous controllers may add some degree of freedom w.r.t. continuous controllers.
- Improve the convergence of the system to the desired behavior.

**Drawback:** the question of existence of solutions in the case of switched system can be hard to resolve. See for example Tucsnak, Sigalotti or Hante.

For the rest of the presentation, we suppose there exists a solution to the closed loop system presented below.
Framework of the study

We are concerned with \( n \times n \) linear hyperbolic systems of conservation laws of the form:

\[
\begin{align*}
\partial_t y(x, t) + \Lambda_{\sigma(t)} \partial_x y(x, t) &= 0, \\
& t \in [0, +\infty), \ x \in (0, 1), \\
y(0, t) &= G_{\sigma(t)} y(1, t), \quad t \in [0, \infty), \\
y(x, 0) &= y^0(x) \ x \in [0, 1],
\end{align*}
\]

(16a) (16b) (16c)

where \( y : [0, 1] \times [0, +\infty) \to \mathbb{R}^n \), \( \sigma : \mathbb{R}_+ \to \mathcal{I} \) is a piecewise constant function, called the Switching signal. \( \mathcal{I} = \{1, \ldots, N\} \) is a finite set of index. \( y^0 \in C_{pw} ([0, 1]) \).
Framework of the study

We are concerned with $n \times n$ linear hyperbolic systems of conservation laws of the form:

\[
\begin{align*}
\partial_t y(x, t) + \Lambda_{\sigma(t)} \partial_x y(x, t) &= 0, \\
& \quad t \in [0, +\infty), \ x \in (0, 1), \\
y(0, t) &= G_{\sigma(t)} y(1, t), \quad t \in [0, \infty), \\
y(x, 0) &= y^0(x) \quad x \in [0, 1],
\end{align*}
\]

(16a)

(16b)

(16c)

where $y : [0, 1] \times [0, +\infty) \to \mathbb{R}^n$, $\sigma : \mathbb{R}_+ \to \mathcal{I}$ is a piecewise constant function, called the Switching signal. $\mathcal{I} = \{1, \ldots, N\}$ is a finite set of index. $y^0 \in C_{pw} ([0, 1])$.

For all $i \in \mathcal{I}$, $\Lambda_i$ is a positive diagonal matrix in $\mathbb{R}^{n \times n}$.
Space of solutions

Definition
A piecewise continuous function $y : [0, 1] \rightarrow \mathbb{R}^n$ is a continuous function on $[0, 1]$ except maybe on a finite number of points $0 = x_0 < x_1 < \cdots < x_p = 1$ such that for all $l \in \{0, \ldots, p - 1\}$ there exists $y_l$ continuous on $[x_l, x_{l+1}]$ and $y_l = y|_{(x_l, x_{l+1})}$. The set of all piecewise continuous functions is denoted by $C_{pw}([0, 1])$. 
Remark

Considering positive definite matrices $\Lambda_i$ is made for the sake of simplicity.
Remark

Considering positive definite matrices $\Lambda_i$ is made for the sake of simplicity. In reality the matrices $\Lambda_i$ are of the form

$$\Lambda_i = \begin{pmatrix} \Lambda_i^+ & 0_{n-m,m} \\ 0_{m,n-m} & \Lambda_i^- \end{pmatrix},$$

where $\Lambda_i^+$ and $\Lambda_i^-$ are respectively diagonal positive definite matrices and diagonal negative definite matrices.
Remark

Considering positive definite matrices $\Lambda_i$ is made for the sake of simplicity. In reality the matrices $\Lambda_i$ are of the form

$$\Lambda_i = \begin{pmatrix} \Lambda_i^+ & 0_{n-m,m} \\ 0_{m,n-m} & \Lambda_i^- \end{pmatrix},$$

where $\Lambda_i^+$ and $\Lambda_i^-$ are respectively diagonal positive definite matrices and diagonal negative definite matrices. We introduce the notations

$$y^+ = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \quad y^- = \begin{pmatrix} y_{m+1} \\ \vdots \\ y_n \end{pmatrix}$$

such that $y = \begin{pmatrix} y^+ \\ y^- \end{pmatrix}$. 

P.-O. Lamare, LJK Switched Hyperbolic Systems
Remark

Considering positive definite matrices \( \Lambda_i \) is made for the sake of simplicity. In reality the matrices \( \Lambda_i \) are of the form

\[
\Lambda_i = \begin{pmatrix}
\Lambda_i^+ & 0_{n-m,m} \\
0_{m,n-m} & \Lambda_i^-
\end{pmatrix},
\]

where \( \Lambda_i^+ \) and \( \Lambda_i^- \) are respectively diagonal positive definite matrices and diagonal negative definite matrices. We introduce the notations

\[
y^+ = \begin{pmatrix}
y_1 \\
\vdots \\
y_m
\end{pmatrix} \quad y^- = \begin{pmatrix}
y_{m+1} \\
\vdots \\
y_n
\end{pmatrix}
\]
such that \( y = \begin{pmatrix} y^+ \\ y^- \end{pmatrix} \).

