Fouad El Hachemi: Phd Student

Jamal Daafouz: Thesis supervisor
Mario Sigalotti: Thesis co-supervisor
Overview

- Motivation
- Singly perturbed systems
- Singly perturbed switched systems
- Planar linear switched systems
- Planar singularly perturbed switched systems
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Motivation

Hot strip mill: A global view
Motivation

Tail end phase: The strip leaves the stands

**Fig.**: HSM lateral view: Dethreading instant on the third stand
Two time scale switched linear model

After linearization, the subsystem corresponding to the $i^{th}$ mode of the switched system can be written in the standard singular perturbation form:

\[
\begin{align*}
\varepsilon \dot{x}^i_1(t) &= M^i_{11}x^i_1(t) + M^i_{12}x^i_2(t) + N^i_{u,1}u(t) + N^i_{d,1}d(t) \\
\dot{x}^i_2(t) &= M^i_{21}x^i_1(t) + M^i_{22}x^i_2(t) + N^i_{u,2}u(t) + N^i_{d,2}d(t) \\
y(t) &= C^i_{y,2}x^i_2(t),
\end{align*}
\]

where $M^i_{11}$ is assumed to be Hurwitz for any $i \in I$ and $\varepsilon = 0.05$. 

Consider the two time scale LTI system:

\[ \dot{x}(t) = M(\varepsilon)x(t), \]  \hspace{1cm} (1)

with

\[ M(\varepsilon) = \begin{bmatrix} \varepsilon^{-1}M_{11} & \varepsilon^{-1}M_{12} \\ M_{21} & M_{22} \end{bmatrix}. \]

\( \varepsilon > 0 \) is a scalar parameter and

\[ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' \in \mathbb{R}^n \]

is the state vector, for all \( t \geq 0 \).
The system (1) may be decoupled into the fast and slow subsystems:

\[ \varepsilon \dot{x}_f(t) = (M_{11} + O(\varepsilon))x_f(t) \]  \hspace{1cm} (2)

\[ \dot{x}_s(t) = (M_s + O(\varepsilon))x_s(t), \]  \hspace{1cm} (3)

with \( x_f(t) = x_1(t) + M_{11}^{-1}M_{12}x_2(t), \) \( x_s(t) = x_2(t) \) and \( M_s = M_{22} - M_{21}M_{11}^{-1}M_{12}. \)

Stability of the slow and fast dynamics \( \Rightarrow \) Stability of the two time scale system \( \dot{x}(t) = M(\varepsilon)x(t) \) for any \( \varepsilon \in (0, \varepsilon_{\text{max}}] \) (Kokotovic et al., 1986).

This property was found to be true for two time scale switched systems under dwell time constraints (Alwan et al., 2008).
Consider the two time scale switched system:

\[ \dot{x}(t) = M^{\sigma(t)}(\varepsilon)x(t), \]

where

\[ \{ M^i(\varepsilon) = \begin{bmatrix} \varepsilon^{-1}M_{11}^i & \varepsilon^{-1}M_{12}^i \\ M_{21}^i & M_{22}^i \end{bmatrix} : i \in \mathcal{I} = \{1, \ldots, N\} \} \]

is a family of matrices and \( \sigma : \mathbb{R}^+ \rightarrow \mathcal{I} \) is the switching signal, which is assumed to be arbitrary.
Switched singularly perturbed systems

In particular, let $\mathcal{I} = \{1, 2\}$, $\varepsilon_{\text{max}} = 0.076$ and

$$M_1^1(\varepsilon) = \begin{bmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 0 & -1 \end{bmatrix}, \quad M_2^2(\varepsilon) = \begin{bmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 5 & -1 \end{bmatrix}.$$ 

Since $M_{11}^1 = M_{11}^2 = -1$, the fast switched system

$$\varepsilon \dot{x}_f(t) = M_{11}^{\sigma(t)} x_f(t)$$

is asymptotically stable for any switching rule.

Since $M_s^1 = M_s^2 = -1$, the slow switched system

$$\dot{x}_s(t) = M_s^{\sigma(t)} x_s(t),$$

is asymptotically stable for any switching rule.
Example of destabilizing switching rule

Consider the following periodic switching rule \((T = \varepsilon \text{ sec})\):

![Diagram showing periodic switching rule]

We obtain a periodic dynamical system characterized by the matrix 
\[ D(\varepsilon) = e^{M_1(\varepsilon)T} e^{M_2(\varepsilon)T} . \]
The computation of the spectral radius of \(D(\varepsilon)\) yields

\[ \rho(D(\varepsilon)) = 1 + 9.5529\varepsilon - 28.7211\varepsilon^2 + O(\varepsilon^3) > 1 \]
for any period \(2T\) and any \(\varepsilon \in (0, \varepsilon_{\text{max}}]\).

