Observer based stabilization of linear system under sparse measurement

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Road Map

- Introduction
- A Stability Results
- Stable unmeasured subspace
- Unstable unmeasured subspace
- Example
- Example of base
Questions

**Figure**: (a) The traditional paradigm to capture information; (b) The CS paradigm to capture information, where $\Phi$ is a random sensing matrix.
How use Compressive Sensing in Control System theory?

This question generates several important questions:

- How to pass from signal to system?
- What is the appropriate base for system?
- How to verify the RIP condition for system?
- How to bypass the optimization algorithm in order to deal with real time algorithms?
Answers

Answer to question a :
- Remember Kalman-Bucy filter
- Signal is generated by a system
- Cluster, Bayesian approach,... can be see as the first step of a modelization

Answer to question b :
- Bases in CS are Frequency, Wavelet, Curvelet,...
- Oscillations are generated by system
- Some normal forms may be use as base?
- Some other assumption than $s < m << N$ are generally done in CS (approach signal processing). In system theory for example diagnostic, only few systems are selected.

An example of base will be given at the end of this presentation.
Answer to question c:
In Compressive Sensing

\[ y = \Phi v \]

with \( y \in \mathbb{R}^m \) and \( v \in \mathbb{R}^n \). The RIP condition is: There exists \( \delta \in ]0, 1[ \) such that

\[
(1 - \delta) \|v\|_2^2 < \|\Phi v\|_2^2 < (1 + \delta) \|v\|_2^2
\]

With respect to linear system undersampling this condition is closed to a random sampling in order to preserve the system observability.
Answer to question d:

In this presentation an answer to question d is given, but adding a stability study of the observer based control. In the next, it is firstly recalled and introduced some stability results before to present our main results.
References

- Some references in Signal Processing


Some references in Signal Processing

- ...
Some references in Control Systems Theory


References

- Some references in Control Systems Theory
  - ...
Nonlinear case

Let us consider the following system:

$$\begin{cases}
\dot{x}_1(t) = f_1(x_1(t), x_2(t)); & t \neq t_k \\
\dot{x}_2(t) = f_2(x_2(t)); & t \neq t_k \\
x_1(t_k^+) = R x_1(t_k) \\
x_2(t_k^+) = x_2(t_k)
\end{cases}$$  \hspace{1cm} (1)

where $x_1(t) \in \mathbb{R}^p$, $x_2(t) \in \mathbb{R}^{n-p}$, $f_1 : \mathbb{R}^n \to \mathbb{R}^p$ and $f_2 : \mathbb{R}^{n-p} \to \mathbb{R}^{n-p}$.

**Assumption**

$f_1$ is at least locally lipschitz, where $l_1$ and $l_2$ are respectively the Lipschitz constants with respect to $x_1$ and $x_2$. 
Nonlinear case

Assumption

The sampling sequence \( t_k \in T = \{ t_i : i \in \mathbb{N} \} \subset \mathbb{R} \) verifies that there exists \( \tau_{\text{max}} \) and \( \tau_{\text{min}} \) with \( \tau_{\text{max}} > \tau_{\text{min}} > 0 \) such that \( \forall i > 0 \)

\[
t_{i+1} \geq t_i + \tau_{\text{min}} \quad \text{and} \quad t_{i+1} \leq t_i + \tau_{\text{max}}
\]

Moreover we define:

\[
\theta_k \triangleq t_{k+1} - t_k \\
x(t^+_k) \triangleq \lim_{h \to 0} x(t_k + h) \\
x(t^-_k) \triangleq \lim_{h \to 0} x(t_k - h) = x(t_k)
\]
Theorem

If the system (1) verifies the previous assumptions and satisfies the following conditions:

1. there exists a strictly positive definite function $V_2 : \mathbb{R}^{n-p} \mapsto \mathbb{R}^+$ such that $V_2(0) = 0$ and $\dot{V}_2(0) = 0$.

