Min-switching stabilization for discrete-time switching systems with nonlinear modes

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Discrete-time Lur’e system:

\[ x_{k+1} = Ax_k + F \varphi(y_k), \quad (1) \]
\[ y_k = Cx_k, \quad (2) \]

where \( x_k \in \mathbb{R}^n \), \( y_k \in \mathbb{R}^p \), \( (k \in \mathbb{N}) \).

Assumption:

- The nonlinearity \( \varphi(y) \) verifies the cone bounded sector condition:
  \[ \text{SC}(\varphi(\cdot), y, \Lambda) = \varphi'(y)\Lambda[\varphi(y) - \Omega y] \leq 0, \quad (3) \]

with \( \Lambda \in \mathbb{R}^{p \times p} \) diagonal positive definite.

Aim:

- Consider a suitable Lur’e-like Lyapunov function in order to:
  - propose sufficient conditions for the global stability analysis problem (Lur’e problem);
  - cover a wider range of cone bounded nonlinearities;
  - relax the assumptions of the classical literature of the Lur’e problem.
Discrete-time switched system composed of Lur’e subsystems:

\[
x_{k+1} = A_{\sigma(k)} x_k + F_{\sigma(k)} \varphi_{\sigma(k)}(y_k),
\]
\[
y_k = C_{\sigma(k)} x_k,
\]

where \( x_k \in \mathbb{R}^n, y_k \in \mathbb{R}^p, (k \in \mathbb{N}) \).

Particularities:

- The function \( \sigma(\cdot) : \mathbb{N} \to \mathcal{I}_N = \{1, \ldots, N\} \) is the switching rule;
- The nonlinearities \( \varphi_i(\cdot) \) are mode dependent and switch;
- The modal nonlinearity can be a source of additional instability.

Aim:

- Extend our appropriate Lur’e-like Lyapunov function in order to:
  - incorporate the case of switching nonlinearities;
  - study the Lur’e problem in the switched systems framework;
  - provide a min-switching strategy which renders the origin of a family of discrete-time Lur’e subsystems asymptotically stable.
Outline of the talk

Concept of Lur’è problem

Switched Lur’è systems

Global stabilization

Switched Lur’è system with control saturation

Local stabilization

Conclusions
Concept of Lur’e problem

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Conclusions
Standard discrete-time Lur’e problem

Classical Lyapunov functions:
- The quadratic function with respect to the state (circle criterion): $x'Px$;
- Lur’e-type Lyapunov function (Popov criterion):
  \[ v(x) = \alpha x^2 + 2\eta \int_{0}^{C'x} \varphi(s)ds, \alpha > 0, \eta \geq 0; \]
  - $\varphi(\cdot)$ must be time-invariant to have: $\int_{0}^{C'x} \varphi(s)ds \geq 0$;
  - $v(x)$ is inspired from the continuous-time;
  - In continuous-time case, only (3) is needed to conclude $\dot{v} < 0$;
- In discrete-time, an extra assumption is necessary to bound $\int_{y_k}^{y_{k+1}} \varphi(s)ds$. Ex: $\frac{d\varphi(y)}{dy} \leq K_{\max}$.

A counterexample: half-circle allowing vertical tangents.
A Lur’ě-like Lyapunov function for discrete-time

Definitions

\[ V : \begin{cases} \mathbb{R}^n \times \mathbb{R}^p & \rightarrow \mathbb{R}, \\ (x; \varphi(Cx)) & \mapsto x'Px + 2\varphi(Cx)\Delta \Omega Cx, \end{cases} \]  

- with \( 0_n < P = P' \in \mathbb{R}^{n \times n} \) and \( 0_p \leq \Delta \in \mathbb{R}^{p \times p} \) diagonal.

Bounding quadratic functions:

\[ \underline{V}(x) \leq V(x; \varphi(Cx)) \leq \overline{V}(x). \]  

where \( \underline{V}(x) = x'Px \) and \( \overline{V}(x) = x'(P + 2C'\Omega\Delta \Omega C)x. \)
Basic properties

Candidate Lyapunov function:

- \( V(x; \varphi(Cx)) \geq 0 \) due to \( P > 0_n \) and the sector condition (3) of \( \varphi(\cdot) \).
- \( V(x; \varphi(Cx)) = 0 \iff x = 0 \), due to \( P > 0_n \).
- Relation (7) implies that function (6) is radially unbounded.
- Lyapunov difference: \( \delta_k V = V(x_{k+1}; \varphi(Cx_{k+1})) - V(x_k; \varphi(Cx_k)) \).