By the change of variable \( z(x, t) = \begin{pmatrix} y^+(x,t) \\ y^-(1-x,t) \end{pmatrix} \) we obtain a new equation in the same form as (16).
Stability and Stabilization

**Exponential Stability**

There exist $c > 0$ and $\alpha > 0$ such that all solutions of (16) satisfy the inequality

$$|y(\cdot, t)|_{L^2([0,1])} \leq c e^{-\alpha t} |y^0|_{L^2([0,1])},$$

(17)

for all $t \geq 0$.

**Stabilization**

Design $\sigma : \mathbb{R}^+ \to \mathcal{I}$ such that the solutions of (16) satisfy (17)
Based on Lyapunov techniques.

The candidate Lyapunov function considered is defined by

\[ V(y) = \int_{0}^{1} y^\top Q y e^{-\mu x} \, dx, \quad (18) \]

for all \( y \in C_{pw}([0, 1]) \). \( Q \) is a diagonal positive definite matrix \( Q \in \mathbb{R}^{n \times n} \).
Stability

Based on Lyapunov techniques.
The candidate Lyapunov function considered is defined by

\[ V(y) = \int_0^1 y^\top Q y e^{-\mu x} dx, \quad (18) \]

for all \( y \in C_{pw}([0,1]) \). \( Q \) is a diagonal positive definite matrix \( Q \in \mathbb{R}^{n \times n} \).

2 advantages with this approach:
- no need to know the solution
Stability

Based on Lyapunov techniques.

The candidate Lyapunov function considered is defined by

\[ V(y) = \int_{0}^{1} y^\top Q y e^{-\mu x} \, dx , \]  

(18)

for all \( y \in C_{pw}([0, 1]) \). \( Q \) is a diagonal positive definite matrix \( Q \in \mathbb{R}^{n \times n} \).

2 advantages with this approach:

- no need to know the solution
- computing quadratic Lyapunov functions may be done by solving matrix inequalities
Stabilizability

The time derivative of $V$ along the solutions of the switched system of conservation laws (16) is

$$
\dot{V} = -2 \int_0^1 y(x, t)^\top Q \Lambda_i \partial_x y(x, t) e^{-\mu x} \, dx
$$

$$
= - \left[ y(x, t)^\top Q \Lambda_i y(x, t) e^{-\mu x} \right]_0^1 - \mu \int_0^1 y(x, t)^\top Q \Lambda_i y(x, t) e^{-\mu x} \, dx
$$

$$
\leq y(1, t)^\top \left[ G_i^\top Q \Lambda_i G_i - Q \Lambda_i e^{-\mu} \right] y(1, t) - \mu \lambda V .
$$

where $\lambda = \min_{i \in \{1, \ldots, N\}} \left( \min_{j \in \{1, \ldots, n\}} \lambda_{i,j} \right)$.
Let the sensor value defined as \( w(t) = y(1, t) \).
Let the sensor value defined as \( w(t) = y(1, t) \) and \( \alpha = \frac{1}{2} \lambda \mu \) we obtain:

**Lemma**

\[
\dot{V} \leq -2\alpha V + q_i(w(t))
\]

where \( q_i(w(t)) = q_i(w(t)) = w(t)^\top \left[ G_i^\top Q \Lambda_i G_i - Q \Lambda_i e^{-\mu} \right] w(t) \).
Let the sensor value defined as \( w(t) = y(1, t) \) and \( \alpha = \frac{1}{2} \lambda \mu \) we obtain:

**Lemma**

\[
\dot{V} \leq -2\alpha V + q_i(w(t))
\]

where \( q_i(w(t)) = q_i(w(t)) = w(t)^\top \left[ G_i^\top Q \Lambda_i G_i - Q \Lambda_i e^{-\mu} \right] w(t) \).

Thanks this Lemma it is natural to consider the following switching rule:

\[
\sigma[w](t) = \arg\min_{i \in \{1, \ldots, N\}} q_i(w(t))
\]
Let the sensor value defined as \( w(t) = y(1, t) \) and \( \alpha = \frac{1}{2} \lambda \mu \) we obtain:

**Lemma**

\[
\dot{V} \leq -2\alpha V + q_i(w(t))
\]

where \( q_i(w(t)) = q_i(w(t)) = w(t)^\top \left[ G_i^\top Q \Lambda_i G_i - Q \Lambda_i e^{-\mu} \right] w(t) \).

