Implicit Euler Numerical Simulation and Implementation of Sliding Mode Systems

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GDR MACS Systèmes Hybrides

07 octobre 2010
Objectives

▶ Study the implicit Euler discretization of a class of differential inclusions with sliding surfaces (relay systems)
▶ Show that this numerical method permits a smooth stabilization on the sliding surface, in a finite number of steps
▶ Show how this may be used in real-time implementations of sliding mode control
To start with we consider the simplest case:

\[ \dot{x}(t) \in -\text{sgn}(x(t)) = \begin{cases} 
  1 & \text{if } x(t) < 0 \\
  -1 & \text{if } x(t) > 0 \\
  [-1,1] & \text{if } x(t) = 0 
\end{cases} \]

with \( x(t) \in \mathbb{R} \). This system possesses a unique Lipschitz continuous solution for any \( x_0 \). The backward Euler discretization of (1) reads as:

\[ \begin{align*}
  x_{k+1} - x_k &= -hs_{k+1} \\
  s_{k+1} &\in \text{sgn}(x_{k+1})
\end{align*} \]
Remark: As is known the explicit Euler discretization of such discontinuous systems yields spurious oscillations around the switching surface [Galias et al, IEEE T AC and CAS 2006, 2007, 2008].

\[ \rightsquigarrow \text{this means that the derivative of the switching function while sliding occurs, is very badly estimated.} \]

Both the explicit and the implicit methods converge (the approximated solution \( x^N(\cdot) \) tends to the Filippov’s solution as \( h \to 0 \)). However or the backward Euler method the following holds:

**Lemma**

*For all \( h > 0 \) and \( x_0 \in \mathbb{R} \), there exists \( k_0 \) such that \( x_{k_0+n} = 0 \) and \*

\[ \frac{x_{k_0+n+1} - x_{k_0+n}}{h} = 0 \] *for all \( n \geq 1 \).*
On this simple case this has the following graphical interpretation, as the intersection of two graphs:

Figure: Iterations of the backward Euler method.
An interesting property is that the smooth stabilization and the finite-time convergence on the switching surface, hold (more or less) independently of the step $h > 0$:

\[(a) \quad h = 0.2 \quad \quad (b) \quad h = 0.02 \quad \quad (c) \quad h = 0.01\]

**Figure:** A simple example for $x_0 = 1.01$ at $t_0 = 0$. 
EXTENSIONS

We shall focus on inclusions of the form:

\[
\begin{cases}
\dot{x}(t) \in f(t, x(t)) - B \operatorname{Sgn}(Cx(t) + D), \text{ a.e. on } (0, T) \\
x(0) = x_0
\end{cases}
\]  

with

\[B \in \mathbb{R}^{n \times m}\]

and

\[\operatorname{Sgn}(Cx(t) + D) \overset{\Delta}{=} (\operatorname{sgn}(C_1x + D_1), \ldots, \operatorname{sgn}(C_mx + D_m))^T \in \mathbb{R}^m,\]

where \(\operatorname{sgn}(\cdot)\) is multivalued at 0.
Example (Equivalent-Control-Based Sliding-Mode-Control)

Consider the system \( \dot{x}(t) = Fx(t) + Gu \), with an ECB-SMC:

\[ u(x) = -(HG)^{-1}HFx - \alpha(HG)^{-1}\text{Sgn}(Hx) \quad (4) \]

with \( \alpha > 0 \). The closed-loop system is:

\[ \dot{x}(t) \in (F - G(HG)^{-1}HF)x(t) - \alpha G(HG)^{-1}\text{Sgn}(Hx(t)) \quad (5) \]

that belongs to the class of differential inclusions in (3).
Well-posedness of the differential inclusions (3)

Proposition

Consider the differential inclusion in (3). Suppose that

- There exists \( L \geq 0 \) such that for all \( t \in [0, T] \), for all \( x_1, x_2 \in \mathbb{R}^n \), one has \( \|f(t, x_1) - f(t, x_2)\| \leq L \|x_1 - x_2\| \).

- There exists a function \( \Phi(\cdot) \) such that for all \( R \geq 0 \):

\[
\Phi(R) = \sup \left\{ \left\| \frac{\partial f}{\partial t}(\cdot, v) \right\|_{L^2((0,T);\mathbb{R}^n)} \| v \|_{L^2((0,T);\mathbb{R}^n)} \leq R \right\} < +\infty.
\]

If there exists an \( n \times n \) matrix \( P = P^T > 0 \) such that

\[
P B_{\cdot i} = C_{\cdot i}^T
\]  

(6)

for all \( 1 \leq i \leq m \), then for any initial data the differential inclusion (3) has a unique solution \( x : (0, T) \rightarrow \mathbb{R}^n \) that is Lipschitz continuous with essentially bounded derivative.
Proof of Proposition 1: The proof uses the change of state variables $z = Rx$, where $R = R^T > 0$ and $R^2 = P$, as introduced in [Brogliato, S&CL 2004]. After some manipulations the system is rewritten as

$$\dot{z}(t) \in Rf(t, R^{-1}z(t)) - \sum_{i=1}^{m} \partial f_i(z(t))$$

(7)

where $f_i(z) = |C_i \cdot R^{-1}z + D_i|$.

