Contribution to the control of nonlinear systems under aperiodic sampling

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Sampled-data systems

Digital/Networked Control Systems
Applications of sampled-data
The HYCON2 Project

Highly-complex and networked control systems (HYCON2)

- FP7 project coordinated by Françoise Lamnabhi-Lagarrigue (CNRS).
- WP2 is related to research on Networked Control Systems.
Challenges in sampled-data control

Processor: limited calculation power
Network: finite bandwidth
Sampler: minimum responding time

⇒ finite number of samples per time unit

How fast SHOULD we sample? ↔ How fast CAN we sample?
Challenges in sampled-data control

**Sampler clock**: jitter
**Network**: packet dropouts
**Scheduling**: interaction between algorithms
**Real-time computing**: microprocessor latency

$\Rightarrow$ sampling is not necessarily periodic

How to ensure robustness with respect to asynchronous sampling?
Challenges in sampled-data control

**Aperiodic sampling may cause instability**

\[
\dot{x}(t) = A_0 x(t) + B_0 u(t), \quad u(t) = K x(t_k)
\]

\[
A_0 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \quad K = \begin{bmatrix} -1 & -6 \end{bmatrix}.
\]

Periodic sampling \( T_1 = 0.18, T_2 = 0.54 \)  

Aperiodic sampling \( T_1 \rightarrow T_2 \rightarrow T_1 \cdots \)
Problem under study

Continuous-time controller

\[ \dot{x}(t) = f(x(t), u(t)) \]

\[ u(t) = K(x(t)) \]
**Problem under study**

*Digital implementation under asynchronous sampling (emulation approach)*

\[
\dot{x}(t) = f(x(t), u(t))
\]

\[
u(t) = K(x(t_k)), \quad \forall t \in [t_k, t_{k+1}), \quad 0 < \epsilon \leq t_{k+1} - t_k \leq h_{\text{max}}, \quad \forall k \in \mathbb{N}.
\]

*Find stability criteria for nonlinear sampled-data control systems, which provide a computable estimate of the Maximum Allowable Sampling Period (MASP).*
Existing results

**The linear time-invariant case:** Realistic model?

- Input delay approach *(Fridman et al 2004), (Fridman 2010), (Michiels 2005)*
- Robust control based analysis *(Mirkin 2007), (Fujioka 2009)*
- Impulsive modelling *(Naghshtabrizi 2008)*
- Discrete-time approaches & convex embedding *(Hetel, Daafouz et al 2007)*
- Sum of squares *(Seurét 2011)*

→ Bilinear case

**The nonlinear case:** Constructive?

- Input delay approach *(Mazenc et al 2013)*
- Hybrid system modelling *(Nešić et al 2009), (Burlion et al 2006)*
- Single/vector Lyapunov functions *(Karafyllis et al 2007)*
- $L_p$ stability *(Zaccarian et al 2003)*
Existing results

**The linear time-invariant case: Realistic model?**

- Input delay approach (Fridman et al 2004), (Fridman 2010), (Michiels 2005)
- Robust control based analysis (Mirkin 2007), (Fujioka 2009)
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- Discrete-time approaches & convex embedding (Hetel, Daafouz et al 2007)
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→ **Bilinear case**

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- Input delay approach (Mazenc et al 2013)
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- Single/vector Lyapunov functions (Karafyllis et al 2007)
- $L_p$ stability (Zaccarian et al 2003)
Outline

1. Introduction
2. Stability of bilinear sampled-data systems - hybrid systems approach
3. Stability of bilinear sampled-data systems - dissipativity approach
4. Stability of input-affine nonlinear systems with sampled-data control
5. Conclusions and perspectives
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Bilinear systems

\[ \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} [u(t)]_i N_i x(t) + B_0 u(t) \]

- The simplest class of nonlinear systems.
- Offer a more accurate approximation of nonlinear systems than the classical linear ones.
- Allows to address the problem for a simple nonlinear system.
- Several applications: power electronics, mechanical systems, chemical processes.
- Continuous-time stabilization techniques: quadratic; division; sliding and linear state feedback.
Bilinear systems

\[ \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} [u(t)]_i N_i x(t) + B_0 u(t) \]
Problem formulation