Thanks this Lemma it is natural to consider the following switching rule:

\[
\sigma[w](t) = \arg\min_{i \in \{1, \ldots, N\}} q_i(w(t))
\]

Let us define the simplex:

\[
\Gamma := \left\{ \gamma \in \mathbb{R}^N \middle| \sum_{i=1}^{N} \gamma_i = 1, \gamma_i \geq 0 \right\}.
\]
Assumption

There exist $\gamma \in \Gamma$, a diagonal definite positive matrix $Q$ and a coefficient $\mu > 0$ such that

$$\sum_{i=1}^{N} \gamma_i \left( G_i^T Q \Lambda_i G_i - e^{-\mu} Q \Lambda_i \right) \leq 0_n .$$

(21)
**Assumption**

There exist $\gamma \in \Gamma$, a diagonal definite positive matrix $Q$ and a coefficient $\mu > 0$ such that

$$\sum_{i=1}^{N} \gamma_i \left( G_i^T Q \Lambda_i G_i - e^{-\mu} Q \Lambda_i \right) \leq 0_n .$$

(21)

**Theorem**

Under Assumption 1, system (16) with switching rule (19) is globally exponentially stable. Letting $V$ has proposed, there exist $c > 0$ such that all solutions of (16) satisfy the inequality

$$|y(., t)|_{L^2([0,1])} \leq c e^{-\mu \lambda t} |y^0|_{L^2([0,1])} ,$$

(22)

for all $t \geq 0$. 

P.-O. Lamare, LJK Switched Hyperbolic Systems
Proof

\[ \dot{V} \leq -2\alpha V + q_{\sigma[w](t)}(w(t)) = -2\alpha V + \min_{i \in \{1, \ldots, N\}} q_i(w(t)). \]
Proof

\[ \dot{V} \leq -2\alpha V + q_{\sigma[w](t)}(w(t)) = -2\alpha V + \min_{i\in\{1,...,N\}} q_i(w(t)). \]

By Assumption 1 there exists \( \gamma \in \Gamma \) such that

\[ \sum_{i=1}^{N} \gamma_i \left( G_i^T Q \Lambda_i G_i - e^{-\mu Q \Lambda_i} \right) \leq 0_n. \]
Proof

\[ \dot{V} \leq -2\alpha V + q_{\sigma[w](t)}(w(t)) = -2\alpha V + \min_{i \in \{1, \ldots, N\}} q_i(w(t)). \]

By Assumption 1 there exists \( \gamma \in \Gamma \) such that

\[ \sum_{i=1}^{N} \gamma_i \left( G_i^T Q \Lambda_i G_i - e^{-\mu} Q \Lambda_i \right) \leq 0_n. \]

Therefore \( \forall t > 0, \sum_{i=1}^{N} \gamma_i q_i(w(t)) \leq 0. \)
Proof

\[ \dot{V} \leq -2\alpha V + q_{\sigma[w](t)}(w(t)) = -2\alpha V + \min_{i \in \{1, \ldots, N\}} q_i(w(t)). \]

By Assumption 1 there exists \( \gamma \in \Gamma \) such that

\[ \sum_{i=1}^{N} \gamma_i \left( G_i^\top Q \Lambda_i G_i - e^{-\mu} Q \Lambda_i \right) \leq 0_n. \]

Therefore \( \forall t > 0, \sum_{i=1}^{N} \gamma_i q_i(w(t)) \leq 0 \). Thanks to the above inequality one gets that there exists at least one \( i \) for which \( q_i(w(t)) \leq 0 \).

Hence \( \forall t > 0, \exists i \in I : q_i(w(t)) \leq 0 \), which gives \( \dot{V} \leq -2\alpha V \).

Finally \( V \) satisfies

\[ V \leq e^{-2\alpha t} V (y^0). \]
Example:

\[
y^0(x) = \begin{pmatrix} \sqrt{2} \sin(3\pi x) \\ \sqrt{2} \sin(4\pi x) \end{pmatrix}
\]

\[
\Lambda_{i \in \{1,2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix}
\]

\[
G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix}
\]

\[
Q = l_2
\]

\[
\mu = 0.1
\]

\[
\gamma_1 = \frac{3}{4} \text{ et } \gamma_2 = \frac{1}{4}
\]
Example:

\[ y^0(x) = \left( \begin{array}{c} \sqrt{2} \sin(3\pi x) \\ \sqrt{2} \sin(4\pi x) \end{array} \right) \]

\[ \Lambda_{i \in \{1,2\}} = \left( \begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right) \]

\[ G_1 = \left( \begin{array}{ccc} 1.1 & 0 \\ -0.3 & 0.1 \end{array} \right) \]

\[ G_2 = \left( \begin{array}{cc} 0 & 0.2 \\ 0.1 & -1 \end{array} \right) \]

\[ Q = I_2 \]

\[ \mu = 0.1 \]

\[ \gamma_1 = \frac{3}{4} \text{ et } \gamma_2 = \frac{1}{4} \]
Example:

- \( y^0(x) = \begin{pmatrix} \sqrt{2} \sin(3\pi x) \\ \sqrt{2} \sin(4\pi x) \end{pmatrix} \)
- \( \Lambda_{i \in \{1, 2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \)
- \( G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix} \)
- \( G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix} \)
- \( Q = I_2 \)
- \( \mu = 0.1 \)
- \( \gamma_1 = \frac{3}{4} \) et \( \gamma_2 = \frac{1}{4} \)
Example:

\[ y^0(x) = \left( \sqrt{2} \sin(3\pi x) \right) \]

\[ \Lambda_{i \in \{1,2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix} \]

\[ G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix} \]

\[ Q = I_2 \]

\[ \mu = 0.1 \]

\[ \gamma_1 = \frac{3}{4} \text{ et } \gamma_2 = \frac{1}{4} \]
Example :