The multivalued mapping $z \mapsto \sum_{i=1}^{m} \partial f_i(z(t))$ is maximal monotone.

We then use a result in [Bastien-Schatzman ESAIM M2AN 2002] to conclude.
The existence of a positive definite $P$ such that $PB = C^T$ is satisfied in many instances of sliding-mode control: observer-based sliding-mode control, Lyapunov-based discontinuous robust control.

This is an “input-output” constraint on the system, constraining the relative degree of the triple $(A, B, C)$.

It is satisfied when $(A, B, C)$ is positive real (dissipative).

We admit $B \in \mathbb{R}^{n \times m}$ with $m > n$ (more “inputs” than states).
Example (Observer-based SMC)

An observer is sometimes used to reduce chattering. The closed-loop dynamics of an OB-SMC [Young et al, IEEE Trans. Cont. Syst. Tech, 1999] is given by:

\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{e}(t) \\
\dot{x}_s(t) \\
\dot{x}_s(t)
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & 0 \\
k & -k & -k & 0 \\
0 & 0 & 0 & 1 \\
\frac{1}{\tau^2} & 0 & -\frac{1}{\tau^2} & -\frac{2}{\tau}
\end{pmatrix}
\begin{pmatrix}
x(t) \\
e(t) \\
x_s(t) \\
\dot{x}_s(t)
\end{pmatrix} -
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}
\text{sgn}(Cx(t))
\]

with \( C = (1 \quad -1 \quad 0 \quad 0) \).
The OB-SMC closed-loop satisfies \( PB = C^T \) with

\[
P = \begin{pmatrix}
1 & -1 & 0 & 0 \\
-1 & p_{22} & 0 & 0 \\
0 & 0 & p_{33} & 0 \\
0 & 0 & 0 & p_{44}
\end{pmatrix}
\]

with \( p_{22} > 1, \ p_{33} > 0, \ p_{44} > 0 \).

Another example is the Lyapunov-based discontinuous robust control:

\[
\dot{x}(t) = f(x(t), t) + \sum_{i=1}^{m} B_{i} u_i + \sum_{i=1}^{m} B_{i} \gamma_i(t)
\]

with \( u_i(x) = -\rho_i \text{sgn}(\nabla V^T(x) B_{i}), \ |\gamma_i(t)| < \rho_i, \ V(x) = \frac{1}{2} x^T P x, \ P = P^T > 0 \).
Remark

- Such relay systems are Filippov’s differential inclusions if the right-hand-side is defined properly.
- The result states existence and uniqueness of solutions for arbitrary co-dimension of the switching surface (at the price of considering a narrow subclass of switching systems).
- The underlying “philosophy” is that sliding mode control is not switching discontinuous control: it is multivalued, hence graph-continuous (the actuators have to “fill-in” the graph!)
Time-discretization of (3)

The differential inclusion in (3) is therefore discretized as follows:

\[
\begin{cases}
\frac{x_{k+1} - x_k}{h} \in f(t_k, x_k) - BS\text{sgn}(Cx_{k+1} + D), \text{ a.e. on } (0, T) \\
x(0) = x_0
\end{cases}
\]  (9)

From [Bastien-Schatzman ESAIM M2AN 2002] we have that:

**Proposition**

*Under Proposition 1 conditions, there exists \( \eta \) such that for all \( h > 0 \) one has*

\[
\forall t \in [0, T], \quad \|x(t) - x^N(t)\| \leq \eta \sqrt{h} \]  (10)

*Moreover*

\[
\lim_{h \to 0^+} \max_{t \in [0, T]} \|x(t) - x^N(t)\|^2 + \int_0^t \|x(s) - x^N(s)\|^2 ds = 0.
\]
However we have more: the discrete state reaches the sliding surface (when it exists) in a finite number of steps, and stabilizes on it in a smooth way.

Let $y(t) \triangleq Cx(t) + D$.

**Lemma**

Let us assume that a sliding mode occurs for the index $\alpha \subset \{1 \ldots m\}$, that is $y_\alpha(t) = 0$, $t > t_*$. Let $C$ and $B$ be such that (6) holds and $C_\alpha \cdot B_\alpha > 0$. Then there exists $h_c > 0$ such that $\forall h < h_c$, there exists $k_0 \in \mathbb{N}$ such that $y_{k_0+n} = Cx_{k_0+n+1} + D = 0$ for all integers $n \geq 1$.

Such algorithms are similar to proximal algorithms which possess finite-time stabilization properties [Baji and Cabot, Set-Valued Analysis 2006].
Remarks

Contrarily to other methods that reduce (not suppress...) chattering, the discrete-time sliding surface is equal to the continuous-time sliding surface: $\Sigma = \{x \in \mathbb{R}^2 | Cx + D = 0\}$ and $\Sigma_d = \{x_k \in \mathbb{R}^2 | Cx_k + D = 0\}$.