Continuous-time control of bilinear systems with linear state feedback (local stabilization):

\[ \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} u_i(t)N_i x(t) + B_0 u(t) \]

LMI based synthesis (Amato et al 2007), (Tarbouriech et al 2009).
Problem formulation

Sampled-data implementation (emulation approach):

\[
\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} u_i(t) N_i x(t) + B_0 u(t)
\]

\[
u(t) = F x(t_k), \quad \forall t \in [t_k, t_{k+1}),
\]

\[
0 < \epsilon \leq t_{k+1} - t_k \leq \frac{h_{\text{max}}}{\text{MASP}}, \quad \forall k \in \mathbb{N}
\]

**Problem:** Constructive method to estimate the MASP, and the domain of attraction using LMI.
Hybrid system model for Networked Control Systems: (Walsh et al 2002) and (Nešić et al 2004).

\[
e(t) = x(t_k) - x(t)
\]

\[
f(x, e) = \tilde{A}[x(t), e(t)]x(t) + Be(t),
\]

\[
g(x, e) = -\tilde{A}[x(t), e(t)]x(t) - Be(t),
\]

\[
\tilde{A}[x, e] := A[x(t_k)] = A_0 + B_0 F + \sum_{i=1}^{m} [Fx(t_k)]_i N_i,
\]

\[
B = B_0 F.
\]
Hybrid system model

\[
\begin{align*}
\dot{x} &= f(x, e) = \tilde{A}[x, e]x + Be \\
\dot{e} &= g(x, e) = -\tilde{A}[x, e]x - Be \\
\dot{\tau} &= 1 \\
x^+ &= x \\
e^+ &= 0 \\
\tau^+ &= 0
\end{align*}
\]

\[
\tau \in [0, h_{\text{max}})
\]

\[
\tau \in [\epsilon, h_{\text{max}}]
\]

Hybrid system model:

\[
\xi := [x^T, e^T, \tau]^T
\]

\[
F(\xi) := [f(x, e)^T, g(x, e)^T, 1]^T
\]

\[
G(\xi) := [x, 0, 0]^T
\]

\[
C := \{\xi : \tau \in [0, h_{\text{max}})\}
\]

\[
D := \{\xi : \tau \in [\epsilon, h_{\text{max}}]\}
\]

\[
\begin{align*}
\dot{\xi} &= F(\xi), & \xi \in C, \\
\xi^+ &= G(\xi), & \xi \in D.
\end{align*}
\]
Proposed solutions

Local Analysis:

\[ P = \text{conv}\{x_1, x_2, \ldots, x_p\} \]

\[ A[x(t_k)] \in P_A = \text{conv}\{A_1, A_2, \ldots, A_p\} \]

\[ A_i = A_0 + B_0 F + \sum_{j=1}^{m} [Fx_i]j N_j, \quad \forall i \in \{1, 2, \ldots, p\} \]
Method 1

Result for the general nonlinear case (Nešić et al 2009):

Lyapunov function $U(\xi) = V(x) + \gamma \phi(\tau) W^2(e)$

\[
\left\langle \frac{\partial W(e)}{\partial e}, g(x,e) \right\rangle \leq LW + H(x,e) \\
\langle \nabla V(x), f(x,e) \rangle < -\varrho(|x|) - \varrho(W(e)) - H^2(x,e) + \gamma^2 W^2(e)
\]

$\Rightarrow \langle \nabla U(\xi), F(\xi) \rangle < -\varrho(|x|) - \varrho(W(e))$

Question: How to find $V(\cdot), W(\cdot), H(\cdot, \cdot), \varrho(\cdot), \gamma$ and $L$? It is stated in (Nešić et al 2009) that:

"We note that finding these functions may be hard for general nonlinear systems".
Method 1

Result for the general nonlinear case (Nešić et al 2009):

Lyapunov function $U(\xi) = V(x) + \gamma \phi(\tau) W^2(e)$

$$\left\langle \frac{\partial W(e)}{\partial e}, g(x, e) \right\rangle \leq LW + H(x, e)$$

$$\left\langle \nabla V(x), f(x, e) \right\rangle < -\phi(|x|) - \phi(W(e)) - H^2(x, e) + \gamma^2 W^2(e)$$

$$\Rightarrow \left\langle \nabla U(\xi), F(\xi) \right\rangle < -\phi(|x|) - \phi(W(e))$$