- $y^0(x) = \left( \frac{\sqrt{2}}{\sqrt{2}} \sin(3\pi x) \right)$
- $\Lambda_{i \in \{1,2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
- $G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix}$
- $G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix}$
- $Q = I_2$
- $\mu = 0.1$
- $\gamma_1 = \frac{3}{4}$ et $\gamma_2 = \frac{1}{4}$
Example:

- $y^0(x) = \left( \frac{\sqrt{2} \sin(3\pi x)}{\sqrt{2} \sin(4\pi x)} \right)$

- $\Lambda_{i \in \{1,2\}} = \left( \begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right)$

- $G_1 = \left( \begin{array}{cc} 1.1 & 0 \\ -0.3 & 0.1 \end{array} \right)$

- $G_2 = \left( \begin{array}{cc} 0 & 0.2 \\ 0.1 & -1 \end{array} \right)$

- $Q = I_2$

- $\mu = 0.1$

- $\gamma_1 = \frac{3}{4}$ et $\gamma_2 = \frac{1}{4}$
**Example:**

\[
y^0(x) = \begin{pmatrix} \sqrt{2} \sin(3\pi x) \\ \sqrt{2} \sin(4\pi x) \end{pmatrix}
\]

\[
\Lambda_{i \in \{1,2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix}
\]

\[
G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix}
\]

\[
Q = I_2
\]

\[
\mu = 0.1
\]

\[
\gamma_1 = \frac{3}{4} \text{ et } \gamma_2 = \frac{1}{4}
\]
Example:

- \( y^0(x) = \begin{pmatrix} \sqrt{2} \sin(3\pi x) \\ \sqrt{2} \sin(4\pi x) \end{pmatrix} \)
- \( \Lambda_{i \in \{1,2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \)
- \( G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix} \)
- \( G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix} \)
- \( Q = I_2 \)
- \( \mu = 0.1 \)
- \( \gamma_1 = \frac{3}{4} \) et \( \gamma_2 = \frac{1}{4} \)
Example:

- $y^0(x) = \begin{pmatrix} \sqrt{2} \sin(3\pi x) \\ \sqrt{2} \sin(4\pi x) \end{pmatrix}$

- $\Lambda_{i \in \{1,2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

- $G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix}$

- $G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix}$

- $Q = I_2$

- $\mu = 0.1$

- $\gamma_1 = \frac{3}{4}$ et $\gamma_2 = \frac{1}{4}$
Example:

- $y^0(x) = \begin{pmatrix} \sqrt{2} \sin(3\pi x) \\ \sqrt{2} \sin(4\pi x) \end{pmatrix}$

- $\Lambda_{i \in \{1,2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

- $G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix}$

- $G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix}$

- $Q = l_2$

- $\mu = 0.1$

- $\gamma_1 = \frac{3}{4}$ et $\gamma_2 = \frac{1}{4}$
Example:

\[ y^0(x) = \begin{pmatrix} \sqrt{2} \sin(3\pi x) \\ \sqrt{2} \sin(4\pi x) \end{pmatrix} \]

\[ \Lambda_{i \in \{1, 2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix} \]

\[ G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix} \]

\[ Q = I_2 \]

\[ \mu = 0.1 \]

\[ \gamma_1 = \frac{3}{4} \text{ et } \gamma_2 = \frac{1}{4} \]
Example:

\[ y^0(x) = \left( \frac{\sqrt{2} \sin(3\pi x)}{\sqrt{2} \sin(4\pi x)} \right) \]

\[ \Lambda_{i \in \{1, 2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix} \]

\[ G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix} \]

\[ Q = I_2 \]

\[ \mu = 0.1 \]

\[ \gamma_1 = \frac{3}{4} \text{ et } \gamma_2 = \frac{1}{4} \]

Assumption 1 holds:

\[ A = \sum_{i=1}^{2} \gamma_i \left( G_i^\top Q \Lambda_i G_i - e^{-\mu} Q \Lambda_i \right) \]

\[ = \begin{pmatrix} -1.1747 & -0.0825 \\ -0.0825 & -0.0923 \end{pmatrix} \]

\[ \text{spec} (A) = \{-1.1809; -0.0861\} \]
Example:

\[ y^0(x) = \begin{pmatrix} \sqrt{2} \sin(3\pi x) \\ \sqrt{2} \sin(4\pi x) \end{pmatrix} \]

\[ \Lambda_{i \in \{1,2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix} \]

\[ G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix} \]

\[ Q = I_2 \]

\[ \mu = 0.1 \]

\[ \gamma_1 = \frac{3}{4} \text{ et } \gamma_2 = \frac{1}{4} \]
Example:

\[ y^0(x) = \begin{pmatrix} \sqrt{2} \sin(3\pi x) \\ \sqrt{2} \sin(4\pi x) \end{pmatrix} \]

\[ \Lambda_{i \in \{1,2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix} \]

\[ G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix} \]

\[ Q = I_2 \]

\[ \mu = 0.1 \]

\[ \gamma_1 = \frac{3}{4} \text{ et } \gamma_2 = \frac{1}{4} \]
Example:

- \( y^0(x) = \begin{pmatrix} \sqrt{2}\sin(3\pi x) \\ \sqrt{2}\sin(4\pi x) \end{pmatrix} \)
- \( \Lambda_{i \in \{1,2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \)
- \( G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix} \)
- \( G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix} \)
- \( Q = I_2 \)
- \( \mu = 0.1 \)
- \( \gamma_1 = \frac{3}{4} \) et \( \gamma_2 = \frac{1}{4} \)
Example:

- $y^0(x) = \left( \begin{array}{c} \sqrt{2} \sin(3\pi x) \\ \sqrt{2} \sin(4\pi x) \end{array} \right)$
- $\Lambda_i \in \{1, 2\} = \left( \begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right)$
- $G_1 = \left( \begin{array}{cc} 1.1 & 0 \\ -0.3 & 0.1 \end{array} \right)$
- $G_2 = \left( \begin{array}{cc} 0 & 0.2 \\ 0.1 & -1 \end{array} \right)$
- $Q = I_2$
- $\mu = 0.1$
- $\gamma_1 = \frac{3}{4}$ et $\gamma_2 = \frac{1}{4}$

$\Rightarrow$ Fast switching, this behavior is most of the time undesirable!
Example:

\[ y^0(x) = \left( \sqrt{2} \sin(3\pi x), \sqrt{2} \sin(4\pi x) \right) \]

\[ \Lambda_{i \in \{1,2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix} \]

\[ G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix} \]

\[ Q = l_2 \]

\[ \mu = 0.1 \]

\[ \gamma_1 = \frac{3}{4} \text{ et } \gamma_2 = \frac{1}{4} \]

=> Fast switching, this behavior is most of the time undesirable!
Seek strategies to avoid this...
Argmin with hysteresis

\[ \text{Inv:} \quad q_{\sigma[w](t^-)}(w(t)) < 0 \]

\[ q_{\sigma[w](t^-)}(w(t)) = 0; \]
\[ \sigma[w](t) := \begin{cases} \arg\min_{i \in \{1, \ldots, N\}} q_i(w(t)) & \text{if } q_{\sigma[w](t^-)}(w[t]) < 0 \\ \arg\min_{i \in \{1, \ldots, N\}} q_i(w(t)) & \text{if } q_{\sigma[w](t^-)}(w(t)) = 0 \end{cases} \]

**Figure:** Argmin with hysteresis automaton.

\[ \sigma[w](t) = \begin{cases} \sigma[w](t^-) & \text{if } q_{\sigma[w](t^-)}(w[t]) < 0 \\ \arg\min_{i \in \{1, \ldots, N\}} q_i(w(t)) & \text{if } q_{\sigma[w](t^-)}(w(t)) = 0 \end{cases} \quad \text{(23)} \]
Argmin with hysteresis

Figure: Comparison between argmin and argmin with hysteresis.
Theorem
Under Assumption 1, system (16) with switching rule (23) is globally exponentially stable. Letting $V$ has proposed, there exist $c > 0$ such that all solutions of (16) satisfy the inequality

$$|y(\cdot, t)|_{L^2([0,1])} \leq ce^{-\mu \lambda t} |y^0|_{L^2([0,1])},$$

(24)

for all $t \geq 0$.

The proof follows the line than with the argmin rule.
Argmin, hysteresis and filter

Thanks to the Lemma it holds:

\[ \dot{V} \leq -2\alpha V + q_{\sigma[w]}(t)(w(t)). \]
Argmin, hysteresis and filter

Thanks to the Lemma it holds:

\[ \dot{V} \leq -2\alpha V + q_{\sigma[w]}(t)(w(t)). \]

Instead imposing that \( q_{\sigma[w]}(t)(w(t)) \leq 0 \) at any time \( t > 0 \), we just imposing that a weighted averaged value of \( q_{\sigma[w]}(s)(w(s)) \) is negative or zero.
Argmin, hysteresis and filter

Thanks to the Lemma it holds:

\[ \dot{V} \leq -2\alpha V + q_{\sigma[w]}(t)(w(t)). \]

Instead imposing that \( q_{\sigma[w]}(t)(w(t)) \leq 0 \) at any time \( t > 0 \), we just imposing that a weighted averaged value of \( q_{\sigma[w]}(s)(w(s)) \) is negative or zero.

With Gronwall Lemma:

\[ V \leq e^{-2\alpha t} V(y^0) + e^{-2\alpha t} \int_0^t e^{2\alpha s} q_{\sigma[w]}(s)(w(s)) ds \]  (25)
Argmin, hysteresis and filter

Thanks to the Lemma it holds:

$$\dot{V} \leq -2\alpha V + q_{\sigma[w]}(t)(w(t)).$$

Instead imposing that $q_{\sigma[w]}(t)(w(t)) \leq 0$ at any time $t > 0$, we just imposing that a weighted averaged value of $q_{\sigma[w]}(s)(w(s))$ is negative or zero.

