Reachability Analysis of Linear Systems using Support Functions[☆]

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Abstract

This work is concerned with the algorithmic reachability analysis of continuous-time linear systems with constrained initial states and inputs. We propose an approach for computing an over-approximation of the set of states reachable on a bounded time-interval. The main contribution over previous works is that it allows us to consider systems whose sets of initial states and inputs are given by arbitrary compact convex sets represented by their support functions. We actually compute two over-approximations of the reachable set. The first one is given by the union of convex sets with computable support functions. As the representation of convex sets by their support function is not suitable for some tasks, we derive from this first over-approximation a second one given by the union of polyhedrons. The overall computational complexity of our approach is comparable to the complexity of the most competitive available specialized algorithms for reachability analysis of linear systems using zonotopes or ellipsoids. The effectiveness of our approach is demonstrated on several examples.

Key words: reachability analysis, support functions, computational methods, linear systems

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1. Introduction

Computers have become ubiquitious in control systems design, offering the opportunity for the development of new techniques for synthesis and analysis. One of these approaches, inspired by the algorithmic verification of discrete systems (i.e. model checking [1]), has emerged from hybrid systems research and is based on reachability analysis. It consists in computing the set of states reachable by a system; thus making it possible to examine all its possible be-

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haviours. This information can then be used either for algorithmic verification or controller synthesis (see e.g. [2, 3]).

Numerous techniques have been developped in the latest decade for reachability analysis of continuous and hybrid systems (see e.g. [4, 5, 6, 7, 8]). The standard hybrid reachability algorithm alternates computations of the sets reachable under the discrete dynamics and of the sets reachable under the continuous dynamics. Reachability under the continuous dynamics is often considered as the most challenging part of the job and it has been the main focus of the work on hybrid systems reachability.

In this paper, we consider the computation of the set of states reachable by a linear system with constrained initial states and inputs. More precisely, we consider continuous-time systems of the form:

$$\dot{x}(t) = Ax(t) + Bu(t),\ u(t) \in U,\ x(0) \in X_0$$

where U and X_0 are compact convex sets denoting the sets of inputs U and initial states X_0, respectively. Several approaches have been proposed for reachability analysis of this class of systems assuming that the sets U and X_0 belong to the class of ellipsoids [9, 7, 12], polytopes [4, 5, 6] or zonotopes [10, 11, 12]. Our work extends and unifies these techniques as we do not assume that U and X_0 belong to some special class of sets. U and X_0 are just assumed to be compact and convex sets specified by their support function. Support functions are classical tools of convex analysis and can be computed efficiently for a fairly large class of sets including ellipsoids, polytopes and zonotopes.

We propose an approach for the computation of an over-approximation of the set of states reachable on a bounded time interval $[0, T]$: $\mathcal{R}_{[0,T]}(X_0)$. It is based on a discretization of time; given a time step $\tau = T/N$ with $N \in \mathbb{N}$, we compute convex over-approximations Ω_i of the sets of states reachable on time intervals of the form $[i\tau, (i+1)\tau]$. The result is an over-approximation of $\mathcal{R}_{[0,T]}(X_0)$ given by the union of compact convex sets with computable support functions. Furthermore, the over-approximation can be made arbitrarily close by choosing the time step τ small enough.

As the representation of convex sets by their support function is not suitable for some tasks, especially when an explicit representation is needed, we propose a method for computing tight polyhedral over-approximations $\overline{\Omega}_i$ of the convex sets Ω_i. Thus, our approach allows us to compute an over-approximation of $\mathcal{R}_{[0,T]}(X_0)$ given by the union of polyhedrons. The faces of the polyhedrons can be made arbitrarily close from the actual reachable set by choosing the time step τ small enough. Moreover, the overall computational complexity of our approach is comparable to the complexity of the most competitive available specialized algorithms for reachability analysis of linear systems using zonotopes [11] or ellipsoids [12]. The effectiveness of our approach is demonstrated on several examples.

The results presented in this paper, appeared in preliminary form in [13].

2. Convex Sets and Support Functions

The support function of a convex set is a classical tool of convex analysis. In the following, we shall use support functions as a representation of arbitrary complex convex sets. In this section, we present some properties of support functions and show how they can be used for the computation of polyhedral approximations of convex sets. The results are stated without the proofs that are quite straightforward and can be found in several textbooks on convex analysis (see e.g. [14, 15, 16]).

2.1. Support Functions

Definition 1. *Let $\Omega \subseteq \mathbb{R}^d$ be a compact convex set; the support function of Ω, denoted ρ_Ω, is defined as:*

$$\begin{array}{rccl} \rho_\Omega : & \mathbb{R}^d & \to & \mathbb{R} \\ & \ell & \mapsto & \max_{x\in\Omega} \ell \cdot x \end{array}$$

The notion of support function is illustrated in Figure 1. It can be shown that the support function of a compact convex set is a convex function.

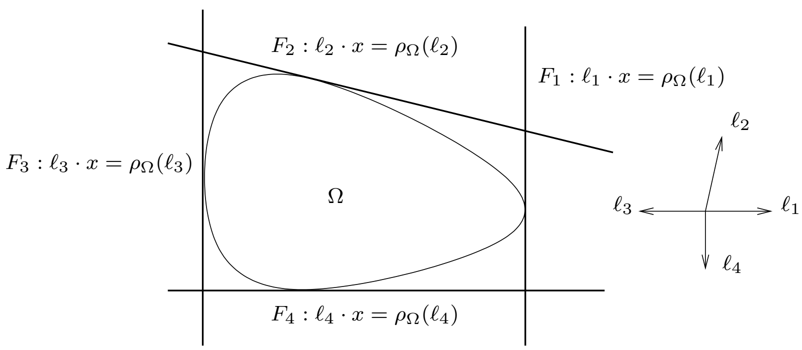

Figure 1: Illustration of the notion of support function of a convex set Ω.

