Nonlinear reachability computation with complex systems

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Outline

1. Introduction

2. Nonlinear hybridization approach to reachability

3. Nonlinear hybridization with order preserving dynamical systems

4. Extension to hybrid systems

5. Future work
Outline

1. Introduction
   - Hybrid systems and reachability analysis
   - Guaranteed set integration and reachable sets

2. Nonlinear hybridization approach to reachability

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5. Future work
Cyber-Physical systems

- Interaction discrete + continuous dynamics
- Safety-critical embedded systems
Cyber-Physical systems

- Service robots. (Physical) Human-Robot Interaction
Cyber-Physical systems

Nonlinear continuous + discrete hybrid dynamics

- Invariants, guards and reset functions are **non linear**.
- Continuous dynamics **non linear**, of high dimension, and **uncertain**.
Cyber-Physical systems

Hybrid automaton (Alur, et al., 95)

\[ H = (Q, D, P, \Sigma, A, \text{Inv}, \mathcal{F}), \]

**continuous dynamics**

\[
\begin{align*}
\text{flow}(q) & : \quad \dot{x}(t) = f_q(x, p, t), \\
\text{Inv}(q) & : \quad \nu_q(x(t), p, t) < 0,
\end{align*}
\]

**discrete dynamics**

\[
\begin{align*}
A \ni e & : \quad (q \rightarrow q') = (q, \text{guard}, \sigma, \rho, q'), \\
\text{guard}(e) & : \quad \gamma_e(x(t), p, t) = 0,
\end{align*}
\]

\[ t_0 \leq t \leq t_N, \quad x(t_0) \in X_0 \subseteq \mathbb{R}^n, \quad p \in P \]
Study of hybrid automata

Verification and synthesis of autonomous and embedded systems

- Air traffic management systems.
- Ground transportation systems.
- Flight control systems ...
- Service robots sharing human living space.

Modelling and analysis of complex physical systems

- Hybrid and networked systems in engineering, biology and economics.

→ Hybrid reachability analysis
Hybrid Reachability Computation

Set reachable in finite time → Safety verification

\[ \rho(x_e) \]

\[ \nu_0(.) < 0 \]

\[ \nu_1(.) < 0 \]

Bounded Model Checking
- Prove correctness,
- or
- Exhibit incorrectness via counter-example.

Check satisfiability of SAT modulo theories formulae.
Hybrid Reachability Computation

Set reachable in finite time $\rightarrow$ Synthesis

- Model-based control ...
- Parameter synthesis in Systems Biology ...
Hybrid Reachability Computation

Set reachable in finite time $\rightarrow$ Bounded Error State Estimation

- Predictor-Corrector approach to bounded-error state estimation
- Bounded error moving horizon state estimator
- Initial state reconstruction
Introduction

Hybrid systems and reachability analysis

Hybrid Reachability Computation

Set reachable in finite time $\rightarrow$ Interval Observers

\[
\begin{bmatrix}
\dot{x} \\
\dot{x}
\end{bmatrix} = \left[ A(t) - K(t)C(t) \right] \begin{bmatrix}
x \\
x
\end{bmatrix} + \begin{bmatrix}
\Phi(x, x, \underline{p}, \bar{p}, t) \\
\Phi(x, \bar{x}, \underline{p}, \bar{p}, t)
\end{bmatrix} + K(t) \begin{bmatrix}
\bar{y}_M(t) \\
\bar{y}_M(t)
\end{bmatrix}
\]

$\forall x(t_0) \in [x(t_0), \bar{x}(t_0)], \forall p \in [\underline{p}, \bar{p}], \Rightarrow \left\{ \begin{array}{l}
\forall t \geq t_0, \quad x(t) \leq x(t) \leq \bar{x}(t), \\
\lim_{t \to +\infty} \|\bar{x}(t) - \bar{x}(t)\| = w(\|\bar{p} - \underline{p}\|, \|\bar{y}(t) - y(t)\|)
\end{array} \right.$
Continuous Reachability Computation

Uncertain nonlinear dynamical system

\[
\dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in X_0 \subseteq \mathbb{D} \subseteq \mathbb{R}^n, \quad p \in \mathbb{P}
\]

Reachable set

\[
\mathcal{R}([t_0, t]; X_0) = \left\{ x(\tau), \ t_0 \leq \tau \leq t \mid \dot{x}(\tau) = f(x, p, \tau) \land x(t_0) \in X_0 \land p \in \mathbb{P} \right\}
\]
Reachability computation: State-of-the-art

Affine systems
- time discretization + set integration + computational geometry
  - zonotopes (Girard, 2005)
  - ellipsoids (Botchkarev and Tripakis, 2000), (Kurzhanski and Varaiya, 2000, 2005)
- Hybrid abstraction (Guéguen and Zaytoon, 2004), (Lefebvre and Guéguen, 2006), (Doyen, et al., 2005), (Kloetzer and Belta, 2006)
Reachability computation: State-of-the-art

Nonlinear systems

- Level set methods, viscosity solutions to Hamilton-Jacobi-Isaacs PDE
  (Bayen, et al., 2002), (Tomlin, et al., 2003), (Mitchell, et al., 2005),...
- Linear hybridization: simplified dynamics ...
  (Tiwari and Khanna, 2002), (Maler, et al., 2006),
  (Batt, et al., 2007), (Asarin, et al., 2007)
- Set integration, interval analysis
  (Henzinger, et al., 2000)
- Constraint propagation, abstraction refinement, interval analysis
  (Ratschan and She, 2007)
- ODE enclosure, constraint solving, interval analysis
  (Eggers, Fränzle & Herde, 2008; 2009)
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Our methods: Guaranteed set integration
A nonlinear hybridization approach to reachability computation

Tools
- Set computation via \textit{interval analysis}
- Set integration via
  - \textit{Interval Taylor methods}
  - Müller existence theorem and differential inequalities
  - Theory of order preserving monotone dynamical systems
- Bounding systems as \textit{hybrid automata}

Results
1. No linearization. Truly non linear.
2. Integration time step can be varying
3. \textbf{Analytical solution} for reachable set boundaries
4. A scalable method
Interval analysis
(Dwyer,51) (Warmus,56) (Sunaga,58) (Moore,59)

Extension of real arithmetics to intervals

\[ \circ \in \{+, -, \cdot, /\}, [x] \circ [y] = \{ x \circ y \mid x \in [x], y \in [y] \} \]

Inclusion function

\[ \forall [x] \subseteq \mathbb{D}, f([x]) \subseteq [f([x])] \]