2. $\|R\|_1 e^{l_1 \theta_k} < 1$ for all $k > 0$.

then, $\forall \epsilon^* > 0$, the ball $B_{2\epsilon^*}$ is globally asymptotically stable considering the sequence of time $t_k^+$ (i.e. the time just after the reset instants).
Nonlinear case

Proposition

Assume that the conditions of the theorem hold for system (1), then for any $\epsilon^* > 0$ (of theorem 1) the ball $B_\beta$ is globally asymptotically stable with $\beta = \frac{\epsilon^*}{\|R\|_1} + \epsilon^*$, if $\|R\|_1 \neq 0$, and the system (1) converges globally asymptotically to zero if $\|R\|_1 = 0$. 
Let us consider the following class of linear system:

\[
\begin{align*}
\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t); \quad t \neq t_k \\
\dot{x}_2(t) &= A_{22}x_2(t) \\
x_1(t_k^+) &= Rx_1(t) \\
x_2(t_k^+) &= x_2(t)
\end{align*}
\]

Where \( x(t) \in \mathbb{R}^n \), \( A_{11} \in \mathbb{R}^{p \times p} \), \( A_{12} \in \mathbb{R}^{p \times (n-p)} \), \( A_{22} \in \mathbb{R}^{(n-p) \times (n-p)} \) and \( R \in \mathbb{R}^{p \times p} \).
Linear case

**Theorem**

*If the system (3) verifies the following condition:

1. $A_{22}$ is Hurwitz continuous
2. $\|Re^{A_{11}^\theta_k}\|^2 < 1 \quad \forall k > 0$

Then, $\forall \epsilon > 0$, the state of system (3) converges to a ball of radius $\epsilon$.***
Now, we consider the following class of system :

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t_k) &= Cx(t_k)
\end{align*}
\]

(4)

Where \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R}^m \) is the input and \( y(t_k) \in \mathbb{R}^p \) is the output. The matrixes \( A, B \) and \( C \) are constant and of appropriate dimension. Moreover the system is assumed detectable and stabilizable.
Assumption

*There exist a regular matrix* $T$ *which transform the system 4 into the system 5 described below (with* $\tilde{x} = Tx$).

\[
\begin{aligned}
\dot{\tilde{x}}_1(t) &= \tilde{A}_{11}\tilde{x}_1(t) + \tilde{A}_{12}\tilde{x}_2(t) + \tilde{B}_1u(t) \\
\dot{\tilde{x}}_2(t) &= \tilde{A}_{22}\tilde{x}_2(t) + \tilde{B}_2u(t) \\
y(t_k) &= \tilde{x}_1(t_k)
\end{aligned}
\]

*Where* $\tilde{x}(t) = (\tilde{x}_1^T, \tilde{x}_2^T)^T$ *with* $\tilde{A}_{11} \in \mathbb{R}^{p \times p}$, $\tilde{A}_{12} \in \mathbb{R}^{p \times (n-p)}$, $\tilde{A}_{22} \in \mathbb{R}^{(n-p) \times (n-p)}$, $\tilde{B}_1 \in \mathbb{R}^{p \times m}$, $\tilde{B}_2 \in \mathbb{R}^{(n-p) \times m}$ *and* $\tilde{A}_{22}$ *Hurwitz continuous.*
Stable unmeasured subspace

For the system (5) the proposed observer base control is:

a- The control

\[ u(t) = -K\hat{x} \triangleq (-K_1, -K_2) \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \] (6)

b- the Impulsive observer

\[
\begin{align*}
\dot{\hat{x}}_1(t) &= \hat{A}_{11}\hat{x}_1(t) + \hat{A}_{12}\hat{x}_2(t) + \hat{B}_1 u(t) \\
\dot{\hat{x}}_2(t) &= \hat{A}_{22}\hat{x}_2(t) + \hat{B}_2 u(t) \\
\hat{x}_1(t_k^+) &= R\hat{x}_1(t_k) + (I_p - R)\hat{x}_1(t_k) \\
\hat{y}(t_k) &= \hat{x}_1(t_k)
\end{align*}
\] (7)

Where \( K \in \mathbb{R}^{m \times n} \) is a pole placement matrix and \( R \in \mathbb{R}^{m \times n} \) is the reset matrix.
Stable unmeasured subspace

**Theorem**

*If the system (4) verifies the previous assumptions, then it is possible, for considered sampling sequence (2), to find an observer based control (6)-(7) which stabilize practically asymptotically the system.*
Unstable unmeasured subspace

We consider again the system 4, but with:

**Assumption**

*There exist a regular matrix* $T$ *which transform the system 4 into the system 8 described below (with* $\tilde{x} = Tx$).

\[
\begin{align*}
\dot{\tilde{x}}_1(t) &= \tilde{A}_{11}\tilde{x}_1(t) + \tilde{A}_{12}\tilde{x}_2(t) + \tilde{B}_1 u(t) \\
\dot{\tilde{x}}_2(t) &= \tilde{A}_{22}\tilde{x}_2(t) + \tilde{B}_2 u(t) \\
y(t_k) &= \tilde{x}_1(t_k)
\end{align*}
\]