The level set of our function (6)

\[
L_V(\gamma) = \{ x \in \mathbb{R}^n; V(x; \varphi(Cx)) \leq \gamma \}. \tag{8}
\]

- The set \( L_V(\gamma) \) may be non-convex and disconnected.
Global Stability Analysis

If there exists a matrix $G \in \mathbb{R}^{n \times n}$, a matrix $0_n < P = P^{'} \in \mathbb{R}^{n \times n}$, a diagonal matrix $0_p \leq \Delta \in \mathbb{R}^{p \times p}$ and diagonal matrices $0_p < T, W \in \mathbb{R}^{p \times p}$, such that the LMI

$$
\begin{bmatrix}
P - G' - G & G' A & G' F & 0_{n \times p} \\
\star & -P & C' \Omega [T - \Delta] & A' C' \Omega [W + \Delta] \\
\star & \star & -2T & F' C' \Omega [W + \Delta] \\
\star & \star & \star & -2W
\end{bmatrix} < 0_{2n+2p},
$$

is verified, then the function $V(x; \varphi(Cx))$ is a Lyapunov function and the origin of system (1)-(2) is globally asymptotically stable.

Main idea:

$$
\delta_k V - 2SC(\varphi(\cdot), y_{k+1}, W) - SC(\varphi(\cdot), y_k, T) < 0 \ \forall x_k \neq 0.
$$

No assumption about the variation of $\varphi(\cdot)$.

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Example: global stability analysis

- Discrete-time Lur’e system with \( n = 2, \ p = 1, \ \Omega = \frac{1}{\sqrt{2}} \):

\[
A = \begin{bmatrix} 0.5 & 0.1 \\ 0.3 & -0.4 \end{bmatrix}; \quad F = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}; \quad C' = \begin{bmatrix} 1 \\ 0 \end{bmatrix};
\]

- The nonlinearity is \( \varphi(y) = 0.5\Omega y(1 + \cos(10y)) \);

- Our Lyapunov function (6) exists for this system, and the parameters are given by

\[
P = \begin{bmatrix} 0.9825 & -0.0846 \\ -0.0846 & 0.9476 \end{bmatrix}; \quad \Delta = 0.7503.
\]
Asymptotic stability

Contractivity of the level set $L_{V}(\gamma = V(x_k, \varphi(y_k)))$, for $k = 0, \ldots, 15.$
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![Graph showing contractivity of level set](image-url)
Asymptotic stability

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\[ V(x_k, \varphi(y_k)) \]

$V(x_k, \varphi(y_k))$ for $k = 0, \ldots, 15$. 

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\begin{align*}
&\text{Asymptotic stability} \\
&\text{Contractivity of the level set } L_V(\gamma = V(x_k, \varphi(y_k))), \text{ for } k = 0, \ldots, 15.
\end{align*}
Asymptotic stability

Contractivity of the level set $L_V(\gamma = V(x_k, \varphi(y_k)))$, for $k = 0, \ldots, 15$. 

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Asymptotic stability

Contractivity of the level set $L_V(\gamma = V(x_k, \varphi(y_k)))$, for $k = 0, \ldots, 15$. 
Concept of Lur’e problem

Switched Lur’e systems

Global stabilization

Switched Lur’e system with control saturation

Local stabilization

Conclusions
Switched Lur’ë system

Discrete-time switched system composed of Lur’ë subsystems:

\[
    x_{k+1} = A_{\sigma(k)} x_k + F_{\sigma(k)} \varphi_{\sigma(k)}(y_k), \\
    y_k = C_{\sigma(k)} x_k,
\]

where \( x_k \in \mathbb{R}^n, y_k \in \mathbb{R}^p, \sigma(\cdot) : \mathbb{N} \to \mathcal{I}_N = \{1, \ldots, N\} \).

Motivation:

- The active nonlinearity is defined by the switching rule.
- Each mode is associated with a nonlinearity;
- The sector conditions are mode-dependents, \( \forall i \in \mathcal{I}_N : 
  \)
  \[
  SC(\varphi_i(\cdot), y, \Lambda_i) = \varphi_i'(y) \Lambda_i [\varphi_i(y) - \Omega_i y] \leq 0
  \]
Problem formulation

Assumptions:
- The system (10)-(11) verifies the assumption, \( \forall i \in \mathcal{I}_N : \)
  - The modal nonlinearities \( \varphi_i(\cdot) \) are available in real time by model estimation or measuring.