Verified numerical implementation

Directed rounding → Outward rounding
Guaranteed set integration with Taylor methods
(Moore, 66) (Eijgenraam, 81) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)

\[
\dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \ x(t_0) \in [x_0], \ p \in [p]
\]

Time grid \( t_0 < t_1 < t_2 < \cdots < t_N \)

- Proof of existence
- Yield a priori solution \([\tilde{x}_j] : \forall \tau \in [t_j, t_{j+1}] \ x(\tau) \in [\tilde{x}_j]\)
Guaranteed set integration with Taylor methods

\[
\dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \ x(t_0) \in [x_0], \ p \in [p]
\]

Time grid \( t_0 < t_1 < t_2 < \cdots < t_N \)

- Compute tight enclosure \( [x_{j+1}] \supseteq x(t_{j+1}) \)

\[
[x_{j+1}] = [x_j] + \sum_{i=1}^{k-1} (t_{j+1} - t_j)^i f[i]([x_j], [p]) + (t_{j+1} - t_j)^k f[k]([\tilde{x}_j], [p])
\]
Guaranteed set integration with Taylor methods

\[ \dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_{N}, \quad x(t_0) \in [x_0], \quad p \in [p] \]

\[ t_0 < t_1 < t_2 < \cdots < t_{N} \]

Yield an analytical formula for solution set

\[ \forall \tau \in [t_j, t_j + h_j] \quad x(\tau) \in [x](\tau) \]

\[ [x](\tau) = [x_j] + \sum_{i=1}^{k-1} (\tau - t_j)^i f[i] ([x_j], [p]) + (\tau - t_j)^k f[k] ([\tilde{x}_j], [p]) \]
Guaranteed set integration with Taylor methods
(Moore, 66) (Eijgenraam, 81) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)

Need to control wrapping effect
Guaranteed set integration with Taylor methods
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Need to control wrapping effect
Guaranteed set integration with Taylor methods

(Moore, 66) (Eijgenraam, 81) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)

Need to control wrapping effect

- Mean value forms
- Matrice preconditioning
- Linear transforms

⇒ May fail when set size is large!
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1 Introduction

2 Nonlinear hybridization approach to reachability
   - Reachability computation via Müller’s theorem
   - Hybrid automata as bounding systems
   - Example
   - More on the hybrid bounding method

3 Nonlinear hybridization with order preserving dynamical systems

4 Extension to hybrid systems

5 Future work
Guaranteed set integration with Müller’s theorem
(Müller, 27) (Walter, 97) (Singer & Barton, 06)

\[ \dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \ x(t_0) \in [x_0], \ p \in [p] \]

The Müller’s existence theorem

\[ f(x, p, t) \text{ continuous over } \mathcal{Z} : \left\{ \begin{array}{l}
\omega(t) \leq x(t) \leq \Omega(t) \\
p \leq p \leq \overline{p} \\
t_a \leq t \leq t_b
\end{array} \right. \]

- \( \omega(t_a) = x_a \)
- \( \forall i, \ \omega_i(t) \text{ continuous over } [t_a, t_b] \)
- \( \forall i, \ D^\pm \omega_i(t) \leq \min_{\mathcal{Z}_i(t)} f_i(x, p, t) \)

\[ \mathcal{Z}_i(t) : \left\{ \begin{array}{l}
x_i = \omega_i(t), \ \omega_j(t) \leq x_j \leq \Omega_j(t), \ j \neq i, \\
p \leq p \leq \overline{p}, \\
t = t
\end{array} \right. \]
Guaranteed set integration with Müller’s theorem
(Müller, 27) (Walter, 97) (Singer & Barton, 06)

\[ \dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p] \]

The Müller’s existence theorem (cont’d)

- \( \Omega(t_a) = \overline{x}_a \)
- \( \forall i, \Omega_i(t) \) continuous over \( [t_a, t_b] \)
- \( \forall i, D^\pm \Omega_i(t) \geq \max_{\overline{Z}_i(t)} f_i(x, p, t) \)

\[ \overline{Z}_i(t) : \begin{cases} x_i = \Omega_i(t), & \omega_j(t) \leq x_j \leq \Omega_j(t), \ j \neq i, \\ p \leq p \leq \overline{p}, \\ t = t \end{cases} \]
Guaranteed set integration with Müller’s theorem
(Müller, 27) (Walter, 97) (Singer & Barton, 06)

\[\dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p]\]

The Müller’s existence theorem (cont’d 2)

Then, \(\forall x_a \in [x_a, \bar{x}_a], \forall p \in [p, \bar{p}],\) it exists a solution \(x(t)\) that stays in the domain \(\Xi:\)

\[
\begin{cases}
t_a \leq t \leq t_b, \\
\omega(t) \leq x(t) \leq \Omega(t)
\end{cases}
\]

such that \(x(t_a) = x_a\)

Furthermore, if \(\forall p \in [p, \bar{p}], f(x, p, t)\) is Lipschitz wrt \(x\) over \(\mathbb{D}\), then solution is unique for all \(p\).
Guaranteed set integration with Müller’s theorem
(Müller, 27) (Walter, 97) (Singer & Barton, 06)

\[
\dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p]
\]

(i) Build the coupled system

\[
\begin{align*}
\dot{\omega} &= \mathcal{f}(\omega, \Omega, p, \bar{p}, t), \quad \omega(t_0) = x(t_0), \\
\dot{\Omega} &= \mathcal{f}(\omega, \Omega, \bar{p}, \bar{p}, t), \quad \Omega(t_0) = \bar{x}(t_0),
\end{align*}
\]

(ii) Use set integration to solve IVP for coupled ODE
- Interval Taylor methods ...