It is to be noted that the set Ω is uniquely determined by its support function as the following equality holds:

$$\Omega = \bigcap_{\ell\in\mathbb{R}^d} \{x \in \mathbb{R}^d : \ell \cdot x \leq \rho_\Omega(\ell)\} \tag{1}$$

which means that any convex set Ω is the intersection of the infinite set of halfspaces with normal vector $l \in \mathbb{R}^d$ and distance value $\rho_\Omega(l)$.

Proposition 1. *For the following compact convex sets, the support function can be computed.*

- *The unit ball for the* 1*-norm:* $\mathrm{B}_1 = \{x \in \mathbb{R}^d : \|x\|_1 \leq 1\}$*. Then,*

$$\rho_{\mathrm{B}_1}(\ell) = \|\ell\|_\infty.$$

- *The unit ball for the usual Euclidean norm:* $\mathrm{B}_2 = \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$*. Then,*

$$\rho_{\mathrm{B}_2}(\ell) = \|\ell\|_2.$$

- *The unit ball for the* ∞*-norm:* $\mathrm{B}_\infty = \{x \in \mathbb{R}^d : \|x\|_\infty \leq 1\}$*. Then,*

$$\rho_{\mathrm{B}_\infty}(\ell) = \|\ell\|_1.$$

- *An ellipsoid:* $\Omega = \left\{x \in \mathbb{R}^d : x^\top Q^{-1} x \leq 1\right\}$ *where* Q *is a positive definite symmetric matrix. Then,*

$$\rho_\Omega(\ell) = \sqrt{\ell^\top Q \ell}.$$

- *A hyper-rectangle:* $\Omega = [-h_1; h_1] \times \ldots \times [-h_d; h_d]$ *where* $h_1, \ldots, h_d \in \mathbb{R}^+$*. Then,*

$$\rho_\Omega(\ell) = \sum_{j=1}^{d} |h_j \ell_j|$$

  where ℓ_j *denotes the* j^{th} *coordinate of* ℓ*.*

- *A zonotope:* $\Omega = \{\alpha_1 g_1 + \cdots + \alpha_r g_r : \alpha_j \in [-1, 1],\ j = 1, \ldots, r\}$ *where the generators* $g_1, \ldots, g_r \in \mathbb{R}^d$*. Then,*

$$\rho_\Omega(\ell) = \sum_{j=1}^{r} |g_j \cdot \ell|.$$

- *A polytope:* $\Omega = \left\{x \in R^d : Cx \leq d\right\}$ *where* C *and* d *are a matrix and vector of compatible dimension. Then, computing* $\rho_\Omega(\ell)$ *is equivalent to solving the linear program:*

$$\begin{cases} \text{Maximize } \ell \cdot x \\ \text{Subject to } Cx \leq d \end{cases}$$

Thus, support functions and support vectors can be computed efficiently for a quite large class of sets. Further, more complex sets can be considered using operations on elementary convex sets. Given a compact convex set $\Omega \subseteq \mathbb{R}^d$ and a matrix A, $A\Omega$ denotes the image of Ω by A. Given a positive scalar λ, $\lambda\Omega = (\lambda I)\Omega$ where I is the identity matrix. Let Ω, $\Omega' \subseteq \mathbb{R}^d$, $\mathrm{CH}(\Omega, \Omega')$ denotes the convex hull of Ω and Ω' and $\Omega \oplus \Omega'$ denotes the Minkowski sum of Ω and Ω':

$$\Omega \oplus \Omega' = \{x + x' : x \in \Omega, x' \in \Omega'\}.$$

The support function of sets defined using these operations can be computed using the following properties:

Proposition 2. *For all compact convex sets Ω, $\Omega' \subseteq \mathbb{R}^d$, for all matrices A, all positive scalars λ, and all vectors $\ell \in \mathbb{R}^d$, we have:*

$$\begin{aligned}
\rho_{A\Omega}(\ell) &= \rho_\Omega(A^\top \ell) \\
\rho_{\lambda\Omega}(\ell) &= \rho_\Omega(\lambda\ell) \;=\; \lambda\rho_\Omega(\ell) \\
\rho_{\mathrm{CH}(\Omega,\Omega')}(\ell) &= \max(\rho_\Omega(\ell), \rho_{\Omega'}(\ell)) \\
\rho_{\Omega\oplus\Omega'}(\ell) &= \rho_\Omega(\ell) + \rho_{\Omega'}(\ell).
\end{aligned}$$

Using these properties, one can easily consider convex sets of unusual shape. Figure 2 illustrates how the support function of the Minkowski sum of a ball with the convex hull of two polytopes can be computed.

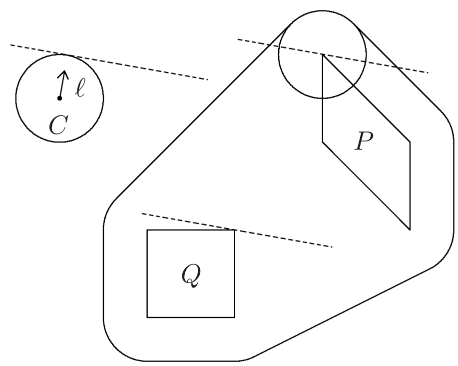

Figure 2: Computation of the support function of the Minkowski sum of a ball C with the convex hull of two polytopes P and Q: $\rho_{\mathrm{CH}(P,Q)\oplus C}(\ell) = \max(\rho_P(\ell), \rho_Q(\ell)) + \rho_C(\ell)$.