At each step one has to solve a generalized equation with unknown $x_{k+1}$ that takes the form of a mixed linear complementarity system (MLCP).

Specific MLCP solvers are needed to implement the method.
NUMERICAL EXPERIMENTS

All the simulations have been obtained with the SICONOS software package of the INRIA, that is an open-source platform dedicated to the simulation of nonsmooth dynamical systems.

More informations and download at

http://siconos.gforge.inria.fr
Let us consider the following two examples:

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -c_1 \end{bmatrix} x - \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \text{sgn}(\begin{bmatrix} c_1 & 1 \end{bmatrix} x).
\]  
(11) (codimension one sliding surface)

\[
B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad D = 0, \quad f(x(t), t) = 0
\]  
(12) (codimension two sliding surface)
(a) $h = 0.3$. Explicit Euler  
(b) $h = 0.1$. Explicit Euler  
(c) $h = 1$. Implicit Euler

Figure: Equivalent control based SMC, $c_1 = 1$, $\alpha = 1$ and $x_0 = [0, 2.21]^T$. State $x_1(t)$ versus $x_2(t)$. 
Figure: Equivalent control based SMC, $c_1 = 1$, $\alpha = 1$ and $x_0 = [0, 2.21]^T$. State $x_1(t)$ versus $x_2(t)$. 
(a) state $x_1(t)$ and $x_2(t)$  (b) phase portrait $x_2(t)$  (c) sgn function $s_1(t)$ and $s_2(t)$ versus time

**Figure:** Multiple Sliding surface. $h = 0.02$, $x(0) = [1.0, -1.0]^T$

The system reaches firstly the sliding surface $2x_2 + x_1 = 0$ without any chattering. The system then slides on the surface up to reaching the second sliding surface $2x_1 - x_2 = 0$ and comes to rest at the origin.
The Filippov’s example with switches accumulation

\[ B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 0, \quad f(x(t), t) = 0. \]

(13)

The trajectories may slide on the codimension 2 surface given by \( Cx = 0 \). The origin is attained after an infinite number of switches in finite time.
(a) state $x_1(t)$ and $x_2(t)$ versus time

(b) phase portrait $x_2(t)$ versus $x_1(t)$

(c) sgn function $s_1(t)$ and $s_2(t)$

**Figure:** Multiple Sliding surface. Filippov Example. $h = 0.002$, $x(0) = [1.0, -1.0]^T$

*The results show that the system reaches the origin without any chattering.*
The case of Zero Holding discretization (ZOH) for control implementation

The ZOH discretization of linear time invariant systems
\[ \dot{x}(t) = Fx(t) + Gu(t) \] with an ECB-SMC controller,
\[ u(x) = -(CG)^{-1}(CFx + \alpha \text{Sgn}(Cx)), \alpha > 0 \]
results in a discrete-time system of the form:

\[ x_{k+1} = \Phi x_k - \Gamma s_k \text{ for all } t \in [kh, (k + 1)h) \] (14)

where \( h > 0 \) is the sampling period, and

\[ \Phi = \exp(Fh) - \int_0^h \exp(F\tau) G CG^{-1} CF \] (15)

\[ \Gamma = \int_0^h \exp(F\tau) G CG^{-1} d\tau \] (16)

with \( G \in R^{n \times m}, C \in R^{m \times n} \), when an explicit Euler implementation of the control is performed.
Similarly to the explicit Euler method, the explicit ZOH discretization may yield spurious oscillations as shown in [Galias-Yu, IEEE CAS 2008].

For an implicit Euler implementation, let us set

\[
\begin{align*}
    u_k &= -(C G)^{-1}(C F x_k + s_{k+1}) \\
    s_{k+1} &= \text{Sgn}(C x_{k+1}),
\end{align*}
\]

which corresponds to the implicit discrete time version of the ECB-SMC controller. We therefore get on each sampling period:

\[
x_{k+1} = \Phi x_k - \Gamma s_{k+1} \quad \text{for all} \quad t \in [kh, (k + 1)h]
\]
At each time–step, one has to solve

\[
\begin{align*}
x_{k+1} &= \Phi x_k - \Gamma s_{k+1} \\
y_{k+1} &= Cx_{k+1} + D \\
s_{k+1} &\in \text{Sgn}(y_{k+1})
\end{align*}
\]  

(19)

Inserting the first line of (19) into the second line we obtain the following one–step system (MLCP)

\[
\begin{align*}
y_{k+1} &= C\Phi x_k + D - C\Gamma s_{k+1} \\
s_{k+1} &\in \text{Sgn}(y_{k+1})
\end{align*}
\]  

(20)
A numerical example

The LTI system with an ECB-SMC controller is defined by the following data,

\[
F = \begin{bmatrix}
0 & 1 \\
-a_1 & -a_2 \\
\end{bmatrix}, \quad G = \begin{bmatrix}
0 \\
1 \\
\end{bmatrix}, \quad C = \begin{bmatrix}
c_1 & 1 \\
\end{bmatrix}.
\] (21)

Starting from the initial data, \( x_0 = [0.55, 0, 55]^T \), [Galias-Yu, IEEE CAS 2008] have shown that the Explicit ZOH discretization of the system with \( a_1 = -2, a_2 = 2, c_1 = 1 \) and \( h = 0.3 \) exhibits a period–2 orbit.
Figure: Equivalent control based SMC, $a_1 = -2$, $a_2 = 2$, $c_1 = 1$ and $h = 0.3$. $x_0 = [0.55, 0, 55]^T$ State $x_1(t)$ versus $x_2(t)$.