Aim:
- Design the following class of state-dependent switching law
  \[
  \sigma(k) = g(x_k)
  \]  (13)

which globally asymptotically stabilizes the origin of the system (10)-(11).
Main idea

Discrete-time switched linear system

\[ x_{k+1} = A_{\sigma(k)} x_k, \] (14)

where \( x_k \in \mathbb{R}^n \), \( \sigma \in \mathcal{I}_N \). Consider \( V(i; x_k) = x_k' P_i x_k \).

Min-switching strategy:
If the Lyapunov-Metzler inequalities are satisfied

\[ A_i' \left( \sum_{\ell \in \mathcal{I}_N} \pi_{\ell i} P_\ell \right) A_i - P_i < 0 \quad \forall i \in \mathcal{I}_N, \] (15)

then \( \sigma(k) = g(x_k) = \arg \min_{i \in \mathcal{I}_N} V(i; x_k) \) stabilizes (14).

Set of Metzler matrices:
The matrix \( \Pi \in \mathcal{M}_d \), where \( \mathcal{M}_d \) is the Metzler matrices set:

\[ \mathcal{M}_d = \left\{ \Pi \in \mathbb{R}^{N \times N}, \pi_{ii} \geq 0, \sum_{\ell \in \mathcal{I}_N} \pi_{\ell i} = 1, \forall i \in \mathcal{I}_N \right\}. \]
Main idea

Example: Discrete-time switched linear system with two modes \((N = 2)\)

\[ A_1 = \begin{bmatrix} 1.08 & 0 \\ 0 & 0.72 \end{bmatrix}; \ A_2 = \begin{bmatrix} -0.55 & 1.1 \\ 0 & -1.1 \end{bmatrix}; \]

The min-switching strategy imposes: \( \min_{j \in \mathcal{I}_N} V(j; x_{k+1}) \leq \min_{i \in \mathcal{I}_N} V(i; x_k) \).
Main tool:

- The extension of our function (6) to the switched systems framework:\n\[
V : \mathcal{I}_N \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}, \quad \begin{cases} 
(i, x, \varphi_i(C_i x)) & \mapsto x' P_i x + 2(\varphi_i(C_i x))' \Delta_i \Omega_i C_i x, 
\end{cases} \quad (16)
\]

- Consider the function \( V_{\text{min}}(x_k) = \min_{i \in \mathcal{I}_N} V(i, x_k, \varphi_i(C_i x_k)) \)

  - inherits all the basic properties of function (16).

Auxiliary notation:

- Extended system matrices and state vector:

\[
\mathbb{A}_i = \begin{bmatrix} A_i & F_i & 0_{n \times Np} \end{bmatrix} \in \mathbb{R}^{n \times (n+(N+1)p)};
\]
\[
\mathbb{E}_i = \begin{bmatrix} 0_{p \times (n+ip)} & I_p & 0_{p \times (N-i)p} \end{bmatrix} \in \mathbb{R}^{p \times (n+(N+1)p)};
\]
\[
Z'_k = \begin{pmatrix} x'_k & \varphi'_i(C_i x_k) & \varphi'_1(C_1 x_{k+1}) & \ldots & \varphi'_N(C_N x_{k+1}) \end{pmatrix}' \in \mathbb{R}^{(n+(N+1)p)}.
\]

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Theorem : Min-switching strategy based on $V(i, x_k, \varphi_i(C_ix_k))$

Assume there exist a matrix $\Pi \in \mathcal{M}_d$; matrices $0_n < P_i = P_i' \in \mathbb{R}^{n \times n}$ and diagonal matrices $0_p < T_i, W_i, 0_p \leq \Delta_i \in \mathbb{R}^{p \times p}$, $(i \in \mathbb{I}_N)$, such that the Lyapunov-Metzler inequalities are satisfied $\forall i \in \mathbb{I}_N$

\[
\begin{bmatrix}
P_i & * & * \\
(\Delta_i - T_i)\Omega_i C_i & 2T_i & * \\
0_{Np \times n} & 0_{Np \times p} & 0_{Np} \\
\end{bmatrix} < 0_{n+(N+1)p}, \quad (17)
\]

where $(P)_{p,i} = \sum_{\ell \in \mathbb{I}_N} \pi_{\ell i} P_{\ell}$, then the min-switching strategy

\[
\sigma(k) = u(x_k) = \arg \min_{i \in \mathbb{I}_N} V(i, x_k, \varphi_i(C_ix_k)) \quad (18)
\]

globally asymptotically stabilizes the system (10)-(11).
Sketch of the proof

The matrix inequalities (17) are formulated in order to:

- Consider the sum of:
  - the sector condition at time $k + 1$:
    \[
    \varphi'_q(C_q x_{k+1}) W_q[\varphi_q(C_q x_{k+1}) - \Omega_q C_q x_{k+1}] \leq 0,
    \]  
  - written in the equivalent form:
    \[
    -z'_k \left( 2E'_q W_q E_q - \text{He}(E'_q W_q \Omega_q C_q A_i) \right) z_k \geq 0,
    \]  
    with $0_p < W_q \in \mathbb{R}^{p \times p}$ diagonal.

- Upper-bound the function $V_{\min}(x_{k+1}) = \min_{j \in \mathcal{I}_N} V(j, x_{k+1}, \varphi_j(C_j x_{k+1}))$ by the aid of these sector conditions;

- Guarantee, due to properties of the Metzler matrix $\Pi \in \mathcal{M}_d$, that $V_{\min}(x_{k+1}) - V_{\min}(x_k) < 2SC(\varphi_{\sigma(k)}(\cdot), y_k, T_{\sigma(k)}) \leq 0.$
State space partition:

- Let the sets $S_i$ allowing to activate the mode $i \in \mathcal{I}_N$:
  \[ S_i = \{ x \in \mathbb{R}^n, \ V_{\text{min}}(x) = V(i, x, \varphi_i(C_ix)) \} , \ \forall i \in \mathcal{I}_N. \] (20)

- Some remarks about these sets $S_i$:
  - $0 \in S_i, \forall i \in \mathcal{I}_N$;
  - $\cup_{i \in \mathcal{I}_N} S_i = \mathbb{R}^n$, at least one mode reaches the minimum of our function;
  - the sets $S_i$ are not necessarily disjoint.

Remark: Feasibility of Inequalities (17) implies inclusions $\pi_{ii}^\frac{1}{2} A_i$ and $\pi_{ii}^\frac{1}{2} (A_i + B_i \Omega_i C_i)$ stable, $\forall i \in \mathcal{I}_N$. 
Illustration

Example: global stabilization

- Switched Lur’e system with $N = n = 2, p = 1, \Omega_1 = 0.6; \Omega_2 = 0.4$:

  $A_1 = \begin{bmatrix} 1.08 & 0 \\ 0 & -0.72 \end{bmatrix}; F_1 = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}; C_1' = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix};$

  $A_2 = \begin{bmatrix} -0.48 & 0.8 \\ 0 & 0.8 \end{bmatrix}; F_2 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}; C_2' = \begin{bmatrix} 0.4 \\ 1 \end{bmatrix}.$

- The nonlinearities are: $\varphi_1(y) = 0.5\Omega_1y(1 + \cos(2y))$ and $\varphi_2(y) = 0.5\Omega_2y(1 - \sin(2.5y)).$

- The numerical results are obtained:

  $P_1 = \begin{bmatrix} 1.1490 & -0.0832 \\ -0.0832 & 1.9764 \end{bmatrix}; P_2 = \begin{bmatrix} 0.3508 & -0.4489 \\ -0.4489 & 3.1440 \end{bmatrix};$

  $\Delta_1 = 0.2585; \Delta_2 = 1.0509; \text{ with the Meztler matrix } \Pi = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}.$
Illustration

State space partition and a trajectory for $x_0 = (14; 11)'$

Set $S = S_1 \cap S_2$ and bounding cones $C_1; C_2$. Trajectory $x_k$ and the modes selected at each instant $k$.

With $\Delta_i \neq 0_p$, the state partition exhibits ripples.
Concept of Lur’e problem

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Conclusions
Discrete-time switched Lur’e system with input saturation

Discrete-time switched system composed of Lur’e subsystems and control saturation:

\[
x_{k+1} = A_{\sigma(k)}x_k + F_{\sigma(k)}\varphi_{\sigma(k)}(y_k) + B_{\sigma(k)}\text{sat}(u_k), \quad (21)
\]
\[
y_k = C_{\sigma(k)}x_k, \quad (22)
\]

where \(x_k \in \mathbb{R}^n\), \(y_k \in \mathbb{R}^p\) and \(u_k \in \mathbb{R}^m\).

Assumptions:
- The state and the modal nonlinearities are available in real time;
- The switched feedback control law is considered:
  \[
u_k = K_{\sigma(k)}x_k + \Gamma_{\sigma(k)}\varphi_{\sigma(k)}(y_k).
  \]

Input saturation:
- Only local stability can be assured;
- The basin of attraction \(B_0\) may be non-convex and disconnected.