2.2. Polyhedral Approximations of Convex Sets

In this paper, we shall consider two notions of approximation of sets based on two distances. Given Ω and Ω' two compact subsets of $\mathbb{R}^d$, we define the distance between Ω and Ω':

$$d(\Omega, \Omega') = \inf_{x\in\Omega} \inf_{x'\in\Omega'} \|x - x'\|.$$

We also define the Hausdorff distance between Ω and Ω':

$$d_H(\Omega, \Omega') = \max\left(\sup_{x\in\Omega} \inf_{x'\in\Omega'} \|x - x'\|, \sup_{x'\in\Omega'} \inf_{x\in\Omega} \|x - x'\| \right).$$

Let us remark that only the Hausdorff distance is a metric in the usual sense. Particularly, $d_H(\Omega, \Omega') = 0$ if and only if $\Omega = \Omega'$ whereas $d(\Omega, \Omega') = 0$ if and only if $\Omega \cap \Omega'$ is not empty.

From equation (1), it is easy to see that polyhedral over-approximation of an arbitrary compact convex set can be obtained by "sampling" its support function.

Proposition 3. *Let Ω be a compact convex set and $\ell_1, \dots, \ell_r \in \mathbb{R}^d$ be arbitrarily chosen vectors; let us define the following halfspaces:*

$$H_k = \{x \in \mathbb{R}^d : \ell_k \cdot x \leq \rho_\Omega(\ell_k)\},\ k = 1, \dots, r$$

and the polyhedron

$$\overline{\Omega} = \bigcap_{k=1}^{r} H_k.$$

Then, $\Omega \subseteq \overline{\Omega}$. Moreover, we say that this over-approximation is tight as Ω touches the faces $F_1, \dots, F_r$ of $\overline{\Omega}$:

$$d(\Omega, F_k) = 0,\ k = 1, \dots, r.$$

An example of such polyhedral over-approximation of a convex set can be seen in Figure 1.

3. Convex Approximations of Reachable Sets

In this paper, we consider continuous-time linear systems of the form:

$$\dot{x}(t) = Ax(t) + Bu(t),\ u(t) \in U$$

where $U \subseteq \mathbb{R}^{d'}$ is a compact convex set specified by its support function ρ_U, A and B are matrices of compatible dimensions. For simplicity of the notations, we shall define the set $V = BU \subseteq \mathbb{R}^d$ with support function $\rho_V(\ell) = \rho_U(B^\top \ell)$ and consider the equivalent system:

$$\dot{x}(t) = Ax(t) + v(t),\ v(t) \in V$$

Given a subset $X \subseteq \mathbb{R}^d$, we denote by $\mathcal{R}_s(X) \subseteq \mathbb{R}^d$ the set of states reachable by the system at time s from states in X:

$$\mathcal{R}_s(X) = \{x(s) :\ \dot{x}(t) = Ax(t) + v(t), v(t) \in V, \forall t \in [0, s] \text{ and } \ x(0) \in X\}.$$

Then, the reachable set on the time interval $[s, s']$ is defined as

$$\mathcal{R}_{[s,s']}(X) = \bigcup_{t \in [s,s']} \mathcal{R}_t(X).$$

Let $X_0 \subseteq \mathbb{R}^d$ be a specified compact convex set of initial states specified by its support function ρ_{X_0}. In the following, we are interested in computing an over-approximation of the reachable set on the time interval $[0, T]$ from the initial states X_0, that is $\mathcal{R}_{[0,T]}(X_0)$. Let $\|.\|$ be a norm. We shall denote

$$R_{X_0} = \max_{x \in X_0} \|x\|,\ D_{X_0} = \max_{x,y \in X_0} \|x - y\|,\ \text{and } R_V = \max_{v \in V} \|v\|.$$

If only the support function of X_0 and V are known, these values might be hard to compute for some norms. Then, the norm $\|.\|$ should be chosen appropriately.

As an example, if $\|.\|$ is the infinity norm, R_{X_0} can be derived easily from the evaluation of ρ_{X_0} at the d canonical generators of $\mathbb{R}^d$ and their opposite.

In this section, we show how the reachable set can be over-approximated by the union of convex sets given by their computable support functions. Further, the Hausdorff distance between the reachable set and its approximation can be made arbitrarily small.

3.1. Time-Discretization Scheme

Our approach is based on a discretization of the time. Let $\tau = T/N$ be the time step (with $N \in \mathbb{N}$). Then, we have:

$$\mathcal{R}_{[0,T]}(X_0) = \bigcup_{i=0}^{N-1} \mathcal{R}_{[i\tau,(i+1)\tau]}(X_0).$$

In order to compute an over-approximation of $\mathcal{R}_{[0,T]}(X_0)$, we shall compute over-approximations of all the sets $\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0)$. The results presented in this part are adapted from [10], though significantly improved to be applied to general convex sets of inputs and initial states. The proofs are quite technical and are stated in appendix for a better readability. We consider the first element of the sequence, $\mathcal{R}_{[0,\tau]}(X_0)$:

Lemma 1. *Let Ω_0 be the convex set defined by:*

$$\Omega_0 = \mathrm{CH}\left(X_0, e^{\tau A} X_0 \oplus \tau V \oplus \alpha_\tau \mathrm{B}\right) \tag{2}$$

where $\alpha_\tau = (e^{\tau\|A\|} - 1 - \tau\|A\|)(R_{X_0} + \frac{R_V}{\|A\|})$ and B denotes the unit ball for the considered norm. Then, $\mathcal{R}_{[0,\tau]}(X_0) \subseteq \Omega_0$ and

$$d_H(\Omega_0, \mathcal{R}_{[0,\tau]}(X_0)) \leq \frac{1}{4}(e^{\tau\|A\|} - 1) D_{X_0} + 2\alpha_\tau. \tag{3}$$