(iii) Inclusion function for solution

\[
\rightarrow \begin{cases} 
[x](t) = \text{ConvexHull} \left( \omega(t), \Omega(t) \right) \\
[\bar{x}](t) = \text{ConvexHull} \left( \tilde{\omega}(t), \tilde{\Omega}(t) \right)
\end{cases}
\]
Tight bracketing functions

Illustrative example

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, p, t), \quad x_1(t_0) \in [x_{1,0}, \bar{x}_{1,0}] \subset \mathbb{R}, \quad p \in [p, \bar{p}] \quad t \geq t_0 \\
\dot{x}_2 &= f_2(x_1, x_2, p, t), \quad x_2(t_0) \in [x_{2,0}, \bar{x}_{2,0}] \subset \mathbb{R},
\end{align*}
\]
Tight bracketing functions

Illustrative example

\[
\begin{cases}
\dot{x}_1 = f_1(x_1, x_2, p, t), & x_1(t_0) \in [x_{1,0}, \bar{x}_{1,0}] \subset \mathbb{R}, \quad p \in [p, \bar{p}] \quad t \geq t_0 \\
\dot{x}_2 = f_2(x_1, x_2, p, t), & x_2(t_0) \in [x_{2,0}, \bar{x}_{2,0}] \subset \mathbb{R}, 
\end{cases}
\]

If \( \forall \ t \geq t_0, \ \forall \ x(t) \in [\omega(t), \Omega(t)] \subset \mathbb{R}^2, \ \forall \ p \in [p, \bar{p}], \)

\[
\frac{\partial f_1}{\partial x_2} > 0 \ \land \ \frac{\partial f_1}{\partial p} > 0
\]
Tight bracketing functions

Illustrative example

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, p, t), \quad x_1(t_0) \in [x_{1,0}, x_{1,0}] \subset \mathbb{R}, \quad p \in [p, p] \quad t \geq t_0 \\
\dot{x}_2 &= f_2(x_1, x_2, p, t), \quad x_2(t_0) \in [x_{2,0}, x_{2,0}] \subset \mathbb{R},
\end{align*}
\]

If \( \forall t \geq t_0, \forall x(t) \in [\omega(t), \Omega(t)] \subset \mathbb{R}^2, \forall p \in [p, p], \)

\[
\frac{\partial f_1}{\partial x_2} > 0 \land \frac{\partial f_1}{\partial p} > 0
\]

\[
\Downarrow
\]

\[
f_1(\omega_1, \omega_2, p) \leq f_1(\omega_1, x_2, p, t) \quad \text{and} \quad f_1(\Omega_1, x_2, p, t) \leq f_1(\Omega_1, \Omega_2, \overline{p})
\]
Nonlinear hybridization approach to reachability

Reachability computation via Müller’s theorem

Tight bracketing functions

Illustrative example

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, p, t), \quad x_1(t_0) \in [x_{1,0}, \bar{x}_{1,0}] \subset \mathbb{R}, \quad p \in [\underline{p}, \bar{p}] \quad t \geq t_0 \\
\dot{x}_2 &= f_2(x_1, x_2, p, t), \quad x_2(t_0) \in [x_{2,0}, \bar{x}_{2,0}] \subset \mathbb{R},
\end{align*}
\]

If \(\forall t \geq t_0, \forall x(t) \in [\omega(t), \Omega(t)] \subset \mathbb{R}^2, \forall p \in [\underline{p}, \bar{p}],\)

\[
\frac{\partial f_1}{\partial x_2} > 0 \land \frac{\partial f_1}{\partial p} > 0
\]

\[\downarrow\]

\[
f_1(\omega_1, \omega_2, p) \leq f_1(\omega_1, x_2, p, t) \quad \text{and} \quad f_1(\Omega_1, x_2, p, t) \leq f_1(\Omega_1, \Omega_2, \bar{p})
\]

\[\downarrow\]

\[
\dot{\omega}_1(t) \equiv f_1(\omega_1, \omega_2, p) \quad \text{and} \quad f_1(\Omega_1, \Omega_2, \bar{p}) \equiv \dot{\Omega}_1(t)
\]
### Tight bracketing functions

#### Illustrative example

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, p, t), \quad x_1(t_0) \in [x_{1,0}, \bar{x}_{1,0}] \subset \mathbb{R}, \quad p \in [p, \bar{p}] \quad t \geq t_0 \\
\dot{x}_2 &= f_2(x_1, x_2, p, t), \quad x_2(t_0) \in [x_{2,0}, \bar{x}_{2,0}] \subset \mathbb{R},
\end{align*}
\]
Tight bracketing functions

Illustrative example

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, p, t), \quad x_1(t_0) \in [x_{1,0}, \overline{x}_{1,0}] \subset \mathbb{R}, \quad p \in [p, \overline{p}] \quad t \geq t_0 \\
\dot{x}_2 &= f_2(x_1, x_2, p, t), \quad x_2(t_0) \in [x_{2,0}, \overline{x}_{2,0}] \subset \mathbb{R},
\end{align*}
\]

If \( \forall t \geq t_0, \ \forall x(t) \in [\omega(t), \Omega(t)] \subset \mathbb{R}^2, \ \forall p \in [p, \overline{p}], \)

\[
\frac{\partial f_2}{\partial x_1} < 0 \land \frac{\partial f_2}{\partial p} > 0
\]

\[
\downarrow
\]

\[
f_2(\Omega_1, \omega_2, p) \leq f_2(x_1, \omega_2, p, t) \quad \text{and} \quad f_2(x_1, \Omega_2, p, t) \leq f_2(\omega_1, \Omega_2, \overline{p})
\]
Tight bracketing functions

Illustrative example

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, p, t), \quad x_1(t_0) \in [x_{1,0}, \bar{x}_{1,0}] \subset \mathbb{R}, \quad p \in [p, \bar{p}] \quad t \geq t_0 \\
\dot{x}_2 &= f_2(x_1, x_2, p, t), \quad x_2(t_0) \in [x_{2,0}, \bar{x}_{2,0}] \subset \mathbb{R},
\end{align*}
\]

A coupled system

\[
\begin{align*}
\dot{\omega}_1 &= f_1(\omega_1, \omega_2, p), \quad \omega_1(t_0) = x_{1,0} \\
\dot{\omega}_2 &= f_2(\Omega_1, \omega_2, p), \quad \omega_2(t_0) = x_{2,0} \\
\dot{\Omega}_1 &= f_1(\Omega_1, \Omega_2, p), \quad \Omega_1(t_0) = \bar{x}_{1,0} \\
\dot{\Omega}_2 &= f_2(\omega_1, \Omega_2, p), \quad \Omega_2(t_0) = \bar{x}_{2,0}
\end{align*}
\]
A rule for writing the bracketing functions

\[ \dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p] \]

**Formal expression of \( \overline{f}_i \):**

- Study monotonicity \( \rightarrow \) Analyze the signs of partial derivatives of \( f_i \)

  For \( l \neq i \),

  \[
  \text{if } \frac{\partial f_i}{\partial x_l} \geq 0 \quad \text{then replace } x_l \leftarrow \Omega_l, \text{ in } f_i
  \]

  \[
  \text{else replace } x_l \leftarrow \omega_l
  \]