Asymptotic stability of the slow and fast switched subsystems \(\not\Rightarrow\) Asymptotic stability of the two time scale switched system.
Example of destabilizing switching rule

Given the following switching rule \( T = 0.35 \text{ sec} \) :

\[ \varepsilon = 0.076 \text{ and } x(0) = [1 \quad 1]' \], we obtain:

\begin{align*}
\text{Graph 1:} & \quad x_1(t) \\
\text{Graph 2:} & \quad x_2(t)
\end{align*}
Assume that there exist matrices $P_f \succ 0$, $Q^i_f \succ 0$, $P_s \succ 0$, $Q^i_s \succ 0$ such that the LMIs

$$M_{11}^i P_f + P_f M_{11}^i ' + Q^i_f \prec 0,$$

$$M_s^i P_s + P_s M_s^i ' + Q^i_s \prec 0,$$

$$\left[
\begin{array}{cc}
Q^i_f & -(M_{11}^i Y^i + P_f M_{21}^i ')
\end{array}
\right]
\left[
\begin{array}{cc}
(\ast)'
Q^i_s - M_{21}^i Y^i - Y^i ' M_{21}^i
\end{array}
\right] \succ 0$$

hold for any $i \in \mathcal{I}$, with $Y^i = - \sum_{h=1, h \neq i}^N M_{11}^{h-1} M_{12}^{h} P_s$. Hence, there exists a positive scalar $\varepsilon_{\text{max}}$ such that the switched system $\dot{x}(t) = M^{\sigma(t)}(\varepsilon)x(t)$ is asymptotically stable for any $\varepsilon \in (0, \varepsilon_{\text{max}}]$ and any switching rule.
Planar SPSS : Problem formulation

Singularly perturbed switched systems (SPSS)

\[
\begin{pmatrix}
\dot{x} \\
\dot{z}
\end{pmatrix} = \begin{pmatrix} A^\varepsilon_{\sigma(t)} & 0 \\
0 & 1/\varepsilon
\end{pmatrix} \begin{pmatrix} x \\
z
\end{pmatrix}
\]

avec \( x \in \mathbb{R}^n, z \in \mathbb{R}^m, A^\varepsilon_{\sigma(t)}, M_{\sigma(t)} \in \mathbb{M}_{n+m}(\mathbb{R}), \varepsilon > 0 \)

\( n = m = 1, \sigma(t) : \mathbb{R} \to \{1, 2\} \)

Let us write

\[
M_i = \begin{pmatrix} a_i & b_i \\
c_i & d_i
\end{pmatrix}, \quad i = 1, 2.
\]

Definition 1: We say that the switched system is globally uniformly asymptotically stable (GUAS, for short) if there exists a class K.L function such that, for every switching signal and every initial condition \( x(0) \), its solution satisfies the inequality

\[
||x(t)|| \leq \beta(||x(0)||, t), \quad t \geq 0.
\]

Definition 2: We say that the SPSS (1) is GUAS as \( \varepsilon \to 0^+ \) if there exists \( \varepsilon_0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \), the switched system described by (1) (with \( \varepsilon \) fixed) is GUAS.
Main Result: Stability Characterization

\[
\delta(X) = \text{tr}(X)^2 - 4 \det(X),
\]
\[
\Gamma(X, Y) = \frac{1}{2} (\text{tr}(X)\text{tr}(Y) - \text{tr}(XY)),
\]
\[
\tau(X, Y) = \begin{cases} 
\frac{\text{tr}(X)}{\sqrt{|\delta(X)|}} & \text{if } \delta(X) \neq 0, \\
\frac{\text{tr}(X)}{\sqrt{|\delta(Y)|}} & \text{if } \delta(X) = 0 \text{ and } \delta(Y) \neq 0, \\
\frac{\text{tr}(X)}{\sqrt{2}} & \text{if } \delta(X) = \delta(Y) = 0,
\end{cases}
\]