This lemma can be roughly understood as follows, $e^{\tau A} X_0 \oplus \tau V$ is an approximation the reachable set at time τ; a bloating operation followed by a convex hull operation gives an approximation of $\mathcal{R}_{[0,\tau]}(X_0)$. The bloating factor α_τ is chosen to ensure over-approximation. The approximation error can be made arbitrarily small by choosing τ small enough. We now consider the other elements of the sequence $\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0)$. Let us remark that we have

$$\mathcal{R}_{[(i+1)\tau,(i+2)\tau]}(X_0) = \mathcal{R}_\tau\left(\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0)\right),\ i = 0, \ldots, N-2.$$

Given a subset $\Omega \subseteq \mathbb{R}^d$, the following lemma provides us with an over-approximation of $\mathcal{R}_\tau(\Omega)$:

Lemma 2. *Let $\Omega \subseteq \mathbb{R}^d$, let Ω' be the set defined by:*

$$\Omega' = e^{\tau A}\Omega \oplus \tau V \oplus \beta_\tau \mathrm{B}$$

where $\beta_\tau = (e^{\tau\|A\|} - 1 - \tau\|A\|)\frac{R_V}{\|A\|}$ and B denotes the unit ball for the considered norm. Then, $\mathcal{R}_\tau(\Omega) \subseteq \Omega'$ and

$$d_H(\Omega', \mathcal{R}_\tau(\Omega)) \leq 2\beta_\tau. \tag{4}$$

The set $e^{\tau A}\Omega \oplus \tau V$ is an approximation the reachable set at time τ; bloating this set using the ball of radius β_τ ensures over-approximation. Again, the approximation error can be made arbitrarily small by choosing τ small enough.

We shall now define the sequence of convex sets Ω_i over-approximating $\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0)$ as follows. Ω_0 is given by equation (2) and

$$\Omega_{i+1} = e^{\tau A}\Omega_i \oplus \tau V \oplus \beta_\tau \mathrm{B},\ i = 0, \dots, N-2. \tag{5}$$

Theorem 1. *Let us consider the sequence of sets Ω_i defined by equations (2) and (5); then, for all $i = 0, \dots, N-1$, $\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0) \subseteq \Omega_i$ and*

$$d_H(\Omega_i, \mathcal{R}_{[i\tau,(i+1)\tau]}(X_0)) \le \tau e^{T\|A\|}\left(\frac{\|A\|}{4}D_{X_0} + \tau\|A\|^2 R_{X_0} + e^{\tau\|A\|}R_V\right).$$

This theorem essentially states that the reachable set $\mathcal{R}_{[0,T]}(I)$ can be over-approximated by the union of convex sets $\Omega_0 \cup \dots \cup \Omega_{N-1}$. Further, the error bound for the Hausdorff distance is in $\mathcal{O}(\tau)$ and thus can be made arbitrarily small by choosing τ small enough.

3.2. Support Functions of the Approximate Reachable Sets

We now consider the computation of the support functions of the approximate reachable sets $\Omega_0 \dots \Omega_{N-1}$ defined in the previous section. For simplicity of the notations, let us introduce the matrix Φ_τ and the set W_τ defined by

$$\Phi_\tau = e^{\tau A},\ W_\tau = \tau V \oplus \beta_\tau \mathrm{B}. \tag{6}$$

Using the proposition 2, it follows that the support function of W_τ is given by

$$\rho_{W_\tau}(\ell) = \tau\rho_V(\ell) + \beta_\tau\rho_{\mathrm{B}}(\ell) \tag{7}$$

where ρ_{B} is the support function of the unit ball for the chosen norm. The following proposition gives the expression of $\rho_{\Omega_0} \dots \rho_{\Omega_{N-1}}$.

Proposition 4. *Let $\Omega_0 \dots \Omega_{N-1}$ be the sets defined by equations (2) and (5). Then, for all ℓ in $\mathbb{R}^d$,*

$$\rho_{\Omega_0}(\ell) = \max\left(\rho_{X_0}(\ell), \rho_{X_0}({\Phi_\tau}^\top \ell) + \tau\rho_V(\ell) + \alpha_\tau\rho_{\mathrm{B}}(\ell)\right) \tag{8}$$

and for $i = 0, \dots, N-1$,

$$\rho_{\Omega_i}(\ell) = \rho_{\Omega_0}\left(({\Phi_\tau}^\top)^i\ell\right) + \sum_{j=0}^{i-1}\rho_{W_\tau}\left(({\Phi_\tau}^\top)^j\ell\right). \tag{9}$$

Proof. Equation (8) is a direct application of proposition 2 to equation (2). Let us prove equation (9) by induction. The equation is true for $i = 0$. Let us assume that it holds for some i; from equation (5), we have $\Omega_{i+1} = \Phi_\tau \Omega_i + W_\tau$. It follows from proposition 2 that

$$\begin{aligned}
\rho_{\Omega_{i+1}}(\ell) &= \rho_{\Omega_i}({\Phi_\tau}^\top \ell) + \rho_{W_\tau}(\ell) \\
&= \left(\rho_{\Omega_0}\left(({\Phi_\tau}^\top)^i {\Phi_\tau}^\top \ell\right) + \sum_{j=0}^{i-1} \rho_{W_\tau}\left(({\Phi_\tau}^\top)^j {\Phi_\tau}^\top \ell\right) \right) + \rho_{W_\tau}(\ell) \\
&= \rho_{\Omega_0}\left(({\Phi_\tau}^\top)^{i+1} \ell\right) + \sum_{j=0}^{i} \rho_{W_\tau}\left(({\Phi_\tau}^\top)^j \ell\right)
\end{aligned}$$

which proves, by induction, equation (9). ■

Hence, we showed that the reachable set of a linear system can be over-approximated arbitrarily close by a union of compact convex sets with computable support functions.