  For \( k = 1, \ldots, n_p \),

  \[
  \text{if } \frac{\partial f_i}{\partial p_k} \geq 0 \quad \text{then replace } p_k \leftarrow \overline{p}_k
  \]

  \[
  \text{else replace } p_k \leftarrow \underline{p}_k
  \]

Repeat for \( i = 1, \ldots, n \), \( \Rightarrow \)

\[
\begin{aligned}
\dot{\omega}(t) &= \overline{f}(\omega, \Omega, \underline{p}, \overline{p}, t), \quad \omega(t_0) = x_0 \\
\dot{\Omega}(t) &= \overline{f}(\omega, \Omega, \underline{p}, \overline{p}, t), \quad \Omega(t_0) = \overline{x}_0
\end{aligned}
\]
Nonlinear hybridization approach to reachability

Reachability computation via Müller’s theorem

Example

Biological System (*Mitogen-Activated Protein Kinase cascades*) (Sontag, 2005)

Nonlinear uncertain system

\[
\begin{align*}
\dot{x}_1 &= -\frac{v_2 x_1}{k_2 + x_1} + v_0 u + v_1 \\
\dot{x}_2 &= \frac{v_6 (y_{tot} - x_2 - x_3)}{k_6 + (y_{tot} - x_2 - x_3)} - \frac{v_3 x_1 x_2}{k_3 + x_2} \\
\dot{x}_3 &= \frac{v_4 x_1 (y_{tot} - x_2 - x_3)}{k_4 + (y_{tot} - x_2 - x_3)} - \frac{v_5 x_3}{k_5 + x_3} \\
\dot{x}_4 &= \frac{v_10 (z_{tot} - x_4 - x_5)}{k_{10} + (z_{tot} - x_4 - x_5)} - \frac{v_7 x_3 x_4}{k_7 + x_4} \\
\dot{x}_5 &= \frac{v_8 x_3 (z_{tot} - x_4 - x_5)}{k_8 + (z_{tot} - x_4 - x_5)} - \frac{v_9 x_5}{k_9 + x_5} \\
u &= g x_5
\end{align*}
\]

Use Interval Taylor Methods
Example
Biological System (*Mitogen-Activated Protein Kinase cascades*) (Sontag, 2005)

Flow pipe

![Flow pipe graph](image-url)
Example
Biological System (*Mitogen-Activated Protein Kinase cascades*) (Sontag, 2005)

Nonlinear uncertain system

\[
\begin{align*}
\dot{x}_1 &= -\frac{v_2 x_1}{k_2 + x_1} + v_0 u + v_1 \\
\dot{x}_2 &= \frac{v_6 (y_{tot} - x_2 - x_3)}{k_6 + (y_{tot} - x_2 - x_3)} - \frac{v_3 x_1 x_2}{k_3 + x_2} \\
\dot{x}_3 &= \frac{v_4 x_1 (y_{tot} - x_2 - x_3)}{k_4 + (y_{tot} - x_2 - x_3)} - \frac{v_5 x_3}{k_5 + x_3} \\
\dot{x}_4 &= \frac{v_{10} (z_{tot} - x_4 - x_5)}{k_{10} + (z_{tot} - x_4 - x_5)} - \frac{v_7 x_3 x_4}{k_7 + x_4} \\
\dot{x}_5 &= \frac{v_8 x_3 (z_{tot} - x_4 - x_5)}{k_8 + (z_{tot} - x_4 - x_5)} - \frac{v_9 x_5}{k_9 + x_5} \\
u &= g x_5
\end{align*}
\]

Find the bracketing systems and use Müller’s theorem?
Example

Biological System (*Mitogen-Activated Protein Kinase cascades*) (Sontag, 2005)

Bracketing systems $\rightarrow$ a coupled systems of ODE

$$\begin{align*}
\dot{x}_1 &= -\frac{\nu_2 x_1}{k_2 + x_1} + \nu_0 u + \nu_1 \\
\dot{x}_2 &= \frac{\nu_6 (y_{tot} - x_2 - x_3)}{k_6 + (y_{tot} - x_2 - x_3)} - \frac{\nu_3 x_1 x_2}{k_3 + x_2} \\
\dot{x}_3 &= \frac{\nu_4 x_1 (y_{tot} - x_2 - x_3)}{k_4 + (y_{tot} - x_2 - x_3)} - \frac{\nu_5 x_3}{k_5 + x_3} \\
\dot{x}_4 &= \frac{\nu_10 (z_{tot} - x_4 - x_5)}{k_10 + (z_{tot} - x_4 - x_5)} - \frac{\nu_7 x_3 x_4}{k_7 + x_4} \\
\dot{x}_5 &= \frac{\nu_8 x_3 (z_{tot} - x_4 - x_5)}{k_8 + (z_{tot} - x_4 - x_5)} - \frac{\nu_9 x_5}{k_9 + x_5} \\
\dot{u} &= \frac{-\nu_2 x_1}{k_2 + x_1} + \nu_0 u + \nu_1 \\
\frac{u}{\dot{u}} &= \frac{g x_5}{g x_5}
\end{align*}$$
Example
Biological System (*Mitogen-Activated Protein Kinase cascades*) (Sontag, 2005)

Flow pipes

![Flow pipes graph](image-url)
Reachability computation

\[ \dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p] \]

Time grid \[ t_0 < t_1 < t_2 < \cdots < t_N \]

→ Analyse partial derivatives
→ Use Müller’s theorem
→ Solve the coupled dynamical system
→ \[ [x](t) = [\operatorname{Inf}(\omega(t)), \operatorname{Sup}(\Omega(t))] \]

Bracketing functions in the general case

Signs of partial derivatives change with integration time
→ Hybridization: Hybrid automata as bounding systems
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Nonlinear hybridization

Illustrative example: \( \dot{x} = f(x, p_1, p_2, t) \quad x(t_0) \in [x_0, \bar{x}_0] \subset \mathbb{R}, \quad p_i \in [\underline{p}_i, \bar{p}_i] \)
Nonlinear hybridization approach to reachability

Hybrid automata as bounding systems

Nonlinear hybridization

Illustrative example: \( \dot{x} = f(x, p_1, p_2, t) \) \( x(t_0) \in [x_0, \bar{x}_0] \subset \mathbb{R} \), \( p_i \in [\underline{p}_i, \bar{p}_i] \)

\[ g_i(.) = \frac{\partial f}{\partial p_i}(.) \]

\[ [g]_1([\bar{x}], [p], [t_j, t_{j+1}]) \]

\[ [g]_2([\bar{x}], [p], [t_j, t_{j+1}]) \]

\[ [g]_1([x], [p], t) \]

\[ [g]_2([x], [p], t) \]