$\Rightarrow$ the chattering on the switching surface is suppressed.
The implicit Euler method allows one to nicely simulate the main features of sliding-mode systems:

- Finite-time stabilization on the switching surface (of codimension $\geq 1$)
- Smooth stabilization on the switching surface

It extends to the discrete-time implementation with ZOH discretization: looks like a promising solution for discrete-time sliding modes, a long-standing issue in control engineering.

Second talk: disturbance attenuation in first and second order systems with sliding mode control.
Chattering-free digital sliding-mode control with state observer and disturbance rejection

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GDR MACS, systèmes hybrides, 07 octobre 2010
In this second talk we focus on the application of the previous ideas (simulation and implementation of sliding mode controllers with implicit Euler methods) to various types of sliding mode systems with disturbance:

- first order
- first order plus disturbance compensation
- second order with disturbance compensation
- twisting and super twisting algorithms

with the Euler and the ZOH (zero order hold) discretisations methods.

**Objectives:**

- Numerical chattering suppression (like previous talk)
- Disturbance attenuation by factor $h$ or $h^2$
Definition

Let $h = t_{k+1} - t_k > 0$ be the sampling period, $k \geq 0$. An $m$-discrete-time sliding surface $\Sigma_d$ is a codimension $m$ subspace of the state space, such that the discrete state vector $x_k \triangleq x(t_k)$ satisfies $x_k \in \Sigma_d$ for all $k_{\min} \leq k \leq k_{\max}$, $k_{\min} < k_{\max} - 1$, $k_{\min} \geq 0$. Moreover this holds whatever $h > 0$.

A very attractive feature of the digital method based on the *implicit* Euler method is that the numerical sliding surface $\Sigma_d$ and the continuous-time sliding surface $\Sigma_c$ satisfy $\Sigma_d = \Sigma_c$.

If $\Sigma_c = \{ x \in \mathbb{R}^n \mid Cx + D = 0 \}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^m$, then $\Sigma_d = \{ x_k \in \mathbb{R}^n \mid Cx_k + D = 0 \}$. 
Let us start by considering the following basic sliding mode system:

\[
\begin{cases}
\dot{x}(t) = -a\tau(t) + \varphi(t) \\
\tau(t) \in \text{sgn}(x(t)),
\end{cases}
\]  

(22)

where \( \varphi(\cdot) \) is the perturbation such that \( \|\varphi\|_\infty < \rho < a \).

- The control input is here \( u(t) = \tau(t) \). It may be seen, in the language of differential inclusions theory, as a selection of the set-valued right-hand-side of the system.
- Choosing correctly this selection is the object of the following discretization.
- The system (22) has \( x = 0 \) as its unique equilibrium point, which is globally asymptotically stable and is reached in finite time (this may be shown with the Lyapunov function \( V(x) = x^2 \)).
The discrete-time sliding mode system is implemented as follows:

\[
\begin{align*}
\tilde{x}_{k+1} &= x_k - ah\tau_{k+1} \\
\tau_{k+1} &\in \text{sgn}(\tilde{x}_{k+1}) \\
x_{k+1} &= x_k - ah\tau_{k+1} + h\varphi_{k+1}
\end{align*}
\]  

(23)

- The first two lines of (23) may be considered as the \textit{nominal unperturbed} plant, from which one computes the input at time \( t_k \).
- The third line is the backward Euler approximation of the plant, on which the disturbance is acting. One has \( u(t) = \tau_{k+1} \) on the time-interval \( [t_k, t_{k+1}) \).
Proposition

Suppose that the initial state in (23) satisfies \(|x_0| > ah > 0\). Then after a finite number of steps \(k_0\) one obtains that \(\tilde{x}_k = 0\) and \(x_k = h\varphi_k\) for all \(k > k_0\). In other words, the disturbance is attenuated by a factor \(h\). Moreover the approximated derivative of the state satisfies \(\frac{x_{k+1} - x_k}{h} = \varphi_{k+1} - \varphi_k\) for all \(k > k_0 + 1\) whereas \(\frac{\tilde{x}_{k+1} - \tilde{x}_k}{h} = 0\) for all \(k > k_0\). The control input takes values inside the sign multifunction multivalued part on the sliding surface for all \(k > k_0\).

The fundamental idea behind such discretisation is that multivaluedness implies the continuity (in the graph sense) of the relay multifunction:

\[\rightsquigarrow\text{sliding mode is neither high frequency nor high gain!}\]
The nominal system attains its sliding surface \( \tilde{x}_k = 0 \) and on this surface the disturbance is attenuated by a factor \( h \) (in the continuous-time version the disturbance is exactly compensated).