4. Polyhedral Approximations of the Reachable Sets

The representation of convex sets by their support function is not suitable for some tasks, especially when an explicit representation is needed. From proposition 3, polyhedral approximations of the sets $\Omega_0, \dots, \Omega_{N-1}$ can be obtained by evaluating their support functions in several directions. These sets provide with polyhedral approximations of the reachable sets:

Theorem 2. *Let $\rho_{\Omega_0}, \dots, \rho_{\Omega_{N-1}}$ be the functions defined in proposition 4. Let $\ell_1, \dots, \ell_r \in \mathbb{R}^d$ be arbitrarily chosen vectors; let us define the following halfspaces:*

$$H_{i,k} = \{x \in \mathbb{R}^d : \ell_k \cdot x \le \rho_{\Omega_i}(\ell_k)\},\ i = 0, \dots, N-1,\ k = 1, \dots, r$$

and the polyhedrons

$$\overline{\Omega}_i = \bigcap_{k=1}^{r} H_{i,k}.$$

Then, for all $i = 0, \dots, N-1$, $\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0) \subseteq \overline{\Omega}_i$. Let $F_{i,1}, \dots, F_{i,r}$ denote the faces of polyhedron Ω_i, then

$$d(\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0), F_{i,k}) \le \tau e^{T\|A\|} \left(\frac{\|A\|}{4} D_{X_0} + \tau \|A\|^2 R_{X_0} + e^{\tau\|A\|} R_V \right).$$

Proof. From theorem 1, $\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0) \subseteq \Omega_i$ and from proposition 3, $\Omega_i \subseteq \overline{\Omega}_i$. Therefore, the first part of the theorem holds. Let us consider a face $F_{i,k}$ of the polyhedron $\overline{\Omega}_i$; from proposition 3, $d(\Omega_i, F_{i,k}) = 0$. Then,

$$\begin{aligned}
d(\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0), F_{i,k}) &\le d(\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0), \Omega_i) + d(\Omega_i, F_{i,k}) \\
d(\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0), F_{i,k}) &\le d_H(\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0), \Omega_i).
\end{aligned}$$

The second part of the theorem follows from theorem 1. ■

Theorem 2 states that by evaluating the functions $\rho_{\Omega_0}, \ldots, \rho_{\Omega_{N-1}}$, we can compute a union of polyhedrons over-approximating the reachable set $\mathcal{R}_{[0,T]}(X_0)$. Moreover, the distance between each face of the approximating polyhedrons and the actual reachable set can be made arbitrarily small. Let us remark that the polyhedral over-approximation $\overline{\Omega}_i$ is not computed from the previous polyhedra of the sequence but from the support function of Ω_i. As a consequence, the proposed algorithm is not subject to the wrapping effect (i.e. the successive over-approximations do not propagate in the computations).

4.1. Efficient Algorithm for the Computation of the Polyhedral Approximations

In this part, we consider the problem of computing efficiently the polyhedral over-approximations of the reachable set defined in the previous paragraph. We present an efficient algorithm that evaluates the support functions $\rho_{\Omega_0}, \ldots, \rho_{\Omega_{N-1}}$ in a given direction ℓ. It is based on the following observation: let us introduce the following auxiliary sequences $r_0, \ldots, r_{N-1} \in \mathbb{R}^d$ and $s_0, \ldots, s_{N-1} \in \mathbb{R}$:

$$\begin{array}{lclclcl} r_0 & = & \ell, & r_{i+1} & = & {\Phi_\tau}^\top r_i, \\ s_0 & = & 0, & s_{i+1} & = & s_k + \rho_{W_\tau}(r_i). \end{array}$$

Equivalently, we have

$$r_i = ({\Phi_\tau}^\top)^i \ell \text{ and } s_i = \sum_{j=0}^{i-1} \rho_{W_\tau}\left(({\Phi_\tau}^\top)^j \ell\right).$$

Therefore,

$$\rho_{\Omega_i}(\ell) = \rho_{\Omega_0}(r_i) + s_i.$$

Algorithm 1 Evaluation of $\rho_{\Omega_0}(\ell), \ldots, \rho_{\Omega_{N-1}}(\ell)$.

Input: The matrix Φ_τ given by (6), the support functions ρ_{Ω_0} and ρ_{W_τ} given by (8) and (7), the vector ℓ and an integer N.

Output: $\rho_i = \rho_{\Omega_i}(\ell)$ for i in $\{0, \ldots, N-1\}$

$r_0 \leftarrow \ell$
$s_0 \leftarrow 0$
$\rho_0 \leftarrow \rho_{\Omega_0}(r_0)$
for i from 0 to $N-2$ **do**
    $r_{i+1} \leftarrow {\Phi_\tau}^\top r_i$
    $s_{i+1} \leftarrow s_i + \rho_{W_\tau}(r_i)$
    $\rho_{i+1} \leftarrow \rho_{\Omega_0}(r_{i+1}) + s_{i+1}$
end for
return $\{\rho_0, \ldots, \rho_{N-1}\}$

Algorithm 1 implements efficiently the evaluation of $\rho_{\Omega_0}(\ell), \ldots, \rho_{\Omega_{N-1}}(\ell)$. It performs, at each of the $N-1$ iterations, the product of a matrix with a vector and the evaluation of the support functions ρ_{Ω_0} and ρ_{W_τ} given by (8) and (7).

The global time complexity of algorithm 1 is therefore $\mathcal{O}(N(d^2+\mathcal{C}_{X_0}+\mathcal{C}_V+\mathcal{C}_\mathrm{B}))$ where $\mathcal{C}_{X_0}$, $\mathcal{C}_V$ and $\mathcal{C}_\mathrm{B}$ denote the complexity of evaluating ρ_{X_0}, ρ_V, and ρ_B, respectively[1].