\( q = 1 \quad q = 0 \quad q = 2 \quad q = 0 \quad q = 3 \quad q = 0 \quad q = 4 \)

N. Ramdani (PRISME)
Illustrative example: \( \dot{x} = f(x, p_1, p_2, t) \) \( x(t_0) \in [x_0, \bar{x}_0] \subset \mathbb{R}, \) \( p_i \in [\underline{p}_i, \overline{p}_i] \)

\[ g_i(\cdot) = \frac{\partial f}{\partial p_i}(\cdot), \quad [\tilde{g}_i]_j = g_i([\tilde{x}_j], [p_1], [p_2], [t_j, t_{j+1}]) \]
Nonlinear hybridization

Illustrative example: \[ \dot{x} = f(x, p_1, p_2, t) \quad x(t_0) \in [x_0, \bar{x}_0] \subset \mathbb{R}, \quad p_i \in [p_i, \bar{p}_i] \]

Mode switching

- **Full interval mode** \( q = 0 \) \( \rightarrow \) **Bracketing systems** mode \( q \neq 0 \)
  \[ \rightarrow \text{switch and carry on} \]

- **Bracketing systems** mode \( q \neq 0 \) \( \rightarrow \) **Full interval** mode \( q = 0 \)
  \[ \rightarrow \text{switch and re-do time step calculation} \]
Nonlinear hybridization

Illustrative example: \( \dot{x} = f(x, p_1, p_2, t) \) \( x(t_0) \in [x_0, \bar{x}_0] \subset \mathbb{R}, \quad p_i \in [p_i-, \bar{p}_i] \)

Time grid \( t_0 < t_1 < t_2 < \cdots < t_N \)

Hybrid Bounding algorithm

1. Initialize (select bounding systems)
2. Do loop
3. Integrate one step ahead \( \rightarrow [\tilde{x}_j], [x_{j+1}] \)
4. Check Switching \( \leftarrow \) \( \text{sign}(\frac{\partial f}{\partial x_l}(.)), \text{sign}(\frac{\partial f}{\partial p_k}(.)), [\tilde{x}_j], [t_j, t_{j+1}] \)
5. Switch mode if necessary (change bounding systems)
   end Do
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Uncertain nonlinear system from bio-reactors

Haldane model. Biotechnological process in a stirred reactor

\[
\begin{align*}
\dot{x} &= f_x(x, s) = (\mu_0 \frac{s}{s + k_s + s^2/k_i} - \alpha d)x \\
\dot{s} &= f_s(x, s) = -k \mu_0 \frac{s}{s + k_s + s^2/k_i} x + (s_{in} - s)d
\end{align*}
\]

Biomass density: \(x\),

Substrate concentration: \(s\),

Concentration of input substrate: \(s_{in}(t) = s^0_{in} + 15 \cos(1/5t)\),

Uncertain parameters: \(\mu_0 = 0.75 \pm 1\%\), \(s^0_{in} = 65 \pm 1.5\%\).

Initial state: \(x(t_0) \times s(t_0) = [9.5, 10.5] \times [36, 44]\).

Coefficients: \(k = 42.14\), \(k_s = 9.28 mmol/l\), \(k_i = 256 mmol/l\), \(\alpha = 0.5\), \(d = 2\).
Uncertain nonlinear system from bio-reactors

Haldane model. Biotechnological process in a stirred reactor

\[
\begin{align*}
\dot{x} &= f_x(x, s) = (\mu_0 \frac{s}{s+k_s+s^2/k_i} - \alpha d)x \\
\dot{s} &= f_s(x, s) = -k\mu_0 \frac{s}{s+k_s+s^2/k_i}x + (s_{in} - s)d
\end{align*}
\]

Signs of partial derivatives

\[
\forall t > t_0, \quad 
\frac{\partial f_x}{\partial \mu_0} > 0 \land \frac{\partial f_s}{\partial x} < 0 \land \frac{\partial f_s}{\partial \mu_0} < 0 \land \frac{\partial f_s}{\partial s_{in}^0} > 0
\]

\[
\text{sign}(\frac{\partial f_x}{\partial s}) = \text{sign}(k_s k_i - s^2)
\]
Nonlinear hybridization approach to reachability

Example

Uncertain nonlinear system from bio-reactors

\[ q = 1, s > \sqrt{k_s k_2}, \partial f_x / \partial s < 0 \]

\[
\begin{align*}
\dot{x} &= \mu_0 \frac{s}{s + k_s + s^2 / k_i} x - \alpha d x \\
\dot{s} &= -k \mu_0 \frac{s}{s + k_s + s^2 / k_i} x + d(s_{in} - s) \\
\dot{x} &= \mu_0 \frac{s}{s + k_s + s^2 / k_i} x - \alpha d x \\
\dot{s} &= -k \mu_0 \frac{s}{s + k_s + s^2 / k_i} x + d(s_{in} - s)
\end{align*}
\]

\[ q = 2, s < \sqrt{k_s k_2}, \partial f_x / \partial s > 0 \]

\[
\begin{align*}
\dot{x} &= \mu_0 \frac{s}{s + k_s + s^2 / k_i} x - \alpha d x \\
\dot{s} &= -k \mu_0 \frac{s}{s + k_s + s^2 / k_i} x + d(s_{in} - s) \\
\dot{x} &= \mu_0 \frac{s}{s + k_s + s^2 / k_i} x - \alpha d x \\
\dot{s} &= -k \mu_0 \frac{s}{s + k_s + s^2 / k_i} x + d(s_{in} - s)
\end{align*}
\]

\[ q = 0. \text{Original uncertain model} \]

\[
\begin{align*}
\dot{x} &= f_x(x, s) = (\mu_0 \frac{s}{s + k_s + s^2 / k_i} - \alpha d) x \\
\dot{s} &= f_s(x, s) = -k \mu_0 \frac{s}{s + k_s + s^2 / k_i} x + (s_{in} - s) d
\end{align*}
\]
Uncertain nonlinear system from bio-reactors

**Numerical implementation**

- C++ Class Libraries

- Interval computation → Profil/BIAS

- Taylor coefficients and differentiation → FADBAD++

- Set integration with mean-value form + QR factorization (Lohner’s and Rihm’s methods)
Uncertain nonlinear system from bio-reactors