The proof is rather easy by analysing by inspection the possible solutions of the generalised equation with unknown \( \tau_{k+1} \).

The controller is given simply by:

\[
\tau_{k+1} - \frac{x_k}{ah} \in - \mathcal{N}_{[-1,1]}(\tau_{k+1})
\]

which is equivalent to

\[
\tau_{k+1} = \text{proj}([-1, 1]; \frac{x_k}{ah})
\]
Let us consider the case with disturbance compensation. Let us define \( \dot{x}(t) = -a\tau_1(t) \), \( \tau_1(t) \in \text{sgn}(x(t)) \), \( e = x - \hat{x} \), and the controller \( u = -a\text{sgn}(x(t)) - \alpha\text{sgn}(e(t)) \), \( a > 0 \), \( \alpha > 0 \) and \( a < \alpha \).

Thus the closed-loop system is given by:

\[
\begin{align*}
\dot{x}(t) &= -a\tau_1(t) - \alpha\tau_2(t) + \varphi(t) \\
\dot{e}(t) &= -\alpha\tau_2(t) + \varphi(t) \\
\tau_1(t) &\in \text{sgn}(x(t)) \\
\tau_2(t) &\in \text{sgn}(e(t))
\end{align*}
\]

where \( \varphi(\cdot) \) is a disturbance such that \( \|\varphi\|_\infty < \rho < \min(a, \alpha) \).
The fixed point \((x, e) = (0, 0)\) of the system may be shown to be globally strongly asymptotically stable with the nonsmooth Lyapunov function \(V(x, e) = |x| + |e|\).

Moreover, the system attains in a finite time the sliding surface \(e = 0\) where it evolves according to the sliding dynamics \(\dot{x}(t) = -a \tau_1(t) + \varphi(t)\).

The condition \(a < \alpha\) implies that the origin is not attained directly, but first the system slides on the surface \(e = 0\). On this surface the dynamics in \(x\) evolves as a disturbance-free system.
The discrete sliding mode system is implemented as follows:

\[
\begin{align*}
\tilde{x}_{k+1} &= x_k - ah\tau_{1,k+1} - \alpha h\tau_{2,k+1} \\
\tilde{e}_{k+1} &= e_k - \alpha h\tau_{2,k+1} \\
\tau_{1,k+1} &\in \text{sgn}(\tilde{x}_{k+1}) \\
\tau_{2,k+1} &\in \text{sgn}(\tilde{e}_{k+1}),
\end{align*}
\]  
(25)

and the update procedure representing the plant dynamics is given by:

\[
\begin{align*}
x_{k+1} &= x_k - ah\tau_{1,k+1} - \alpha h\tau_{2,k+1} + h\varphi_{k+1} \\
e_{k+1} &= e_k - \alpha h\tau_{2,k+1} + h\varphi_{k+1}.
\end{align*}
\]  
(26)
An equivalent formulation of the discrete-time closed-loop system is:

\[
\begin{align*}
\tau_{1,k+1} &= \text{proj}([-1, 1]; \frac{x_k - \alpha h \tau_{2,k+1}}{ah}) \\
\tau_{2,k+1} &= \text{proj}([-1, 1]; \frac{e_k}{\alpha h}) \\
x_{k+1} &= x_k - ah \tau_{1,k+1} - \alpha h \tau_{2,k+1} + h \varphi_{k+1} \\
e_{k+1} &= e_k - \alpha h \tau_{2,k+1} + h \varphi_{k+1}
\end{align*}
\]

(27)

\[\Rightarrow \text{One sees that this is very easily implementable with nested projections.}\]
Proposition

Assume that $|e_0| > \alpha h > 0$. Then after a finite number of steps $k_0$ one obtains $\tilde{e}_k = 0$ and $e_k = h\varphi_{k+1}$ for all $k > k_0$. Let $|x_{k_0} - h\varphi_{k_0+1}| > ah > 0$. Then there exists $k_1 < +\infty$ such that $\tilde{x}_k = 0$ for all $k > k_0 + k_1$ and $x_k = h\varphi_k$ for all $k \geq k_0 + k_1$. 
Extension to higher-order systems

Let us consider the linear time-invariant system with disturbance

\[ \dot{x}(t) = Ax(t) + Bu(t) + D\varphi(t) \]

with \( \|\varphi(t)\|_1 \leq p\varphi_{\text{max}} \) for all \( t \), \( \varphi_{\text{max}} \geq |\varphi_i|_\infty \) for all \( 1 \leq i \leq p \) and \( D \in \mathbb{R}^{n \times p} \).