Then, the evaluation of the support functions in r directions, allowing us to compute polyhedral over-approximations $\overline{\Omega}_0, \dots, \overline{\Omega}_{N-1}$ of the reachable sets $\mathcal{R}_{[0,\tau]}, \dots, \mathcal{R}_{[(N-1)\tau, N\tau]}$ defined as intersections of r halfspaces has time complexity:

$$\mathcal{O}(rN(d^2+\mathcal{C}_{X_0}+\mathcal{C}_V+\mathcal{C}_\mathrm{B})). \tag{10}$$

Let us remark that the complexity is linear in the time horizon N and polynomial in the dimension d; this is comparable to the complexity of the most competitive available algorithms for reachability analysis of linear systems using zonotopes [11] or ellipsoids [12]. In the following, we show how the efficiency of the reachability analysis can be further improved.

4.2. Improvements of the Algorithm

An important advantage of our approach is that it can be trivially parallelized. Indeed, the support function can be evaluated independently in the different directions $\ell_1, \dots, \ell_r$. Thus, running the reachability analysis on α processors makes the overall complexity drops to

$$\mathcal{O}\left(\left\lceil \frac{r}{\alpha} \right\rceil N(d^2+\mathcal{C}_{X_0}+\mathcal{C}_V+\mathcal{C}_\mathrm{B})\right).$$

The second improvement is more sophisticated. Let us assume that the different directions of approximation $\ell_1, \dots, \ell_r$ have been chosen such that:

$$\ell_k = ({\Phi_\tau}^\top)^{p_k}\ell,\ k=1,\dots,r \text{ where } p_1 < \dots < p_r.$$

As an example, one can choose the indices p_k iteratively, taking p_{k+1} such that the angle between $({\Phi_\tau}^\top)^{p_{k+1}}\ell$ and vectors the $({\Phi_\tau}^\top)^{p_i}\ell$ is bigger than some value, or:

$$\forall i \le k,\ \left| \frac{({\Phi_\tau}^\top)^{p_{k+1}}\ell \cdot ({\Phi_\tau}^\top)^{p_i}\ell}{\|({\Phi_\tau}^\top)^{p_{k+1}}\ell\| \|({\Phi_\tau}^\top)^{p_i}\ell\|} \right| \le \varepsilon.$$

Then, from proposition 4, it follows that for all $i = 0, \dots, N-1$, $k = 1, \dots, r$:

$$\begin{aligned}
\rho_{\Omega_i}(\ell_k) &= \rho_{\Omega_0}\left(({\Phi_\tau}^\top)^i({\Phi_\tau}^\top)^{p_k}\ell\right) + \sum_{j=0}^{i-1} \rho_{W_\tau}\left(({\Phi_\tau}^\top)^j({\Phi_\tau}^\top)^{p_k}\ell\right) \\
&= \rho_{\Omega_0}\left(({\Phi_\tau}^\top)^{i+p_k}\ell\right) + \sum_{j=0}^{i-1} \rho_{W_\tau}\left(({\Phi_\tau}^\top)^{j+p_k}\ell\right) \\
&= \rho_{\Omega_0}\left(({\Phi_\tau}^\top)^{i+p_k}\ell\right) + \sum_{j=p_k}^{i+p_k-1} \rho_{W_\tau}\left(({\Phi_\tau}^\top)^{j}\ell\right) \\
&= \rho_{\Omega_0}(r_{i+p_k}) + s_{i+p_k} - s_{p_k}.
\end{aligned}$$

[1]If X_0 and V are ellipsoids or zonotopes with $\mathcal{O}(d)$ generators, and if the chosen norm is one of the classical norms (1, 2, or ∞), the time complexity of algorithm 1 becomes $\mathcal{O}(Nd^2)$.

Thus, it is sufficient to compute the sequences $r_0, \dots, r_{N+p_r-1}$ and $s_0, \dots, s_{N+p_r-1}$. Then, the complexity of the reachability analysis drops to

$$\mathcal{O}((N+p_r)(d^2+\mathcal{C}_{X_0}+\mathcal{C}_V+\mathcal{C}_{\mathrm{B}}+r)).$$

If ℓ is an eigenvector of $\Phi_\tau{}^\top$ associated to a real eigenvalue, it is clear that the vectors $\ell_k = (\Phi_\tau{}^\top)^{p_k}\ell$ will be colinear. In this case, the previous improvement cannot be used. However, evaluating the support function in the direction of an eigenvector ℓ can be interesting as it can be done very efficiently. If $\Phi_\tau{}^\top\ell = \lambda\ell$, with $\lambda \geq 0$ then:

$$\begin{aligned}\rho_{\Omega_i}(\ell) &= \rho_{\Omega_0}(\lambda^i\ell) + \sum_{j=0}^{i-1}\rho_{W_\tau}(\lambda^j\ell)\\ &= \lambda^i\rho_{\Omega_0}(\ell) + \rho_{W_\tau}(\ell)\sum_{j=0}^{i-1}\lambda^j.\end{aligned}$$

Then, only the evaluation of $\rho_{\Omega_0}(\ell)$ and $\rho_{W_\tau}(\ell)$ are needed. Hyperplanes bounding the reachable sets, in the direction given by an eigenvector are computed in only $\mathcal{O}(N+\mathcal{C}_{X_0}+\mathcal{C}_V+\mathcal{C}_B)$. Similarly, if $\lambda < 0$, $\rho_{\Omega_i}(\ell)$ can be computed from $\rho_{\Omega_0}(\ell)$, $\rho_{\Omega_0}(-\ell)$, $\rho_{W_\tau}(\ell)$ and $\rho_{W_\tau}(-\ell)$.

4.3. Comparison with a Related Approach

Reachability analysis of linear systems based on the use of support functions has already been proposed in [9]. We would like to discuss here the differences between the two approaches. In [9], the support functions of the reachable sets are computed recursively using the relation:

$$\rho_{\Omega_{i+1}}(\ell) = \rho_{\Omega_i}(\Phi_\tau{}^\top\ell) + \rho_{W_\tau}(\ell).$$

Then, the polyhedral over-approximation $\overline{\Omega}_i$ is defined as

$$\overline{\Omega}_i = \bigcap_{k=1}^{r}\{x \in \mathbb{R}^d : \ell_{i,k}\cdot x \leq \rho_{\Omega_i}(\ell_{i,k})\}$$

where $\ell_{i,k} = ((\Phi_\tau{}^\top)^{-1})^i\ell_{0,k}$. This means that the directions used for the approximation are not the same for all reachable sets Ω_i. There are two reasons that makes this point potentially problematic.