**Question:** How to find $V(\cdot), W(\cdot), H(\cdot, \cdot), \phi(\cdot), \gamma$ and $L$? It is stated in (Nešić et al 2009) that:

"We note that finding these functions may be hard for general nonlinear systems".
Theorem (CDC’12)

Assume that the MASP is strictly bounded $h_{\text{max}} < \mathcal{T}(L, \gamma)$:

$$
\mathcal{T}(L, \gamma) := \begin{cases} 
\arctan(r)/(Lr) & \gamma > L \\
1/L & \gamma = L \\
\arctanh(r)/(Lr) & \gamma < L 
\end{cases}
$$

where $L$ is given by

$$
L = \frac{1}{2} \max\{-\lambda_{\text{min}}(B^T + B), 0\}
$$

and $\gamma$ is the solution to the optimization problem $\gamma = \min \gamma'$ under the constraints:

$\exists P \in \mathbb{R}^n, P > 0, \text{ and } \alpha > 0$ such that

$$
M_{ij} = \begin{bmatrix} 
A_i^T P + PA_i + \alpha l \left( A_i^T A_j + A_j^T A_i \right) + \alpha l & PB \\
& \left( \alpha - \gamma'^2 \right) l
\end{bmatrix} < 0, \quad \forall i, j \in \{1, 2, ..., p\}.
$$

Then the bilinear sampled-data system is locally uniformly asymptotically stable.
Method 2

Method 1: Conservatism due to upper estimations of the derivative of a Lyapunov function.

Consider studying directly the Lyapunov function:

\[ U'(\xi) = x^T P x + \exp\left(\frac{-\tau}{h_{\text{max}}}\right) e^T Q e \]

Theorem (CDC’12)

Assume that there exist symmetric positive definite matrices \( P, Q, X, Y \in \mathbb{R}^{n \times n} \), such that the following LMIs are satisfied:

\[
\begin{bmatrix}
A_i^T P + PA_i + X \\
 PB - A_i^T Q \\
- B^T Q - QB - \frac{1}{h_{\text{max}}} Q + Y
\end{bmatrix} < 0,
\forall i \in \{1, 2, ..., p\}.
\]

\[
\begin{bmatrix}
A_i^T P + PA_i + X \\
 PB - A_i^T Q \exp(-1) \\
[- B^T Q - QB - \frac{1}{h_{\text{max}}} Q] \exp(-1) + Y
\end{bmatrix} < 0,
\forall i \in \{1, 2, ..., p\}.
\]

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\end{bmatrix} < 0, \quad \forall i \in \{1, 2, \ldots, p\}.
\]

Then the bilinear sampled-data system is locally uniformly asymptotically stable.
Numerical example

Consider the example from (Amato et al 2007) and (Tarbouriech et al 2009):

\[
A_0 = \begin{bmatrix} -0.5 & 1.5 & 4 \\ 4.3 & 6.0 & 5.0 \\ 3.2 & 6.8 & 7.2 \end{bmatrix}; \quad B_0 = \begin{bmatrix} -0.7 & -1.3 \\ 0 & -4.3 \\ 0.8 & -1.5 \end{bmatrix}
\]

\[
N_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
F = \begin{bmatrix} 0.0016 & 0.0035 & 0.0034 \\ 2.2404 & 3.2676 & 5.9199 \end{bmatrix}
\]

\[
\mathcal{P} = [-1.35, +1.35] \times [-0.5, +0.5] \times [-0.5, +0.5]
\]
Numerical example

<table>
<thead>
<tr>
<th>Method 1 (CDC'12)</th>
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<tr>
<td>$h_{\text{max}}$</td>
<td>$5 \times 10^{-3}$</td>
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</table>

$t_{k+1} - t_k = 88 \times 10^{-3}$

$t_{k+1} - t_k = 90 \times 10^{-3}$
Summary

- **Stability of bilinear sampled-data systems was addressed using a hybrid systems approach.**

- **Method 1**: a constructive method to apply results from (Nešić et al 2009) for the bilinear case.

  - **Method 2**: a direct search of a Lyapunov function for the hybrid system.