Let us choose a sliding surface

\[ \Sigma = \{ x \in \mathbb{R}^n \mid Cx = 0, C \in \mathbb{R}^{m \times n} \} \]

where \( m \) is the dimension of the input vector \( u(t) \). The ECB-SMC takes the form

\[ u \in -(CB)^{-1}CAx - \alpha(CB)^{-1}\text{Sgn}(Cx) \]

provided \( CB \) is full-rank.
Let $z \triangleq Cx$. The reduced closed-loop dynamics is

$$\dot{z}(t) = -\alpha \tau + CD \varphi(t), \quad \tau \in \text{Sgn}(z)$$

which is globally asymptotically stable and $\Sigma$ is reached in finite time provided $\alpha > p \|CD\| \varphi_{\text{max}}$

(this can be shown with the Lyapunov function

$$V(z) = \frac{1}{2} z^T z$$

that satisfies along the closed-loop trajectories

$$\dot{V}(t) \leq \|z\|_1 ( -\alpha + p \|CD\| \varphi_{\text{max}})$$)
The ZOH discretization of the ECB-SMC controller on \([t_k, t_{k+1}]\) takes the form:

\[ x_{k+1} = A^*(h)x_k - \alpha B^*(h)\tau_{k+1} + \varphi^*(h), \quad (28) \]

with

\[ A^*(h) = e^{Ah} - \int_0^h e^{At} dt \, B(CB)^{-1}CA \]

\[ B^*(h) = \int_0^h e^{At} dt \, B(CB)^{-1} \]

\[ \varphi_k^*(h) = \int_0^h e^{At} D\varphi((k + 1)h - \tau)d\tau \]
The implicit discrete-time sliding mode controller is calculated from the generalised equation:

\[
\begin{align*}
\begin{cases}
\dot{x}_{k+1} &= A^*(h)x_k - \alpha B^*(h)\tau_{k+1} \\
\tau_{k+1} &\in \text{Sgn}(C\dot{x}_{k+1}) \\
x_{k+1} &= A^*(h)x_k - \alpha B^*(h)\tau_{k+1} + C\varphi_k^*(h)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\downarrow
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
C\dot{x}_{k+1} &= CA^*(h)x_k - \alpha CB^*(h)\tau_{k+1} \\
\tau_{k+1} &\in \text{Sgn}(C\dot{x}_{k+1}) \\
Cx_{k+1} &= CA^*(h)x_k - \alpha CB^*(h)\tau_{k+1} + C\varphi_k^*(h)
\end{cases}
\end{align*}
\]
Suppose that the matrix $CB^*(h)$ is symmetric positive definite (since $CB^* = hl_m + O(h^2)$ it follows that for $h$ small enough $CB^* > 0$ is guaranteed if $CB$ is invertible). Then:

$$CA^*(h)x_k - \alpha CB^*(h)\tau_{k+1} \in N_{[-1,1]^m}(\tau_{k+1})$$

$$\uparrow$$

$$\tau_{k+1} = \text{proj}_{CB^*(h)}([−1, 1]^m; \frac{1}{\alpha}(CB^*(h))^{-1}CA^*(h)x_k)$$ (30)

$$\uparrow$$

$$\tau_{k+1} = \arg\min_{z\in[−1, 1]^m} \frac{1}{2}(z - z_k)^T CB^*(h)(z - z_k),$$

with $z_k = \frac{1}{\alpha}(CB^*(h))^{-1}CA^*(h)x_k$ and where $\text{proj}_{CB^*(h)}$ is the projection in the metric defined by $CB^*(h)$. 
The next results hold:

**Lemma**

Let \( C\tilde{x}_{k+1} = 0 \) for some \( k \geq 0 \). Then

\[
\|Cx_{k+1}\|_1 = \|C\varphi^*_k(h)\| = \leq hp \|C\| \|D\|\varphi_{\text{max}}
\]

**Proposition**

Let \( h > 0 \) be given. Suppose that the solution of (29) (a) satisfies \( \|x_k\| \leq M \) for all \( k \geq 0 \) and some \( M < +\infty \), and that \( CB^*(h) \) is symmetric positive definite, with \( CB^*(h) \geq \gamma I_m > 0 \) for some known \( \gamma \). Then there exists a constant \( \delta(h^2, M) \) such that if \( \alpha > \frac{m}{\gamma} \|C\| hp \|D\| \varphi_{\text{max}} + \delta(h^2, M) \), \( C\tilde{x}_{k+1} = 0 \) for some \( k \geq 0 \) implies \( C\tilde{x}_{k+n} = 0 \) for all \( n \geq 2 \).
The discrete state boundedness assumption is a natural assumption if one admits that the controller should first be designed such that the state indeed remains bounded...

Here we focus on the sliding mode behaviour solely.
Numerical simulations

Obtained with the INRIA software package **siconos**. The disturbance is taken as \( \varphi(t) = \phi \sin(\omega t) \) and we simulate the system:

\[
\begin{aligned}
\dot{x}(t) &= -a \tau_1(t) - \alpha \tau_2(t) + \varphi(t) \\
\dot{e}(t) &= -\alpha \tau_2(t) + \varphi(t) \\
\tau_1(t) &\in \text{sgn}(x(t)) \\
\tau_2(t) &\in \text{sgn}(e(t))
\end{aligned}
\]

(31)

The plant dynamics is integrated in all the simulations with the machine precision, whereas the controller sampling time is much larger: \( h = 10^{-1} \) s. This is equivalent to implementing a ZOH method.
(a) state $x_k$ and error $e_k$ vs. time