Main Result

- First condition for the asymptotic stability of (1):

\[ \Rightarrow A_1^\varepsilon, A_2^\varepsilon \] must be Hurwitz as \( \varepsilon \to 0^+ \)

\[ \Lambda = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det(M) > 0 \text{ and } (a < 0 \text{ or } (a = 0 \text{ and } d < 0)) \right\} \]

- Theorem 2: Let \( M_1, M_2 \in \Lambda \) (Hurwitz as \( \varepsilon \to 0^+ \)). The stability of the SPSS (4) is described by the following 5 cases:

(SP1) System (4) is quadratically stable as \( \varepsilon \to 0^+ \) if and only if

\[ \Gamma(M_1, M_2) > -\sqrt{\det(M_1) \det(M_2)} \]

and one of the following conditions is satisfied:

1. \( \Gamma(M_1, M_2) \leq \sqrt{\det(M_1) \det(M_2)} \),
2. \( a_1 a_2 \neq 0 \),
3. \( a_1 a_2 = 0 \) with \( a_1^2 + a_2^2 \neq 0 \), and

\[ b_1 c_2 + b_2 c_1 \geq -2 \sqrt{\det(M_1) \det(M_2)} \].
Main Result

(SP2) If $\Gamma(M_1, M_2) > \sqrt{\det(M_1) \det(M_2)}$, $a_1 a_2 = 0$ with $a_1^2 + a_2^2 \neq 0$, and $b_1 c_2 + b_2 c_1 < -2 \sqrt{\det(M_1) \det(M_2)}$, then (1) is GUAS as $\epsilon \to 0^+$. 

(SP3) If $\Gamma(M_1, M_2) = -\sqrt{\det(M_1) \det(M_2)}$, then for all $\epsilon > 0$ (1) is uniformly stable but not GUAS.

(SP4) If $\Gamma(M_1, M_2) > \sqrt{\det(M_1) \det(M_2)}$ and $a_1 = a_2 = 0$, then (1) is unbounded as $\epsilon \to 0^+$. 

(SP5) If $\Gamma(M_1, M_2) < -\sqrt{\det(M_1) \det(M_2)}$, then for all $\epsilon > 0$ (1) is unbounded.

CNS for asymptotic stability (SP1+SP2)
(SP1) are necessary and sufficient conditions for the quadratic stability as $\epsilon \to 0^+$

Example: Consider the planar SPSS characterized by the following matrices

$$M_1 = \begin{pmatrix} -1 & \alpha \\ 0 & -1 \end{pmatrix} \quad M_2 = \begin{pmatrix} -1 & 0 \\ \alpha & -1 \end{pmatrix}$$

For what values of $\alpha$ is this system quadratically stable?
- using LMIs conditions $\Rightarrow -1 < \alpha < 1$

Applied to example 1, we $\Rightarrow -2 < \alpha < 2$
Transitions

- Transitions triggered by the change of sign of
  \[
  \eta(\epsilon) = \text{tr}(A_1^e A_2^e) + 2\sqrt{\det(A_1^e) \det(A_2^e)}
  \]
  \[
  = \frac{a_1 a_2}{\epsilon^2} + \frac{b_1 c_2 + b_2 c_1 + 2\sqrt{\det(M_1) \det(M_2)}}{\epsilon} + d_1 d_2
  \]

- If \(a_1 a_2 = 0\), only one transition from (S4) to (S1) or (S1) to (S4) is possible.

- If \(a_1 a_2 \neq 0\), \(\eta(\epsilon)\) can change sign 0, 1 or 2 times.

Consider the planar SPSS characterized by the matrices

\[
M_1 = \begin{pmatrix}
-1 & 0.01 \\
-9 & -1
\end{pmatrix} \quad M_2 = \begin{pmatrix}
-1 & 2 \\
-2 & -2
\end{pmatrix}
\]
(a) The dashed and continuous line are the graph of $\epsilon \mapsto \epsilon \text{tr}(A_1^\epsilon A_2^\epsilon)$ and $\epsilon \mapsto -2\epsilon \sqrt{\det(A_1^\epsilon) \det(A_2^\epsilon)}$, respectively.

$(0, \epsilon_0) \cup (\epsilon_1, \infty)$ the system is of the type (S1), while for $\epsilon \in (\epsilon_0, \epsilon_1)$ it is of type (S4)

$\epsilon_0 = 0.0784$ and $\epsilon_1 = 6.3742$. 


Thank you for your attention