Aim:
- For given control gains, not necessarily stabilizing, design the switching law \(\sigma(k) = g(x_k)\) ensuring the local asymptotic stability of the switched system (21)-(22).
Tools

Main tools:
- Consider the function \( V_{\min}(x) = \min_{i \in \mathcal{I}_N} V(i, x, \varphi_i(C_i x)) \) as candidate Lyapunov function,
- whose the level sets are given by:
  \[
  L_{V_{\min}}(\gamma) = \{ x \in \mathbb{R}^n ; V_{\min}(x) \leq \gamma \}
  = \bigcup_{j \in \mathcal{I}_N} \{ x \in \mathbb{R}^n ; V(j; x; \varphi_j(C_j x)) \leq \gamma \}.
  \]
  and the set \( L_{V_{\min}}(1) \) will be considered as an estimate of \( \mathcal{B}_0 \).

Auxiliary tools:
- Model the saturation as the dead-zone:
  \[
  \Psi(u_k) = u_k - \text{sat}(u_k)
  \]
  (23)
- Consider the modal sector condition for dead-zone:
  \[
  SC_{u_k} = \Psi'(u_k) U_i [\Psi(u_k) - J_{1,i} x_k - J_{2,i} \varphi_i(y_k)] \leq 0,
  \]
  (24)
  which is locally verified in the set:
  \[
  \mathcal{T}_i(\hat{K}_i - \hat{J}_i, \rho) = \left\{ \theta \in \mathbb{R}^{n+p} ; -\rho \leq (\hat{K}_i - \hat{J}_i)\theta \leq \rho \right\},
  \]
  (25)
  only for the \( i \)-th active mode.
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Main idea : (1 mode)

For 1 mode :

\[ S(\hat{K} - \hat{J}, \rho) \]

Inclusions as Matrix Inequalities\(^4\)

IM1) Ball of radius \(\frac{1}{\sqrt{\mu}}\) included inside \(L_V(1)\).

IM2) \(L_V(1) \subset T(\hat{K} - \hat{J}, \rho)\) such that \(SC_{u_k} \leq 0\).

IM3) \(\delta_k V - 2SC_{u_k} - 2SC(\varphi(\cdot), y_{k+1}, W) - SC(\varphi(\cdot), y_k, T) < 0\).

Conclusion : on \(L_V(1)\), \(\delta_k V < 0, \forall x \neq 0\).

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Local stabilization

Lemma 1
Assume there exist a matrix \( \Pi \in \mathcal{M}_d \); matrices \( 0_n < P_i = P'_i \in \mathbb{R}^{n \times n} \) and diagonal matrices \( 0_p \leq \Delta_i \in \mathbb{R}^{p \times p} \), diagonal matrices \( 0_p < Q_i, T_i, W_i \in \mathbb{R}^{p \times p}, 0_m < U_i \in \mathbb{R}^{m \times m} \), matrices \( J_{1,i} \in \mathbb{R}^{m \times n}, J_{2,i} \in \mathbb{R}^{m \times p} (i \in \mathcal{I}_N) \), such that the Lyapunov-Metzler inequalities are satisfied \( \forall i \in \mathcal{I}_N : \)

\[
\begin{align*}
A'_i(P)p_iA_i + \operatorname{He}(A'_i(C^\top\Omega\Delta E)p_i) - \sum_{q \in \mathcal{I}_N} (2E'_q W_q E_q - \operatorname{He}(E'_q W_q \Omega_q C_q A_q)) < 0_{n+m+(N+1)p} \quad (26)
\end{align*}
\]

then the switching law \( \sigma(k) \in \arg \min_{i \in \mathcal{I}_N} V(i, x_k, \varphi_i(C_i x_k)) \) ensures

\[
V_{\min}(x_{k+1}) - V_{\min}(x_k) < 2SCu_k < 0 \quad (27)
\]
Local stabilization