- Both methods are constructive via LMIs.

- Still room for improvements.
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5. Conclusions and perspectives
Problem formulation

**Sampled-data implementation (emulation approach):**

\[
\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} u_i(t) N_i x(t) + B_0 u(t)
\]

\[
u(t) = F x(t_k), \quad \forall t \in [t_k, t_{k+1}),
\]

\[
0 < \epsilon \leq t_{k+1} - t_k \leq h_{\text{max}}, \quad \forall k \in \mathbb{N}
\]
**Introduction**

**Hybrid systems approach**

**Dissipativity approach**

**Input-affine systems**

**Conclusions and perspectives**

---

**Dissipativity-based representation**

**Equivalent model**

**Closed-loop system**

\[ \dot{x}(t) = \left[ A_0 + \sum_{i=1}^{m} (Fx(t_k))_i N_i \right] x(t) + B_0 F x(t_k) \]

\[ \dot{x}(t) = \left[ A_0 + B_0 F + \sum_{i=1}^{m} (Fx(t_k))_i N_i \right] x(t) + \frac{B_0 F}{B} \left( x(t_k) - x(t) \right) \]

\[ \tilde{A}[x,e] := A[x(t_k)] \]

The system can be represented by the interconnection of:

\[ G := \begin{cases} \dot{x}(t) = \tilde{A}[x,e]x(t) + Be(t) \\ y(t) = \dot{x}(t) \end{cases} \]

with the operator \( \Delta_{sh} : y \to e \) defined by:

\[ e(t) = -\int_{t_k}^{t} y(\tau) d\tau := (\Delta_{sh} y)(t), \quad \forall t \in [t_k, t_{k+1}) \]
Dissipativity-based representation

\[ G := \begin{cases} \dot{x}(t) = \ddot{A}[x, e]x(t) + Be(t) \\ y(t) = \dot{x}(t) \end{cases} \]

\[ (\Delta_{sh}y)(t) := -\int_{t_k}^{t} y(\tau)d\tau. \]
Properties of the operator

$L_2$-induced norm (Mirkin 2007):

$$\frac{\|e\|}{\|y\|} \leq \delta_0 := \frac{2}{\pi} h_{\text{max}} \quad \text{i.e.} \quad \int_0^\infty e^T(\tau)e(\tau)d\tau \leq \delta_0^2 \int_0^\infty y^T(\tau)y(\tau)d\tau.$$

Passivity (Fujioka 2009):

$$\langle \Delta_{sh}y, y \rangle = \int_0^\infty y^T(\tau)(\Delta_{sh}y)(\tau)d\tau \leq 0.$$
Properties of $\Delta_{sh}$ are used for the case of LTI sampled-data systems using IQC.

Robust stability analysis via frequency based conditions.

Conditions are constructive (LMIs) thanks to KYP lemma.

Among the less conservative stability criteria.

Can not be applied to bilinear systems

Objective: generalization to the bilinear case using dissipativity approach.
Dissipativity

- An abstract extension of the notion of energy.
- Generalization of Lyapunov functions technique, for input-output systems.
- Encompass several properties of dynamical systems such as passivity and $L_2$-gain.

\[
\begin{align*}
\dot{x} &= f(x) + g(x)e \\
y &= h(x) + j(x)e
\end{align*}
\]

\[
\dot{V}(x(t)) \leq S(y(t), e(t))
\]

$V(x)$ is a storage function \quad $S(y, e)$ is a supply rate.
Properties of the operator

**Boundedness property**

For all \( y \in L_2[t_k, t_{k+1}) \) and \( 0 < X^* = X \in \mathbb{R}^{n \times n} \):

\[
\int_{t_k}^{t} (\Delta_{sh} y)^* X (\Delta_{sh} y) \, d\tau - \delta_0^2 \int_{t_k}^{t} y^* Y \, d\tau \leq 0, \quad \forall t \in [t_k, t_{k+1})
\]