Time history of $s$ component

$q = 2$  $q = 0$  $q = 1$
Uncertain nonlinear system from bio-reactors

Improve mode switching using state vector partitioning

Time history of $s$ component
Uncertain nonlinear system from bio-reactors

Time history of $x$ component
Uncertain nonlinear system from bio-reactors

Reachable space
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Complexity analysis

- One single integration step with Hybrid bounding
  - complexity of solving twice dimensional IVP ODE

- Hybrid bounding vs interval Taylor series:
  - requires less integration time steps
  - do not need partitioning
Nonlinear hybridization approach to reachability

More on the hybrid bounding method

Stability analysis

A modified hybrid automaton

- active mode is a bounding mode, enclosure’s size

\[ \xi(t) = \Omega(q)(t) - \omega(q)(t) \]

- dynamics

\[ \dot{\xi}(t) = \overline{f}(q)(\omega, \Omega, \underline{p}, \bar{p}, t) - \underline{f}(q)(\omega, \Omega, \underline{p}, \bar{p}, t) \]

- active mode is an interval mode \( \rightarrow \) modified jump function

\[ J'(q_{k-1}, q_k)(\cdot) \triangleq J(0, q_k) \circ J(q_{k-1}, 0)(\cdot) \]
Stability analysis

$\varepsilon$-practical stability of hybrid systems

Sufficient conditions which keep $\xi(t)$ trajectories within given bounds

(Xu & Zhai, 2005)
Stability analysis

\( \epsilon \)-practical stability of hybrid systems

Sufficient conditions which keep \( \xi(t) \) trajectories within given bounds (Xu & Zhai, 2005)

\[ \forall \mathbf{x} \in X, \forall \mathbf{p} \in P, \ f(\mathbf{x}, \mathbf{p}, t) \triangleq A(t)\mathbf{x} + \psi(\mathbf{x}, \mathbf{p}, t) \]

- Bounding systems and original system share same stability properties, if
  - \( \psi(\mathbf{x}, ., t) \) and \( \psi(. , \mathbf{p}, t) \) Lipschitz continuous;
  - \( A(t) \) are Metzler matrices (\( A_{i,j}(t) \geq 0 \) for all \( i \neq j \))
    or \( A(t) \) are triangular matrices.
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Order preserving monotone dynamical systems

(Smith, 1995), (Hirsch et Smith, 2005), (Angeli et Sontag, 2003)

**Definition:** Order preserving monotone dynamical system

\[ x(t_0) < y(t_0) \Rightarrow \forall t \geq t_0 \quad x(t) < y(t) \quad \in \{<, \leq, \geq, >\} \]

**Example**

\[
\begin{align*}
&\begin{cases}
  x_1(t_0) \leq y_1(t_0) \\
  x_2(t_0) \geq y_2(t_0) \\
  x_3(t_0) < y_3(t_0) \\
  x_4(t_0) > y_4(t_0) \\
  \ldots
\end{cases} \\
\Rightarrow \forall t > t_0, \quad \begin{cases}
  x_1(t) \leq y_1(t) \\
  x_2(t) \geq y_2(t) \\
  x_3(t) < y_3(t) \\
  x_4(t) > y_4(t) \\
  \ldots
\end{cases}
\end{align*}
\]
Order preserving monotone dynamical systems

(Smith, 1995), (Hirsch et Smith, 2005), (Angeli et Sontag, 2003)

Definition: Cooperative system

\[ \dot{x}(t) = f(x, p, t), \quad x(t_0) \in [x_0] \subseteq \mathbb{D} \subseteq \mathbb{R}^n \]

\{ f, X_0 \} is cooperative over \( \mathbb{D} \) if

\[ \forall i \neq j, t \geq 0 \text{ and } \forall x \in \mathbb{D}, \quad \frac{\partial f_i(x, p, t)}{\partial x_j} \geq 0 \]

A cooperative system is monotone order preserving.
Order preserving monotone dynamical systems

(Smith, 1995), (Hirsch et Smith, 2005), (Angeli et Sontag, 2003)

Incidence graph. A graphical test for monotonicity

MAPK

- Incidence graph with non-negative cycles ⇔
  Order preserving w.r.t. orthant cone of $\mathbb{R}^n$ (Kunze & Siegel, 99)
Monotonicity w.r.t orthant cone of $\mathbb{R}^n$

If $\exists \mathbf{D} = diag[(-1)^{\varepsilon_1}, ..., (-1)^{\varepsilon_n}], \varepsilon_i \in \{0, 1\}$

s.t $\mathbf{x}(t, \mathbf{x}_0, t_0)$ and $\mathbf{y}(t, \mathbf{y}_0, t_0)$ satisfy

$$\mathbf{Dy}_0 \geq \mathbf{Dx}_0 \Rightarrow \mathbf{Dy}(t, \mathbf{y}_0, t_0) \geq \mathbf{Dx}(t, \mathbf{x}_0, t_0) \forall t \geq t_0.$$
Cooperative dynamical systems as bounding systems

\[ t_0 \leq t \leq t_N \]
\[ \dot{x}(t) = f(x, p, t), \quad x(t_0) \in [x_0] \subseteq D \subseteq \mathbb{R}^n, \quad p \in [p] \]

\[ \dot{x}_1(t) = g_1(x_1, p, \overline{p}, t), \quad x_1(t_0) \in D, \text{ cooperative over } D \]
\[ \dot{x}_2(t) = g_2(x_2, p, \overline{p}, t), \quad x_2(t_0) \in D, \text{ cooperative over } D \]

Comparison theorem

if \( \forall x \in D, \quad \forall t \geq t_0, \quad g_1(x, p, \overline{p}, t) \leq f(x, [p], t) \leq g_2(x, p, \overline{p}, t) \)
and \( x_1(0) \leq x_0 \leq x_2(0) \)
\[ \Rightarrow \forall t \geq t_0, \quad x_1(t) \leq x(t) \leq x_2(t) \]

\[ \Rightarrow \text{An inclusion function for } [x(t)] = \text{ConvexHull}[x_1(t), x_2(t)] \]
Bracketing functions

\[ \dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p] \]

\( f \) cooperative over \( \mathbb{D} \)

Formal expression of \( \bar{f}_i \)

Study monotonicity → Analyze the signs of partial derivatives \( \frac{\partial f_i}{\partial p_k} \)

for \( k = 1, \ldots, n_p \), \( \text{if } \frac{\partial f_i}{\partial p_k} \geq 0 \) then replace \( p_k \leftarrow \bar{p}_k \)

else replace \( p_k \leftarrow p_k \)