Lemma 2
Assume there exist matrices $0_n < P_i = P_i' \in \mathbb{R}^{n \times n}$ and diagonal matrices $0_p \leq \Delta_i \in \mathbb{R}^{p \times p}$, diagonal matrices $0_p < Q_i \in \mathbb{R}^{p \times p}$, and matrices $J_{1,i} \in \mathbb{R}^{m \times n}$, $J_{2,i} \in \mathbb{R}^{m \times p}$ ($i \in \mathcal{I}_N$), verifying the LMIs,
\[ \forall (i, \ell) \in \mathcal{I}_N \times \{1, \ldots, m\}, \]
\[
\begin{bmatrix}
   P_i & \ast & \ast \\
   (\Delta_i - Q_i)\Omega_i C_i & 2Q_i & \ast \\
   (K_i - J_{1,i})(\ell) & (\Gamma_i - J_{2,i})(\ell) & \rho(\ell)^2 \\
\end{bmatrix} > 0_{n+p+1},
\]
then our proposed min-switching strategy induce the inclusion
\[ L_{V_{\text{min}}}(1) \subset \mathcal{T}_j(\hat{K}_j - \hat{J}_j, \rho) \]
and the modal sector condition $SC_{U_k}$ is verified for the modes $j$.

Inequalities (26)-(28) combined guarantee the asymptotic stability on the set $L_{V_{\text{min}}}(1)$
Optimization problem

Determining the largest $L_{\nu \min}(1)$:

The following optimization problem

\[
\min_{\Pi, P_i, U_i, Q_i, T_i, W_i, R_i, \Delta_i, J_{1,i}, J_{2,i}, \mu} \quad \mu
\]

under the constraints (26)-(28) and

\[
\begin{bmatrix}
\mu I_n - P_i & * \\
-(\Delta_i + R_i)\Omega_i C_i & 2R_i
\end{bmatrix} > 0_{n+p}, \quad \forall i \in \mathcal{I}_N,
\]

provides the largest set $L_{\nu \min}(1)$ by maximizing the radius of the sphere $E(\mu I_n)$.

The particular structure of the Metzler matrix $\Pi \in \mathbb{R}^{N \times N}$ is considered:

\[
\Pi = \begin{bmatrix}
\gamma & 1 - \gamma \\
1 - \gamma & \gamma
\end{bmatrix}.
\]
Illustration

Example: local stabilization

- Switched Lur’e system with input saturation with \( N = n = 2, \rho = 1, \rho = 5; \Omega_1 = 0.7; \Omega_2 = 0.5 \):

\[
A_1 = \begin{bmatrix} 1.4 & 0.4 \\ 0.2 & 1 \end{bmatrix}; \quad F_1 = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix}; \quad B_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}; \quad C_1' = \begin{bmatrix} 0.9 \\ 0.5 \end{bmatrix};
\]

\[
A_2 = \begin{bmatrix} 1.1 & 0.6 \\ 0.3 & 1.5 \end{bmatrix}; \quad F_2 = \begin{bmatrix} 1.2 \\ 1 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix}; \quad C_2' = \begin{bmatrix} 1 \\ 0.7 \end{bmatrix}.
\]

- The nonlinearities \( \varphi_i(y) \) are defined by, \( \forall y \in \mathbb{R} \):
  \[
  \varphi_1(y) = 0.5\Omega_1 y (1 + \cos(20y)); \quad \varphi_2(y) = 0.5\Omega_2 y (1 - \sin(25y)).
  \]

- The control gains are given by:
  \[
  K_1 = \begin{bmatrix} -0.7168 & -1.0136 \end{bmatrix}; \quad \Gamma_1 = -1.2923;
  \]
  \[
  K_2 = \begin{bmatrix} -1.2581 & -0.7326 \end{bmatrix}; \quad \Gamma_2 = -1.4650;
  \]
State-space partition inside $L_{V_{\text{min}}} (1)$
mode 1 is the blue region and mode 2 is the red region.
Illustrations

2 trajectories, one from $x_0$ settled in the disconnected $LV_{\min}(1)$. Red circle (resp. a black star) means the mode 1 is active (resp. mode 2).
Illustrations

Mapped $x_0$ leading to **unstable** trajectories.

Our estimate is adapted to the shape of $B_0$. 
Concept of Lur’e problem

Switched Lur’e systems

Global stabilization

Switched Lur’e system with control saturation

Local stabilization

Conclusions
Conclusions:

- A suitable switched Lur’e-like Lyapunov function is associated with discrete-time switched systems with **nonlinear modes**.
  - This function enables to cope with switched nonlinearities.
  - This function requires only the cone bounded sector condition assumption.

- Min-switching strategy for global stabilization of based on this function is proposed
  - State space partition related to mode activation are not only restricted to conic regions.

- This function and its level set provide a switching strategy enlarging:
  - the estimate of the basin of attraction of switched Lur’e systems with control saturation.
Thank you very much for your attention!