**Passivity property**

For all \( y \in L_2[t_k, t_{k+1}) \) and \( 0 \leq Y^* = Y \in \mathbb{R}^{n \times n} \):

\[
\int_{t_k}^{t} (\Delta_{sh} y)^* Y y \, d\tau + \int_{t_k}^{t} y^* (\Delta_{sh} y) \, d\tau \leq 0, \quad \forall t \in [t_k, t_{k+1})
\]

\[
\Rightarrow \int_{t_k}^{t} \begin{bmatrix} y & e \end{bmatrix}^T \begin{bmatrix} -\delta_0^2 X & Y \\ Y & X \end{bmatrix} \begin{bmatrix} y \\ e \end{bmatrix} \, d\tau \leq 0, \quad \forall t \in [t_k, t_{k+1})
\]

supply rate \( S(y,e) \)
Stability result

Local Analysis:

\[ \mathcal{P} = \text{conv}\{x_1, x_2, \ldots, x_p\} \]

\[ A[x(t_k)] \in \mathcal{P}_A = \text{conv}\{A_1, A_2, \ldots, A_p\} \]

\[ A_i = A_0 + B_0 F + \sum_{j=1}^{m} [F x_i] j N_j, \quad \forall i \in \{1, 2, \ldots, p\} \]
Stability result

**Theorem (Automatica’14)**

If there exist symmetric positive definite matrices $X$, $Y$, $P \in \mathbb{R}^{n \times n}$ and $P_2$, $P_3 \in \mathbb{R}^{n \times n}$, such that the following LMIs are satisfied

\[
\begin{bmatrix}
A_i^T P_2 + P_2^T A_i & P - P_2^T + A_i^T P_3 & P_2^T B \\
* & -P_3 - P_3^T + \left(\frac{2}{\pi} h_{\text{max}}\right)^2 X & P_3^T B - Y \\
* & * & -X \\
\end{bmatrix} < 0
\]

\[\forall i \in \{1, 2, ..., p\},\]

then the sampled-data system is locally asymptotically stable.
Main idea:

dissipativity properties $\Rightarrow$ contraction of sub-level sets defined by $V(x) = x^TPx$:

$$\dot{V} < S(y,e) \Rightarrow$$

![Graph showing the contraction of sub-level sets](image)
Stability result

Main idea:

dissipativity properties \(\Rightarrow\) contraction of sub-level sets defined by \(V(x) = x^T P x:\)

\[
\dot{V} < S(y, e) \Rightarrow 
\]

\[
V(x(t_{k-1})) < V(x(t_k)) 
\]
**Main idea:**

dissipativity properties $\Rightarrow$ contraction of sub-level sets defined by $V(x) = x^T P x$:

$$\dot{V} < S(y, e) \quad \Rightarrow$$

![Graph showing the contraction of sub-level sets](image_url)
Numerical example

Consider the example from (Amato et al 2007) and (Tarbouriech et al 2009):

\[
A_0 = \begin{bmatrix}
-0.5 & 1.5 & 4 \\
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\end{bmatrix} ; \quad B_0 = \begin{bmatrix}
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\[
N_1 = \begin{bmatrix}
-1 & 0 & 0 \\
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\[
F = \begin{bmatrix}
0.0016 & 0.0035 & 0.0034 \\
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\mathcal{P} = [-1.35, +1.35] \times [-0.5, +0.5] \times [-0.5, +0.5]
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Numerical example

Region of attraction:

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<td>Theorem (ADHS’12)</td>
<td>$43 \times 10^{-3}$</td>
</tr>
<tr>
<td>Theorem (Automatica’14)</td>
<td>$51 \times 10^{-3}$</td>
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**Numerical example**

**Simulation:**

*State evolution for the sampled-data bilinear system with $h_{\text{max}} = 51 \times 10^{-3}:*

![State evolution](image1)

![Time variations of the sampling intervals](image2)
Numerical example

**Simulation:**

*State evolution for the sampled-data bilinear system with $t_{k+1} - t_k = 90 \times 10^{-3}$.*
Summary

- Stability of bilinear sampled-data systems was addressed using a robust control theory approach.

- The method is based on the analysis of contractive invariant sets, and it is inspired by the dissipativity theory.

- The proposed stability conditions are constructive via LMIs.