(b) Multiplier $\tau_{1,k+1}$, $\tau_{2,k+1}$ and perturbation $\varphi_k/\alpha$ vs. time

Figure: $a = 1$, $\alpha = 2$, $\phi = 0.1$, $\omega = 5$. 
Figure: $a = 1, \alpha = 2, \varphi(t) = \phi \sin \omega t, \phi = 0.1, \omega = 100.$
Figure: $a = 1$, $\alpha = 2$, $\varphi(t) = \phi \sin \omega t$, $\phi = 0.1$, $\omega = 100$. 
Figure: \( a = 1, \alpha = 2, \varphi(t) = \phi \sin \omega t, \omega = 100. \)
Second order systems

The plant dynamics is given by

\[ \ddot{x}(t) = u(t) + \varphi(x(t), t), \]  

(32)

where \( x(t) \in \mathbb{R} \) is the state vector, \( u(t) \in \mathbb{R} \) is the control input, the disturbance is \( \varphi(x, t) \in \mathbb{R} \). It is assumed that \( \varphi(x, t) \) is an unknown function such that:

\[ |\varphi(x, t)| < \varphi_{\text{max}} \]  

(33)

for almost all \( x, t \in \mathbb{R} \).
The model repeats the structure of the plant and is given by:

\[ \ddot{x}(t) = u(t) + v(t), \]  
\[ (34) \]

where \( v(t) \in \mathbb{R} \) is the model input. The error dynamics is then written as follows:

\[ \ddot{e}(t) = -v(t) + \varphi(x(t), t), \]  
\[ (35) \]

where \( e = x - \hat{x} \) is the deviation of the model state from the plant state. The error dynamics driven by the sliding mode input is given by:

\[ v(t) \in k_e \dot{e}(t) + k_s s_e(t) + M_v \text{sgn}(s_e(t)), \]  
\[ (36) \]

and it is globally asymptotically stabilized provided that \( M_v > \varphi_{\text{max}} \) and \( s_e = \dot{e} + k_e e \) where \( k_e \) and \( k_s \) are positive constants.
The control law

\[ u \in -v - M_x sgn(s_x) - k_x \dot{x}, \]  

(37)

with \( s_x = \dot{x} + k_x x \), asymptotically compensates for the disturbance \( \varphi(x, t) \). Once the sliding mode occurs on the surface \( s_e = 0 \), the plant equation takes the disturbance-free form

\[
\begin{cases}
\dot{s}_x(t) \in -M_x sgn(s_x(t)) - k_e \dot{e}(t) \\
\dot{e}(t) = -k_e e(t).
\end{cases}
\]  

(38)

because on this sliding surface one has \( M_v sgn(s_e(t)) = \varphi(x(t), t) \). Since the dynamics (38) has \( s_x^* = 0 \) as a globally asymptotically stable fixed point, the desired disturbance compensation is thus provided.
Let $z = [e_s e_x s_x]^T$. The coupled plant/error dynamics in the closed-loop system is given by:

$$
\begin{align*}
\dot{z}(t) &= \begin{bmatrix}
-k_e & 1 & 0 & 0 \\
0 & -k_s & 0 & 0 \\
0 & 0 & -k_x & 1 \\
-k_e & -k_s & 0 & 0
\end{bmatrix} z(t) - \begin{bmatrix}
0 & 0 \\
M_v & 0 \\
0 & 0 \\
M_v & M_x
\end{bmatrix} \tau(t) + \begin{bmatrix}
0 \\
\varphi(x(t), t) \\
0 \\
\varphi(x(t), t)
\end{bmatrix} \\
\tau(t) &\in \text{Sgn} \left( \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} z(t) \right),
\end{align*}
$$

(39)

It is noteworthy that the $(e_s)$ subdynamics is decoupled from the $(x_s)$ subdynamics, as expected.
Proposition

Consider the closed-loop system (39) with positive gains $k_e, k_s, M_x, M_v$ and an external disturbance $\varphi(x, t)$ such that (33) holds for almost all $x \in \mathbb{R}, t \in \mathbb{R}$ and $M_v > \varphi_{\text{max}}$. Then after a finite time, this system evolves in the sliding mode along the surfaces $s_e = 0$ and $s_x = 0$, and along these surfaces, the system dynamics is governed by the asymptotically stable, disturbance-free equations (38).
The backward Euler time-discretization

Let us consider the first error dynamics

\[ \dot{s}_e(t) \in -k_s s_e(t) - M_v \text{sgn}(s_e(t)) + \varphi(x(t), t), \quad (40) \]

and discretize it on \([t_k, t_{k+1})\) as:

\[
\begin{aligned}
\dot{s}_{e,k+1} &= s_{e,k} - h k_s s_{e,k} - h M_v \tau_{1,k+1} \\
\tau_{1,k+1} &\in \text{sgn}(\dot{s}_{e,k+1}) \\
s_{e,k+1} &= s_{e,k} - h k_s s_{e,k} - h M_v \tau_{1,k+1} + h \varphi(x_{k+1}, t_{k+1}) \\
e_{k+1} &= e_k + h \dot{e}_{k+1}
\end{aligned}
\quad (41)

for all \(k \geq 0\).
The first two lines are a generalized equation with unknown \( \tilde{s}_{e,k+1} \), which we may rewrite as \( 0 \in F(\tilde{s}_{e,k+1}) \) for some multifunction \( F : \mathbb{R} \to 2^\mathbb{R} \).