Repeat for \( i = 1, \ldots, n \), ⇒ \[
\begin{cases}
\dot{x}(t) = f(x, p, \bar{p}, t), & x(t_0) = x_0 \\
\dot{x}(t) = \bar{f}(x, p, \bar{p}, t), & \bar{x}(t_0) = \bar{x}_0
\end{cases}
\]
Improved Bracketing functions

\[
\dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \; x(t_0) \in [x_0], \; p \in [p]
\]

\( f \) cooperative over \( \mathbb{D} \)

Formal expression of \( \bar{f}_i \)

Study monotonicity \( \rightarrow \) Analyze the signs of partial derivatives \( \frac{\partial f_i}{\partial p_k} \)

for \( k = 1, \ldots, n_p \),

if \( \frac{\partial f_i}{\partial p_k} \geq 0 \) then replace \( p_k \leftarrow \bar{p}_k \)

else if \( \frac{\partial f_i}{\partial p_k} < 0 \) then replace \( p_k \leftarrow \underline{p}_k \)

else replace \( p_k \leftarrow [p_k] \)

\( \Rightarrow \) \( \dot{x}_i(t) = \bar{f}_i(\bar{x}, \ldots, \underline{p}_k, \bar{p}_{k'}, [p_{k''}], t), \quad k, k', k'' \in \{1, \ldots, n_p\} \)

Repeat for \( i = 1, \ldots, n \), \( \Rightarrow \) \[
\begin{align*}
\dot{x}(t) &= f(x, \underline{p}, \bar{p}, [p], t), \quad x(t_0) = x_0 \\
\dot{x}(t) &= \bar{f}(\bar{x}, \underline{p}, \bar{p}, [p], t), \quad \bar{x}(t_0) = \bar{x}_0
\end{align*}
\]
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Reachable set via set integration

\[ \dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p] \]

Time grid \( t_0 < t_1 < t_2 < \cdots < t_N \)

1. Use interval Taylor methods \( \rightarrow [x](t) \)
2. With monotone systems \( \rightarrow \text{Analyse partial derivatives} \)
   \( \rightarrow \text{Use comparison theorem} \rightarrow [x](t) = \text{ConvexHull} [x(t), \bar{x}(t)] \)

Bracketing functions in the general case

Signs of partial derivatives change with integration time

\( \rightarrow \text{Hybridization} : \text{Hybrid automata as bounding systems} \)
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Numerical implementation

- C++ Class Libraries

- Interval computation → Profil/BIAS

- Taylor coefficients and differentiation → FADBAD++

- Set integration via Interval Taylor methods → Nedialkov’s VNODE
Thermal conductivity and diffusivity of composite materials
An infinite-dimensional system

The experimental set-up

Model reduction $\rightarrow$ finite-state model
Nonlinear hybridization with order preserving dynamical systems

Applications

Thermal conductivity and diffusivity of composite materials

Nonlinear ordinary differential equations

\[
\begin{align*}
\dot{x}_1 &= \alpha_1 (x_2 - 2x_1 + u_0 + u(t)) \\
\dot{x}_2 &= 2\alpha_1 (x_1 - (1 + \frac{\rho_1}{\rho_2})x_2 + \frac{\rho_1}{\rho_2} x_3) \\
\dot{x}_3 &= 2(p_0 + p_1 x_3)(x_4 - x_3 + p_2 \frac{\delta_2}{\rho_2} (x_2 - x_3)) \\
\dot{x}_i &= (p_0 + p_1 x_i)(x_{i+1} - 2x_i + x_{i-1}) \quad i = 4, \ldots, 9 \\
\dot{x}_{10} &= 2(p_0 + p_1 x_{10})(x_9 - x_{10} + p_2 \frac{\delta_2}{\rho_2} (x_{11} - x_{10})) \\
\dot{x}_{11} &= 2\alpha_2 (x_{12} - (1 + \frac{\rho_3}{\rho_2})x_{11} + \frac{\rho_3}{\rho_2} x_{10}) \\
\dot{x}_{12} &= \alpha_3 (x_{13} - 2x_{12} + x_{11}) \\
\dot{x}_{13} &= 2\alpha_3 (x_{12} - (1 + \frac{\rho_3}{\rho_4})x_{13} + \frac{\rho_3}{\rho_4} u_0) \\
u(t) &= \sum_{l=1}^{5} u_l \sin(2^{l-1}\omega_0 t + \phi_0)
\end{align*}
\]

Uncertain parameters and initial state vector

\[p = [p_0 \ p_1 \ p_2]^T \in \mathbb{P}_0 \quad p_0 \in [0.7, 1.23] \times [0.01, 0.015] \times [0.23, 0.64] \]
\[x_{0i} \in [90, 110].\]
Thermal conductivity and diffusivity of composite materials

Interval Taylor methods

→ method fails
Thermal conductivity and diffusivity of composite materials

Interval Taylor methods
→ method fails

Comparison theorem + Nonlinear hybridization
→ Analyse the signs of the partial derivatives $\frac{\partial f_i}{\partial p_k}$
Bracketing functions
→ Hybrid automaton with $1 + 3^{10}$ bracketing modes
only few modes activated
Nonlinear hybridization with order preserving dynamical systems

Thermal conductivity and diffusivity of composite materials

Nonlinear hybridization

→ Switching sequence for Upper bounding modes
Nonlinear hybridization with order preserving dynamical systems

Applications

Thermal conductivity and diffusivity of composite materials

Nonlinear hybridization

→ Switching sequence for Lower bounding modes
Thermal conductivity and diffusivity of composite materials

Nonlinear hybridization \( \rightarrow x_{12} \)-component time-history
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   - Hybrid reachability
   - Interval Constraint Propagation Techniques
   - Hybrid Transitions
   - Example

5. Future work
Hybrid Reachability Computation

Hybrid automaton (Alur, et al., 95)

\[ H = (Q, D, P, \Sigma, A, \text{Inv}, \mathcal{F}), \]

flow(q) : \[ \dot{x}(t) = f_q(x, p, t), \]
Inv(q) : \[ \nu_q(x(t), p, t) < 0, \]

e : \[ (q \rightarrow q') = (q, \text{guard}, \sigma, \rho, q'), \]
guard(e) : \[ \gamma_e(x(t), p, t) = 0, \]

t_0 \leq t \leq t_N, \quad x(t_0) \in X_0 \subseteq \mathbb{R}^n, \quad p \in P
Hybrid Reachability Computation

Set reachable in finite time

\[ \rho(x_{e}) = 0 \]

\[ \nu_{0}(.) < 0 \]

\[ \nu_{1}(.) < 0 \]

\[ \rho_{0}(X_{e}) \]

Reach.