The first reason is numerical. Let us fix $\ell_{0,k}$, then the directions used for the approximation of Ω_i are $\ell_{i,k} = ((\Phi_\tau{}^\top)^{-1})^i\ell_{0,k}$. For simplicity, we assume that the eigenvalue of $(\Phi_\tau{}^\top)^{-1}$ with largest modulus is real and denote ℓ^* the associated eigenvector. Then all the vectors $\ell_{i,1}, \dots, \ell_{i,r}$ tend to point towards the direction of ℓ^* when i grows. This means that the polyhedral over-approximation $\overline{\Omega}_i$ is likely to be ill-conditioned for large values of i[2].

[2] One could argue that the second proposed improvement in section 4.2 suffers from the same problem. Indeed, given a direction ℓ, the vectors $(\Phi_\tau{}^\top)^p\ell$ will point towards the same direction as p grows. For that reason the indices $p_1, \dots, p_r$ must be chosen carefully, using for instance a criterium based on angles as suggested in the previous section.

The second reason is more practical. Sometimes, for instance for visualization, we are not interested in approximating the reachable sets but rather the projection of the reachable sets on an output subspace. Let us consider, for instance the single output system:

$$\begin{cases} \dot{x}(t) &= Ax(t) + v(t),\ v(t) \in V,\ x(0) \in X_0 \\ y(t) &= cx(t) \end{cases}$$

where $c^\top \in \mathbb{R}^d$. Then, in order to compute an over-approximation of the interval reachable by $y(t)$ it is sufficient to run Algorithm 1 with $\ell = c^\top$. Similarly, when dealing with hybrid systems with switching conditions given by hyperplanes, it is interesting to choose the directions of approximation given by the normal vectors to the hyperplanes.

The main advantage of the algorithm presented in [9] over algorithm 1 is that it can be extended very easily to time-varying linear systems. Algorithm 1 does not extend to this class of systems as proposition 4 holds only for time-invariant linear systems.

5. Numerical Experiments

In this section, we show the effectiveness of our approach on some examples. Algorithm 1 has been implemented in OCaml, without any of the improvements proposed in section 4.2. All computations were performed on a Pentium IV 3.2GHz with 1GB RAM.

5.1. RLC Model of a Transmission Line

The first example we consider is a verification problem for a transmission line borrowed from [17]. The goal is to check that the transient behavior of a long transmission line is acceptable both in terms of overshoot and of response time. Figure 3 shows a model of the transmission line, which consists of a number of RLC components (R: resistor, L: inductor, C: capacitor) modelling segments of the line. The left side is the sending end and the right side is the receiving end of the transmission line.

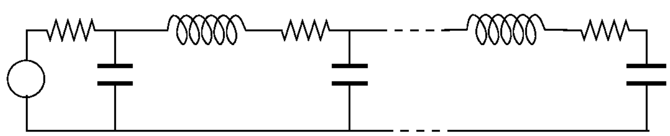

Figure 3: RLC model of a transmission line

The dynamics of the system are given by the single-input single-output linear dynamical system

$$\begin{cases} \dot{x}(t) &= Ax(t) + bu_{in}(t), \quad u_{in}(t) \in U,\ x(0) \in X_0, \\ u_{out}(t) &= cx(t) \end{cases}$$

where $x(t) \in \mathbb{R}^d$ with $d = 81$ is the state vector containing the voltage of the capacitors and the current of the inductors and $u_{in}(t) \in U \subseteq \mathbb{R}$ is the voltage at the sending end. The output of the system is the voltage $u_{out}(t) \in \mathbb{R}$ at the receiving end, since $u_{out}(t) = cx(t)$ we will take $\ell = c^\top$.

Initially, the system is supposed to be in an ε-neighborhood (with $\varepsilon = 0.01$) of the set of steady states for an input voltage inside $[-0.2; 0.2]$. Then, at time $t = 0$, the input voltage is switched to a value in $[0.99; 1.01]$:

$$X_0 = -A^{-1}b\,[-0.2; 0.2] \oplus \varepsilon\mathrm{B},\ U = [0.99; 1.01].$$

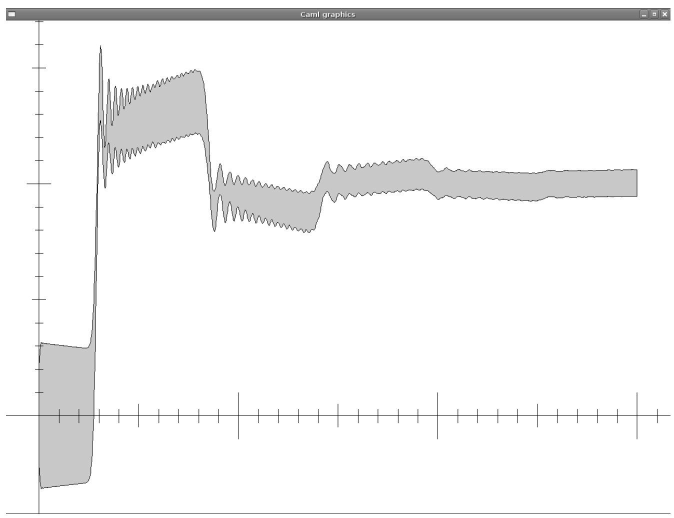

Figure 4: Reachable values by $u_{out}(t)$ against time t.