*Where* $\tilde{x}(t) = (\tilde{x}_1^T, \tilde{x}_2^T)^T$ *with* $\tilde{A}_{11} \in \mathbb{R}^{p \times p}$, $\tilde{A}_{12} \in \mathbb{R}^{p \times (n-p)}$, $\tilde{A}_{22} \in \mathbb{R}^{(n-p) \times (n-p)}$, $\tilde{B}_1 \in \mathbb{R}^{p \times m}$, $\tilde{B}_2 \in \mathbb{R}^{(n-p) \times m}$ *and* $A_{22}$ *is unstable.*
The observer based control for system (8) is quite different in its observer part than the one proposed in the previous section and has the following form:

\[ u(t) = -K\hat{x} \triangleq (K_1, K_2) \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \quad (9) \]
Unstable unmeasured subspace

b- Generalized impulsive observer

\[
\begin{align*}
\dot{x}_1(t) &= \tilde{A}_{11}x_1(t) + \tilde{A}_{12}x_2(t) + \tilde{B}_1u(t) \\
\dot{x}_2(t) &= \tilde{A}_{22}x_2(t) + M(x_2(t) - \hat{x}_2(t)) + \tilde{B}_2u(t) \\
\dot{x}_1(t) &= \tilde{A}_{11}x_1(t) + \tilde{A}_{12}x_2(t) + L_1(x_1(t) - \bar{x}_1(t)) + \tilde{B}_1u(t) \\
\dot{x}_2(t) &= \tilde{A}_{22}x_2(t) + L_2(x_1(t) - \bar{x}_1(t)) + \tilde{B}_2u(t) \\
\hat{x}_1(t^+_k) &= R_{n_1}\hat{x}_1(t_k) + (I_d - R)\bar{x}_1(t_k)
\end{align*}
\]

(10)

Where $K \in \mathbb{R}^{m \times n}$, with $R_{n_1} = \text{diag}\{r_1, \ldots, r_{n_1}\}$ and $-1 < r_i < 1$, for $i \in [1, n_1] \cap N$ and $R$, $W$, $M$, $L_1$ et $L_2$ are matrixes of appropriate dimensions.
Unstable unmeasured subspace

Figure: The block diagram of state feedback with generalized impulsive observer
Unstable unmeasured subspace

Theorem

The system (4), which verified previous assumptions, is practically stabilized, under the sampling 2 by the observer based control (9)-(10), if the following conditions are verified:

1. $\| \text{Re}^{A_{11}\theta_k} \| < 1$.

2. The matrix $\begin{pmatrix} \tilde{A}_{22} - M & 0 & M \\ 0 & \tilde{A}_{11} - L_1 & \tilde{A}_{12} \\ 0 & -L_2 & \tilde{A}_{22} \end{pmatrix}$ is Hurwitz.

3. The matrix $\begin{pmatrix} \tilde{A}_{11} - \tilde{B}_1 K_1 & \tilde{A}_{12} - \tilde{B}_1 K_2 \\ -\tilde{B}_2 K_1 & \tilde{A}_{22} - \tilde{B}_2 K_2 \end{pmatrix}$ is Hurwitz.

4. $\| \hat{x}(0) \| < K_M$, $\| \tilde{x}(0) \| < K_M$, $\| \bar{x}(0) \| < K_M$ and $\forall k > 0$, $\theta_k < \theta_m(K_M)$.
Consider the following linear triangular system:

\[
\begin{align*}
\dot{x}_1 &= 2x_1 + x_2 + 3x_3 + u \\
\dot{x}_2 &= x_2 + x_3 \\
\dot{x}_3 &= -x_3 - u \\
y(t_k) &= x_1(t_k)
\end{align*}
\]

(11)
The goal is to make the system (11) stable. For this, let synthesize the proposed observer base control (10). By choosing $K_1 = 48$, $K_2 = (30, 55)$, $R = 0$, $M = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$, $L_1 = 6$ and $L_2 = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$. The parameters of observer control satisfy the conditions of theorem 3, with corresponding matrix of observer is

\[
\begin{pmatrix}
-4 & 1 & 0 & 5 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -4 & 1 & 3 \\
0 & 0 & -6 & 1 & 1 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}
\]
Simulation results

and the corresponding matrix of control is

\[
\begin{pmatrix}
-48 & 31 & 58 \\
1 & 1 & 0 \\
-30 & -56 & -48
\end{pmatrix}
\]
Simulation results