With Gronwall Lemma:

$$V \leq e^{-2\alpha t} V(y^0) + e^{-2\alpha t} \int_0^t e^{2\alpha s} q_{\sigma[w]}(s)(w(s))ds$$  \hspace{1cm} (25)

If $e^{-2\alpha t} \int_0^t e^{2\alpha s} q_{\sigma[w]}(s)(w(s))ds \leq 0$ then

$$V \leq e^{-2\alpha t} V(y^0)$$
Let us define \( m(t) = e^{-2\alpha t} \int_0^t e^{2\alpha s} q_{\sigma[w](s)}(w(s)) \, ds \).

\[
\dot{m}(t) = -2\alpha m(t) + q_{\sigma[w](t^-)}(w(t)) \quad \text{lnv: } m(t) < 0
\]

\[ m(t) = 0; \quad \sigma[w](t) := \arg\min_{i \in \{1, \ldots, N\}} q_i(w(t)) \]

**Figure**: Argmin, hysteresis and filter automaton.

\[
\sigma[w](t) = \begin{cases} 
\sigma[w](t^-) & \text{if } m(t) < 0 \\
\arg\min_{i \in \{1, \ldots, N\}} q_i(w(t)) & \text{if } m(t) = 0.
\end{cases}
\] (26)
**Theorem**
Under Assumption 1, system (16) with the switching rule (26) is globally exponentially stable. Letting $V$ has proposed, there exist $c > 0$ such that all solutions of (16) satisfy the inequality

$$
|y(\cdot, t)|_{L^2([0,1])} \leq ce^{-\mu \lambda t} |y^0|_{L^2([0,1])},
$$

(27)

for all $t \geq 0$.

The proof follows the line than with the argmin rule.
Comparison of the strategies

\[ y_k(x) = \left( \sqrt{2} \sin((2k - 1)\pi x) \quad \sqrt{2} \sin(2k\pi x) \right)^\top, \quad k = 1, 2, 3, \]
Comparison of the strategies

\[ y_k^0(x) = \left( \sqrt{2} \sin((2k - 1)\pi x) \quad \sqrt{2} \sin(2k\pi x) \right)^\top, \quad k = 1, 2, 3, \]

<table>
<thead>
<tr>
<th>Initial condition ( y_k^0 )</th>
<th>Argmin</th>
<th>Hysteresis</th>
<th>Low-pass filter</th>
</tr>
</thead>
</table>

Theoretical bound on the speed of convergence: 0.05

<table>
<thead>
<tr>
<th>Number of switches by time unit.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 1 )</td>
</tr>
<tr>
<td>( k = 2 )</td>
</tr>
<tr>
<td>( k = 3 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Speed of convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 1 )</td>
</tr>
<tr>
<td>( k = 2 )</td>
</tr>
<tr>
<td>( k = 3 )</td>
</tr>
</tbody>
</table>

**Table**: Comparison of the different switching strategies for the example with three initial conditions in \( L^2([0, 1]) \) basis. Performed during 10 units of time.
Go back to the Saint-Venant equation

The simulation parameters are:

- Length: $L = 100\text{m}$
- Width: $l = 1\text{m}$
- $g = 9.81\ \text{m.s}^{-2}$
- Width: $H_{\text{down}} = 1\text{m}$
- $(H_1^*, H_2^*) = (2.5, 1)\ \text{m}$
- $Q^* = 1\ \text{m}^3\text{.s}^{-1}$
- $(H_1(x, 0), H_2(x, 0)) = (4, 1.4)\ \text{m}$
- $Q(x, 0) = 2\ \text{m}^3\text{.s}^{-1}$
- $(k_1^1, k_2^1) = (1, 0.2)$
- $(k_1^1, k_2^1) = (0.2, 1)$
- $Q = 10^3\ \text{diag} (1.4486, 0.0010, 2.6743, 0.0012)$
- $\mu = 0.775$
Go back to the Saint-Venant equation

The simulation parameters are:

- **Length:** $L = 100\text{m}$
- **Width:** $l = 1\text{m}$
- $g = 9.81 \text{ m.s}^{-2}$
- **Width:** $H_{down} = 1\text{m}$
- $(H_1^*, H_2^*) = (2.5, 1) \text{ m}$
- $Q^* = 1 \text{ m}^3\text{s}^{-1}$
- $(H_1(x, 0), H_2(x, 0)) = (4, 1.4) \text{ m}$
- $Q(x, 0) = 2 \text{ m}^3\text{s}^{-1}$
- $(k_1^1, k_2^1) = (1, 0.2)$
- $(k_1^1, k_2^1) = (0.2, 1)$
- $Q = 10^3\text{diag}(1.4486, 0.0010, 2.6743, 0.0012)$
- $\mu = 0.775$
Go back to the Saint-Venant equation

The simulation parameters are:

- **Length**: \( L = 100 \text{m} \)
- **Width**: \( l = 1 \text{m} \)
- \( g = 9.81 \text{ m.s}^{-2} \)
- **Width**: \( H_{down} = 1 \text{m} \)
- \((H_1^*, H_2^*) = (2.5,1) \text{ m} \)
- \( Q^* = 1 \text{ m}^3\text{s}^{-1} \)
- \((H_1(x,0), H_2(x,0)) = (4,1.4) \text{ m} \)
- \( Q(x,0) = 2 \text{ m}^3\text{s}^{-1} \)
- \((k_1^1, k_2^1) = (1,0.2) \)
- \((k_1^1, k_2^1) = (0.2,1) \)
- \( Q = 10^3 \text{diag} (1.4486, 0.0010, 2.6743, 0.0012) \)
- \( \mu = 0.775 \)
Go back to the Saint-Venant equation

The simulation parameters are:

- Length: \( L = 100\text{m} \)
- Width: \( l = 1\text{m} \)
- \( g = 9.81 \text{ m.s}^{-2} \)
- Width: \( H_{\text{down}} = 1\text{m} \)
- \((H_1^*, H_2^*) = (2.5,1) \text{ m}\)
- \(Q^* = 1 \text{ m}^3\text{s}^{-1}\)
- \((H_1(x,0), H_2(x,0)) = (4,1.4) \text{ m}\)
- \(Q(x,0) = 2 \text{ m}^3\text{s}^{-1}\)
- \((k_1^1, k_2^1) = (1,0.2)\)
- \((k_1^1, k_2^1) = (0.2,1)\)
- \(Q = 10^3 \text{diag}(1.4486, 0.0010, 2.6743, 0.0012)\)
- \(\mu = 0.775\)
Go back to the Saint-Venant equation

The simulation parameters are:

- Length: \( L = 100 \text{m} \)
- Width: \( l = 1 \text{m} \)
- \( g = 9.81 \text{ m.s}^{-2} \)
- Width: \( H_{\text{down}} = 1 \text{m} \)
- \( (H_1^\ast, H_2^\ast) = (2.5,1) \text{ m} \)
- \( Q^\ast = 1 \text{ m}^3 \text{s}^{-1} \)
- \( (H_1(x,0), H_2(x,0)) = (4,1.4) \text{ m} \)
- \( Q(x,0) = 2 \text{ m}^3 \text{s}^{-1} \)
- \( (k_1^1, k_2^1) = (1,0.2) \)
- \( (k_1^1, k_2^1) = (0.2,1) \)
- \( Q = 10^3 \text{diag} (1.4486, 0.0010, 2.6743, 0.0012) \)
- \( \mu = 0.775 \)
Go back to the Saint-Venant equation

The simulation parameters are:

- Length: \( L = 100 \text{m} \)
- Width: \( l = 1 \text{m} \)
- \( g = 9.81 \text{ m.s}^{-2} \)
- Width: \( H_{down} = 1 \text{m} \)
- \((H^*_1, H^*_2) = (2.5,1) \text{ m} \)
- \( Q^* = 1 \text{ m}^3 \text{.s}^{-1} \)
- \((H_1(x,0), H_2(x,0)) = (4,1.4) \text{ m} \)
- \( Q(x,0) = 2 \text{ m}^3 \text{.s}^{-1} \)
- \((k^1_1, k^1_2) = (1,0.2) \)
- \((k^1_1, k^1_2) = (0.2,1) \)
- \( Q = 10^3 \text{diag}(1.4486, 0.0010, 2.6743, 0.0012) \)
- \( \mu = 0.775 \)
Go back to the Saint-Venant equation

The simulation parameters are:

- Length: $L = 100\text{m}$
- Width: $l = 1\text{m}$
- $g = 9.81\ \text{m.s}^{-2}$
- Width: $H_{down} = 1\text{m}$
- $(H_1^*, H_2^*) = (2.5,1)\ \text{m}$
- $Q^* = 1\ \text{m}^3\text{s}^{-1}$
- $(H_1(x,0), H_2(x,0)) = (4,1.4)\ \text{m}$
- $Q(x,0) = 2\ \text{m}^3\text{s}^{-1}$
- $(k_1^1, k_2^1) = (1,0.2)$
- $(k_1^1, k_2^1) = (0.2,1)$
- $Q = 10^3\ \text{diag}(1.4486, 0.0010, 2.6743, 0.0012)$
- $\mu = 0.775$
Go back to the Saint-Venant equation

The simulation parameters are:

- Length: $L = 100m$
- Width: $l = 1m$
- $g = 9.81 \text{ m.s}^{-2}$
- Width: $H_{down} = 1m$
- $(H_1^*, H_2^*) = (2.5, 1) \text{ m}$
- $Q^* = 1 \text{ m}^3\text{s}^{-1}$
- $(H_1(x, 0), H_2(x, 0)) = (4, 1.4) \text{ m}$
- $Q(x, 0) = 2 \text{ m}^3\text{s}^{-1}$
- $(k_1^1, k_2^1) = (1, 0.2)$
- $(k_1^1, k_2^1) = (0.2, 1)$
- \(Q = 10^3 \text{diag} (1.4486, 0.0010, 2.6743, 0.0012)\)
- $\mu = 0.775$
Go back to the Saint-Venant equation