Forbidden

N.Ramdani (PRISME)
Our contributions

Main ideas

- **ODE bounding methods** *(Interval Taylor series. Hybrid bounding approach. ... )*
  - ⇒ Analytical expressions for the boundaries of the continuous flows,

- **Interval constraint propagation techniques**
  - ⇒ Solve event detection/localization problems
  - ⇒ Flow/sets intersection.
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Constraint Satisfaction Problems

Numerical constraint satisfaction problem

CSP : \((\mathcal{Z}, C)\)
- \(\mathcal{Z}\) is a domain for \(z, z \in \mathbb{R}^{n_1}\)
- \(C : \land_{1 \leq i \leq m_1} (h_i(z) < 0), \prec \in \{=, <\},\)

Solution set

\[ S = \{z \in \mathcal{Z} | \land_{1 \leq i \leq m_1} (h_i(z) < 0)\}, \]
Interval Solving

Branch-and-Prune approach

\[ S = \{ z \in \mathbb{Z}, \ | \ h_i(z) < 0 \} \]

\[ \rightarrow S \subseteq S \subseteq S \]

\[ \text{Sup}(h_i([z])) < 0 \Rightarrow [z] \subseteq S \]
\[ \text{Inf}(h_i([z])) \geq 0 \Rightarrow [z] \not\subseteq S \]

otherwise partition . . .
Interval narrowing operators

Prune inconsistent parts

Gauss-Seidel, Newton & Krawczyk intervalles

Interval constraint propagation (Waltz, 75) (Davies, 87) (Cleary, 87) (Jaulin, 01)

Consistency filtering techniques (Collavizza, 99)
Consistency filtering techniques
(Collavizza, 99)

3B consistency. \textit{Shaving}
Technical improvement

**Function monotonicity**

\[ \rightarrow \operatorname{Sup}(h_i([z])) < 0 ? \]

- Use the gradient

\[
\text{if } \operatorname{Inf}(\frac{\partial h_i([z])}{\partial z_k}) > 0 \Rightarrow \operatorname{Sup}(h_i([z])) = h_i(\ldots, \operatorname{Sup}([z_k]), \ldots)
\]

*recursive algorithm* ....
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Computing flow/guards intersection

Time grid → $t_0 < t_1 < t_2 < \cdots < t_N$

Use analytical expression ...
Computing flow/guards intersection

Time grid → $t_0 < t_1 < t_2 < \cdots < t_N$

Compute $[t^*, \overline{t^*}] \times [\mathcal{X}_j^*]$
Computing flow/guards intersection

Time grid \( t_0 < t_1 < t_2 < \cdots < t_N \)

- \([x](t) = \text{Interval Taylor Series (ITS)}(t, [x_j], [\bar{x}_j])\)
- \(\gamma([x](t)) = 0\)

\[\Rightarrow \gamma \circ \text{ITS}(t, x_j, \bar{x}_j) \rightarrow \psi(t, x_j)\]

Solve CSP \(([t_j, t_{j+1}] \times [x_j], \psi(., .) \ni 0)\)
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   - Hybrid reachability
   - Interval Constraint Propagation Techniques
   - Hybrid Transitions
   - Example

5. Future work
Example

flow(1) : $f_1(x_1, x_2) = (x_2, -px_2 - g \sin(x_1))$
inv(1) : $\nu_1(x_1, x_2) = \cos(x_1) - x_2/10 - 0.7$
flow(2) : $f_2(x_1, x_2) = (x_2, -3px_2 - g \sin(x_1))$
inv(2) : $\nu_2(x_1, x_2) = -\nu_1(x_1, x_2)$

guard(1) : $\gamma_1(x_1, x_2) = \nu_1(x_1, x_2)$
reset(1) : $\rho_1(x_1, x_2) = (\alpha_1 x_1, \alpha_2 x_2)$

$x_0 \in [-0.9, -0.8] \times [3, 3.5], \quad \alpha_1 = -1, \quad \alpha_2 \in [-2.05, -2], \quad g = 10, \quad p \in [6, 6.3].$
Example

CPU time = 12.9s PIV 2GHz
Example

![Diagram showing discrete and flow transitions, reachable sets, and frontiers with a time axis. The diagram illustrates the dynamics of a hybrid system over time.]

- Discrete transitions
- Flow transitions
- Reachable sets
- Frontiers

Time (s) vs. $x_2$
Outline

1. Introduction
2. Nonlinear hybridization approach to reachability
3. Nonlinear hybridization with order preserving dynamical systems
4. Extension to hybrid systems
5. Future work
Future work

**Continuous reachability computation**

- Nonlinear hybridization approach to reachability computation with uncertain nonlinear continuous dynamical systems.

  → Develop an open source toolbox package.
  
  → How to improve the rule for building bracketing systems? Use convex relaxations?
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- Use alternate robust integration methods. Bound truncation error?
- Extend hybrid bounding method to DAE.
- Extend hybrid bounding method to PDE and infinite-dimensional systems.
Hybrid reachability computation

- Use of constraint programming for hybrid reachability computation with uncertain nonlinear dynamical systems.

→ Extend nonlinear hybridization approach to truly hybrid systems?
Future work

Future work, cont’d

Hybrid reachability computation

- Use of constraint programming for hybrid reachability computation with uncertain nonlinear dynamical systems.

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→ Better CSP tools?
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→ Embed within state-of-the-art Bounded Model Checkers
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  - SAT modulo ODE formulae
Future work

Future work, cont’d

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→ Use for verification, synthesis and estimation with actual complex and cyber-physical systems
State and Parameter estimation with Hybrid systems

- Bounded-error (hybrid) state estimation with hybrid systems
  - Extend and use hybrid reachability tools
  - Unknown switching sequence ...
State and Parameter estimation with Hybrid systems

- Bounded-error (hybrid) state estimation with hybrid systems
  - Extend and use hybrid reachability tools
  - Unknown switching sequence...
- Bounded-error Error-in-Variables problems for hybrid systems
  - Convex relaxations...
References

Nonlinear Hybridization


Nonlinear Hybridization with Monotone Systems


Nonlinear Hybridization approach to State Estimation


References

Hybrid Reachability Computation