**Figure:** controlled state
Simulation results

Now, consider the following Chaotic system:

\[
\begin{align*}
\dot{x}_1(t) &= a(x_2(t) - x_1(t)) \\
\dot{x}_2(t) &= bx_1(t) + cx_2(t) - x_1(t)x_3(t) + x_4(t) \\
\dot{x}_3(t) &= -dx_3(t) + x_1(t)x_2(t) \\
\dot{x}_4(t) &= -kx_1(t) + x_4(t) \\
y(t_k) &= x_2(t_k)
\end{align*}
\]

(12)

Avec : \( a = 35, b = 7, c = 12, d = 3, k = 5 \) et
\((x_10, x_20, x_30, x_40) = (0.05, 0.01, 0.05, 0.5)\). Les exposants de Lyonapunov sont : \( \lambda_1 = 16.472, \lambda_2 = 1.2729, \lambda_3 = -3 \) et \( \lambda_4 = -39.7443 \). The system (12) is hyperchaotic.
Simulation results

**Figure**: Phase portrait

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Observer based stabilization under sparse measurement
Simulation results

Impulsionnel observer

\[
\begin{align*}
\dot{\hat{x}}_1(t) &= a(\hat{x}_2(t) - \hat{x}_1(t)) \\
\dot{\hat{x}}_2(t) &= b\hat{x}_1(t) + c\hat{x}_2(t) - \hat{x}_1(t)\hat{x}_3(t) + \hat{x}_4(t) \\
\dot{\hat{x}}_3(t) &= -d\hat{x}_3(t) + \hat{x}_1(t)\hat{x}_2(t) \\
\dot{\hat{x}}_4(t) &= -k\hat{x}_1(t) + \hat{x}_4(t) + M(z_4(t) - \hat{x}_4(t)) \\
\hat{x}_2(t^+) &= \hat{x}_2(t_k) + R(x_2(t_k) - \hat{x}_2(t_k))
\end{align*}
\]
Simulation results

Observer correction

\[
\begin{aligned}
  z_4(t) &= z_d^2(t) - b\hat{x}_1(t) - cz_2(t) + \hat{x}_1(t)\hat{x}_3(t) \\
  \dot{z}_2(t) &= z_d^2(t) + \lambda_1|z_2(t) - \hat{x}_2(t)|^{\frac{1}{2}} \text{sign}(z_2(t) - \hat{x}_2(t)) \\
  \dot{z}_d^2(t) &= \lambda_2 \text{sign}(z_2(t) - \hat{x}_2(t))
\end{aligned}
\]
Simulation results

Figure: observer and system states for $\theta = 0$. 

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Observer based stabilization under sparse measurement
Strange attractors identification and state observation

Let us consider the following network of chaotic systems:
1- Lorenz System:

\[
\Sigma_{\text{Lorenz}} = \begin{cases}
\dot{x}_1 &= 13.5x_1 + 10.5x_2 - x_2x_3 \\
\dot{x}_2 &= 22.5(x_1 - x_2) \\
\dot{x}_3 &= -\frac{17}{6}x_3 + x_1x_2
\end{cases}
\] (14)

2- Lu system:

\[
\Sigma_{\text{Lu}} = \begin{cases}
\dot{x}_1 &= 22.2x_1 - x_2x_3 \\
\dot{x}_2 &= 30(x_1 - x_2) \\
\dot{x}_3 &= -\frac{8.8}{3}x_3 + x_1x_2
\end{cases}
\] (15)
Strange attractor identification and state observation

3- Chen system :

\[ \Sigma_{Chen} = \begin{cases} 
\dot{x}_1 = 28x_1 + 7x_2 - x_2x_3 \\
\dot{x}_2 = 35(x_1 - x_2) \\
\dot{x}_3 = -3x_3 + x_1x_2 
\end{cases} \]  \hspace{1cm} (16)

4- Qi system :

\[ \Sigma_{Qi} = \begin{cases} 
\dot{x}_1 = 24(x_1 + x_2) - x_2x_3 \\
\dot{x}_2 = 42.5(x_1 - x_2) + x_1x_3 \\
\dot{x}_3 = -13x_3 + x_1x_2 
\end{cases} \]  \hspace{1cm} (17)

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Observer based stabilization under sparse measurement
Strange attractor identification and state observation

**Figure**: The block diagram of multi-observer base
Strange attractor identification and state observation

**Figure**: Phases Portrait for Strange attractor
Strange attractor identification and state observation

**Figure**: Observation error
Some preliminaries results and relations between CS and Control Theory were highlighted.

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Open Problems

- RIP and 'observability' (new definition)
- Observability normal form with respect to sparse measurement
- Observer and observer based control proof in nonlinear case
- ...