- How to generalize to a more general class of nonlinear systems?
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Problem formulation

**Continuous-time controller**

\[
\dot{x}(t) = f(x(t)) + g(x(t))u(t)
\]

\[
u(t) = K(x(t))
\]

*The (continuous-time) controller \( K \) (of class \( C^1 \)) stabilizes asymptotically the origin of the system.*
**Problem formulation**

*Digital implementation under asynchronous sampling (emulation approach)*

\[
\dot{x}(t) = f(x(t)) + g(x(t))u(t)
\]

\[
u(t) = K(x(t_k)), \quad \forall t \in [t_k, t_{k+1}), \quad 0 < \epsilon \leq t_{k+1} - t_k \leq \frac{h_{\text{max}}}{M}, \quad \forall k \in \mathbb{N}.
\]

**Problem:** Extend the dissipativity-based stability criteria for this more general class of nonlinear systems.
Dissipativity-based representation

**Equivalent model**

Closed-loop system

\[
\dot{x}(t) = f(x(t)) + g(x(t))K(x(t_k))
\]

\[
\dot{x}(t) = \underbrace{f(x(t)) + g(x(t))K(x(t))}_{f_n(x(t))} + \underbrace{g(x(t))(K(x(t_k)) - K(x(t)))}_{g_n(x(t))e(t)}
\]

The system can be represented by the interconnection of:

\[
\begin{cases}
    \dot{x}(t) = f_n(x(t)) + g_n(x(t))e(t) \\
y(t) = \frac{\partial K}{\partial x} \dot{x}(t)
\end{cases}
\]

with the operator \(\Delta_{sh} : y \rightarrow e\) defined by:

\[
e(t) = (\Delta_{sh} y(t)) = - \int_{t_k}^{t} y(\tau) d\tau
\]
Dissipativity-based representation

\[
\begin{align*}
\dot{x}(t) &= f_n(x(t)) + g_n(x(t))e(t) \\
y(t) &= \dot{x}(t)
\end{align*}
\]

\[
(\Delta_{sh}y)(t) := -\int_{t_k}^{t} y(\tau)d\tau.
\]

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Theorem (ECC’13)

Consider a neighbourhood $D \subset \mathbb{R}^n$ of the origin $x = 0$ and differentiable positive definite function $V: D \to \mathbb{R}^+$ such that there exist class $\mathcal{K}$ function $\beta_1$ et $\beta_2$ with:

$$\beta_1(|x|) \leq V(x) \leq \beta_2(|x|), \quad \forall x \in D.$$  

If for $\alpha > 0$ and for any $x(t) \in D$, the function $V$ satisfies:

$$\dot{V}(x(t)) + \alpha V(x(t)) \leq S(y(t), e(t))$$
$$\dot{V}(x(t)) + \alpha V(x(t)) \leq S(y(t), e(t)) \exp(-\alpha h_{\text{max}})$$

Then the equilibrium $x = 0$ of the sampled-data system is locally uniformly asymptotically stable. The maximal sub-level set of $V$ that is contained in $D$:

$$c^* = \max_{\mathcal{L}_c \subset D} c, \quad \mathcal{L}_c := \{x \in \mathbb{R}^n : V(x) \leq c\}.$$  

is an estimate of the domain of attraction.
Polynomial systems

Specialization to the case of polynomial systems

\[
\begin{align*}
F(x, e) & := f(x) + g(x)K(x) + g(x) e \\
G(x, e) & := \frac{\partial K}{\partial x} F(x, e)
\end{align*}
\]

\(f(x), g(x)\) and \(K(x)\) polynomials \(\Rightarrow\) \(F(x, e)\) and \(G(x, e)\) are polynomials
Specialization for the case of polynomial systems

\[
\dot{V}(x(t)) + \alpha V(x(t)) \leq S(y(t), e(t)) \\
\dot{V}(x(t)) + \alpha V(x(t)) \leq S(y(t), e(t)) \exp(-\alpha h_{\text{max}})
\]

\[
0 \leq -\frac{\partial V}{\partial x} F(x, e) - \alpha V(x) + \left[ -\delta_0^2 G^T(x, e) X G(x, e) + 2 G^T(x, e) Y e + e^T X e \right]
\]

\[
0 \leq -\frac{\partial V}{\partial x} F(x, e) - \alpha V(x) + \left[ -\delta_0^2 G^T(x, e) X G(x, e) + 2 G^T(x, e) Y e + e^T X e \right] \exp(-\alpha h_{\text{max}})
\]

The last inequalities are of the form \( p(\xi) \geq 0 \), where \( p(\xi) \in \mathbb{R}[\xi] \), and \( \xi = (x, e) \).