This generalised equation has a unique solution since the sign multifunction is maximal monotone and \( F(\cdot) \) is 2-monotone as the sum of a monotone and a 2-monotone multifunctions.
The next result characterizes the evolution of \( e_k \) on the sliding surface \( \tilde{s}_{e,k} = 0 \).

**Lemma**

*Suppose that the sliding surface \( \tilde{\Sigma}_e = \{\tilde{s}_{e,k} \in \mathbb{R} \mid \tilde{s}_{e,k} = 0\} \) is attained at \( k = k_0 \) and that the system stays on it. Take for simplicity \( k_0 = 0 \). Then:*

\[
e_{k+1} = (1 + hk_e)^{-k-1} e_0 + h^2 (1 + hk_e)^{-1} \sum_{i=0}^{k} (1 + hk_e)^{i-k} \varphi(x_i, t_i).
\]

(42)
The second part of the error dynamics is now discretized as follows:

\[
\begin{align*}
\tilde{s}_{x,k+1} &= s_{x,k} - hk_e \dot{e}_k - hk_s s_{e,k} - hM_v \tau_{1,k+1} - hM_x \tau_{2,k+1} \\
\tau_{2,k+1} &\in \text{sgn}(\tilde{s}_{x,k+1}) \\
s_{x,k+1} &= s_{x,k} - hk_e \dot{e}_k - hk_s s_{e,k} - hM_v \tau_{1,k+1} - hM_x \tau_{2,k+1} + h\varphi_{k+1} \\
x_{k+1} &= x_k + h\dot{x}_{k+1}
\end{align*}
\]

(43)

Thus the following holds:

Proposition

Consider the discrete-time system (43) that represents the system's dynamics on the sliding surface \( \tilde{\Sigma}_e = \{ \tilde{s}_{e,k} \in \mathbb{R} \mid \tilde{s}_{e,k} = 0 \} \).

Suppose that \( M_x > \varphi_{\text{max}} \). There exists \( k_1 < +\infty \), such that for all \( k \geq k_1 \) one has \( \tilde{s}_{x,k} = 0 \). Then \( |s_{x,k+1}| \leq h\varphi_{\text{max}} + |\epsilon_k| + h^2 \alpha \varphi_{\text{max}} \).
The next result characterizes the dynamics of \( x_k \) on the sliding surface \( \tilde{x}_k = 0 \). For simplicity we take \( k_1 = 0 \) in Proposition 7.

**Lemma**

*Suppose that for \( k \geq 0 \) the system evolves on the sliding surface \( \tilde{s}_{x,k} = 0 \), so that (neglecting terms in \( h^2 \)) \( |s_{x,k+1}| \leq h\varphi_{\text{max}} + |\epsilon_k| \). Then*

\[
x_k = (1 + hk_x)^{-1}x_0
- h(1 + hk_x)^{-1}\sum_{i=0}^{k-1}(1 + hk_x)^{-i}(\epsilon_{k-1-i} + h^2\alpha_{k-1-i} + h\varphi_{k-i})
\]

(44)
The disturbance is therefore attenuated by a factor $h^2$ on the state “position” $x_k$. We may rewrite the discrete-time closed-loop system as:

$$
\begin{align*}
 s_{x,k+1} &= s_{x,k} - h k e \dot{e}_k - h k_s s_{e,k} - h M_v \tau_{1,k+1} - h M_x \tau_{2,k+1} + h \varphi(x_{k+1}, e_{k+1}) \\
 s_{e,k+1} &= s_{e,k} - h k_s s_{e,k} - h M_v \tau_{1,k+1} + h \varphi(x_{k+1}, t_{k+1}) \\
 x_{k+1} &= x_k + h \dot{x}_{k+1} \\
 e_{k+1} &= e_k + h \dot{e}_{k+1} \\
 \tau_{1,k+1} &= \text{proj}([-1, 1]; \frac{s_{e,k} - h k_s s_{e,k}}{h M_v}) \\
 \tau_{2,k+1} &= \text{proj}([-1, 1]; \frac{s_{x,k} - h k e \dot{e}_k - h k_s s_{e,k} - h M_x \tau_{1,k+1}}{h M_x}).
\end{align*}
$$

(45)

The controller has a nested-projection structure and is easily implementable at time $t = t_k$ with the knowledge of $x_k$, $x_{k-1}$ and $e_k$, $e_{k-1}$. 

Extensions

The method applies to so-called *twisting* and *super-twisting* controllers, however in such cases there is neither maximal monotonicity, nor decoupling of the closed-loop dynamics.

Results are therefore only partial:

- existence/uniqueness of the discrete-time system fixed point,
- disturbance attenuation on the sliding surface.
- but no proof for finite-time convergence to the fixed point.

*Future works:* relax the matching conditions between controller and disturbance.