The simulation parameters are:

- Length: $L = 100m$
- Width: $l = 1m$
- $g = 9.81 \text{ m.s}^{-2}$
- Width: $H_{down} = 1m$
- $(H_1^*, H_2^*) = (2.5,1) \text{ m}$
- $Q^* = 1 \text{ m}^3.\text{s}^{-1}$
- $(H_1(x,0), H_2(x,0)) = (4,1.4) \text{ m}$
- $Q(x,0) = 2 \text{ m}^3.\text{s}^{-1}$
- $(k_1^1, k_2^1) = (1,0.2)$
- $(k_1^1, k_2^1) = (0.2,1)$
- $Q = 10^3 \text{diag}(1.4486, 0.0010, 2.6743, 0.0012)$
- $\mu = 0.775$
Go back to the Saint-Venant equation

The simulation parameters are:

- Length: \( L = 100\text{m} \)
- Width: \( l = 1\text{m} \)
- \( g = 9.81 \text{ m.s}^{-2} \)
- Width: \( H_{down} = 1\text{m} \)
- \( (H_1^*, H_2^*) = (2.5, 1) \text{ m} \)
- \( Q^* = 1 \text{ m}^3.\text{s}^{-1} \)
- \( (H_1(x, 0), H_2(x, 0)) = (4, 1.4) \text{ m} \)
- \( Q(x, 0) = 2 \text{ m}^3.\text{s}^{-1} \)
- \( (k_1^1, k_2^1) = (1, 0.2) \)
- \( (k_1^1, k_2^1) = (0.2, 1) \)
- \( Q = 10^3 \text{diag}(1.4486, 0.0010, 2.6743, 0.0012) \)
- \( \mu = 0.775 \)
Go back to the Saint-Venant equation

The simulation parameters are:

- Length: \( L = 100 \text{m} \)
- Width: \( l = 1 \text{m} \)
- \( g = 9.81 \text{ m.s}^{-2} \)
- Width: \( H_{\text{down}} = 1 \text{m} \)
- \((H^*_1, H^*_2) = (2.5,1) \text{ m}\)
- \(Q^* = 1 \text{ m}^3.\text{s}^{-1}\)
- \((H_1(x,0), H_2(x,0)) = (4,1.4) \text{ m}\)
- \(Q(x,0) = 2 \text{ m}^3.\text{s}^{-1}\)
- \((k_1^1, k_2^1) = (1,0.2)\)
- \((k_1^1, k_2^1) = (0.2,1)\)
- \(Q = 10^3 \text{diag}(1.4486, 0.0010, 2.6743, 0.0012)\)
- \(\mu = 0.775\)
Go back to the Saint-Venant equation

The simulation parameters are:

- Length: \( L = 100 \text{m} \)
- Width: \( l = 1 \text{m} \)
- \( g = 9.81 \text{ m.s}^{-2} \)
- Width: \( H_{\text{down}} = 1 \text{m} \)
- \((H_1^*, H_2^*) = (2.5,1) \text{ m} \)
- \(Q^* = 1 \text{ m}^3.\text{s}^{-1} \)
- \((H_1(x,0), H_2(x,0)) = (4,1.4) \text{ m} \)
- \(Q(x,0) = 2 \text{ m}^3.\text{s}^{-1} \)
- \((k_1^1, k_2^1) = (1,0.2) \)
- \((k_1^1, k_2^1) = (0.2,1) \)
- \(Q = 10^3 \text{diag}(1.4486, 0.0010, 2.6743, 0.0012) \)
- \(\mu = 0.775 \)
Comparison of the three strategies

Figure: Evolution of the function $V$ for the different strategies.
Conclusion

- Design of three switching rules.
- Sufficient condition for exponential stability with them.
- Application to a network of open channels governed by the Saint-Venant equations.
Conclusion

- Design of three switching rules.
- Sufficient condition for exponential stability with them.
- Application to a network of open channels governed by the Saint-Venant equations.
Conclusion

- Design of three switching rules.
- Sufficient condition for exponential stability with them.
- Application to a network of open channels governed by the Saint-Venant equations.
Perspectives

- Adding a source term in the system.
- Considering switching as a perturbation.
- Considering matrices diagonalizables in different basis.
- Treating the non-linear case.
Perspectives

- Adding a source term in the system.
- Considering switching as a perturbation.
- Considering matrices diagonalizables in different bases.
- Treating the non-linear case.
Perspectives

- Adding a source term in the system.
- Considering switching as a perturbation.
- Considering matrices diagonalizable in different bases.
- Treating the non-linear case.
Perspectives

- Adding a source term in the system.
- Considering switching as a perturbation.
- Considering matrices diagonalizable in different bases.
- Treating the non-linear case.
Thank for your attention!