**Verification of \( p(\xi) \geq 0 \) is a difficult problem! → simplification using SOS**

A multivariate polynomial \( p(\xi) \in \mathbb{R}[\xi] \) is said to be a sum of squares (SOS) (Papachristodoulou 2005) if there exist \( p_i(\xi) \in \mathbb{R}[\xi], i \in \{1, \ldots, M\} \), such that \( p(\xi) = \sum_{i=1}^{M} p_i^2(\xi) \).
Polynomial systems

**Corollary (ECC’13)**

In the case where \( f(x) \), \( g(x) \) and \( K(x) \) are polynomials, let

\[
\mathcal{D} = \{ x \in \mathbb{R}^n : \mu_l(x) \geq 0, \ l = 1, 2, \ldots, s \}
\]

be a neighbourhood of \( x = 0 \). Suppose that there exist a polynomial function \( V(x) \in \mathbb{R}[x] \) and sums of squares \( \sigma_l(\xi) \) and \( \varsigma_l(\xi) \), with \( l \in \{1, \ldots, s\} \) and \( \xi = (x, w) \), such that the following polynomials are SOS:

- \( V(x) - \varphi(x) \),
- \[ -\sum_{l=1}^{s} \sigma_l(\xi)\mu_l(x) - \frac{\partial V}{\partial x} F(x, e) - \alpha V(x) + \left[ -\delta_0 G^T(x, e)XG(x, e) + 2G^T(x, e)Ye + e^T Xe \right], \]
- \[ -\sum_{l=1}^{s} \varsigma_l(\xi)\mu_l(x) - \frac{\partial V}{\partial x} F(x, e) - \alpha V(x) + \left[ -\delta_0 G^T(x, e)XG(x, e) + 2G^T(x, e)Ye + e^T Xe \right]e^{-\alpha h_{\text{max}}}. \]

Then \( x = 0 \) is locally uniformly asymptotically stable. Moreover, \( \mathcal{L}_{c^*} \) is an estimate of the domain of attraction.
Consider the example from (Nešić et al 2009)

\[ \dot{x} = dx^2 - x^3 + u, \quad u = K(x) = -2x, \quad \text{with } |d| \leq 1. \]

<table>
<thead>
<tr>
<th></th>
<th>ECC'13</th>
<th>(Nešić et al 2009)</th>
<th>(Karafyllis et al 2007)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_{\text{max}} )</td>
<td>0.72</td>
<td>0.368</td>
<td>0.1428</td>
</tr>
</tbody>
</table>

Trade-off between the decay rate \( \alpha \) and the MASP:
Numerical example

**Simulation:**

*State evolution for the sampled-data system:*

![Graph showing state evolution for the sampled-data system]
Summary

- **Address a quite general class of systems thanks to exponential dissipativity.**

- **Sufficient conditions for the stability of nonlinear sampled-data systems, which are affine in the control.**

- **The results are numerically tractable for the case of polynomial systems, with the use of SOS.**
Outline

1. Introduction
2. Stability of bilinear sampled-data systems - hybrid systems approach
3. Stability of bilinear sampled-data systems - dissipativity approach
4. Stability of input-affine nonlinear systems with sampled-data control
5. Conclusions and perspectives
Conclusions

- A contributions to the stability analysis of nonlinear systems under aperiodic sampling.

- A particular attention has been given to the case of bilinear systems:
  Hybrid system modeling approach
  Dissipativity approach

- Extend the dissipativity-based results developed for bilinear systems to a more general class of nonlinear systems.

- The methods provide quantitative estimates of the MASP.
Conclusions
Perspectives

- Include other network-imposed imperfections such as: time-varying delays, protocols .. etc.

- Improve the numerical solvability of the proposed conditions in order to decrease the conservatism.

- Controlled sampling: event-based control, self-triggered control and state-dependent sampling \cite{Fiter et al 2012}, for nonlinear systems.
Journals:


Conferences:

Thank you for your attention

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