Figure 4 shows the reachable values of the output voltage for a time horizon of 3ns, it was computed in 0.10s using 0.234MB.

5.2. Extensive Experiments

Our implementation has also been tested on randomly generated examples of different dimension. Tables 1 and 2 summarize the results of our experimentations. We computed polyhedral over-approximations of the reachable sets $\mathcal{R}_{[0,\tau]}(X_0), \ldots, \mathcal{R}_{[(N-1)\tau, N\tau]}(X_0)$ with $N = 100$, for random matrices A of dimension d. We either used the algorithm from [11] (denoted as *direct* in the tables), or algorithm 1 (denoted as *sf* in the tables); initial and inputs set were given either as zonotopes of order 1 (Z) or ellipsoids (E); and the computed tight over-approximation consisted in the intersection of 1, d, or d^2 half-spaces. The program was terminated after 90s.

We can see that the new algorithm has great performances for systems with a single output: it can compute exact bounds on this output for the first 100 timesteps in less than a third of a second for a 500 dimensionnal system, while

$d =$	10	20	50	100	200	500
direct Z 1	< 0.01	0.01	0.13	1.00	5.44	85.9
sf Z 1	< 0.01	< 0.01	0.01	0.01	0.05	0.28
direct E 1	< 0.01	0.02	0.27	1.71	11.8	
sf E 1	< 0.01	< 0.01	< 0.01	0.02	0.05	0.31
direct Z d	< 0.01	0.02	0.27	1.86	11.4	
sf Z d	< 0.01	0.02	0.23	1.5	11.1	
direct E d	0.01	0.04	0.41	2.82	21.9	
sf E d	< 0.01	0.02	0.19	1.48	8.98	
direct Z d^2	0.04	0.35	7.38	90.6		
sf Z d^2	0.04	0.36	9.83			
direct E d^2	0.03	0.26	6.69			
sf E d^2	0.03	0.32	9.16			

Table 1: Execution time (in seconds) for $N = 100$ for several linear systems of different dimensions

$d =$	10	20	50	100	200	500
direct Z 1	0.234	0.234	0.234	0.703	2.258	13.43
sf Z 1	0.234	0.234	0.234	0.469	1.480	8.707
direct E 1	0.234	0.234	0.469	1.172	4.961	
sf E 1	0.234	0.234	0.234	0.703	2.332	12.53
direct Z d	0.234	0.234	0.469	1.172	3.195	
sf Z d	0.234	0.234	0.234	0.703	2.184	
direct E d	0.234	0.469	0.937	3.281	7.332	
sf E d	0.234	0.234	0.234	0.703	3.035	
direct Z d^2	0.703	2.812	18.28	77.81		
sf Z d^2	0.234	0.469	3.75			
direct E d^2	0.703	3.047	18.98			
sf E d^2	0.234	0.469	3.75			

Table 2: Memory consumption (in MB) for $N = 100$ for several linear systems of different dimensions

the fastest previously known algorithm, to the best of the authors knowledge, takes more than a minute. For a larger number of directions of tightness, Algorithm 1 compares well to one of the most competitive available algorithms. Its theoretical complexity (10), linear in the number of timesteps and the number of directions of tightness, is confirmed by table 1.

6. Conclusion

We have presented an approach for the computation of over-approximations of the reachable set of a continuous-time linear system with contrained initial states and inputs. We showed that it can handle arbitrary compact convex sets.

We actually compute two over-approximations of the reachable set, the first one given by the union of convex sets defined by their support functions, the second one given by the union of polyhedrons. The complexity is comparable to the complexity of the most competitive available specialized algorithms for reachability analysis of linear systems using zonotopes or ellipsoids. The effectiveness of our approach is demonstrated on several examples.

Future work should focus on the integration of our techniques in an algorithm for reachability analysis of hybrid systems.

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Appendix: Proofs of Section 3.1

Proof of Lemma 1

Proof. To prove the lemma, we need to introduce the following sets:

$$P = \left\{\lambda x + (1-\lambda)(e^{\tau A}x + \tau v) : \ \lambda \in [0,1], x \in X_0, v \in V\right\}$$

and $Q = \mathrm{CH}\left(X_0, e^{\tau A}X_0 \oplus \tau V\right)$. Let us remark that $P \subseteq Q$, which implies that

$$d_H(P,Q) = \sup_{z \in Q} \inf_{z' \in P} \|z - z'\|.$$

Let $z \in Q$, then there exists $x \in X_0$, $y \in X_0$, $v \in V$ and $\lambda \in [0,1]$ such that

$$z = \lambda x + (1-\lambda)(e^{\tau A}y + \tau v).$$

Since X_0 is convex, $\lambda x + (1-\lambda)y \in X_0$. Then, let us consider the point $z' \in P$ defined

$$z' = \lambda(\lambda x + (1-\lambda)y) + (1-\lambda)\left(e^{\tau A}(\lambda x + (1-\lambda)y) + \tau v\right).$$

Then, we have

$$z - z' = \lambda(1-\lambda)(e^{\tau A} - I)(y - x).$$

We showed that for all $z \in Q$, there exists $z' \in P$ such that

$$\|z - z'\| \le \frac{1}{4}(e^{\tau\|A\|} - 1)D_{X_0}.$$

It follows that

$$d_H(P,Q) \leq \frac{1}{4}(e^{\tau\|A\|}-1)D_{X_0}.$$

Now, let us consider $x(.)$ a trajectory of the system, then there exists an initial state $x_0 \in X_0$ and an input $v(.)$ such that for all t, $v(t) \in V$ and

$$\begin{aligned} x(t) &= e^{tA}x_0 + \int_0^t e^{(t-s)A}v(s)ds \\ &= e^{tA}x_0 + \int_0^t v(s)ds + \int_0^t (e^{(t-s)A}-I)v(s)ds \\ &= e^{tA}x_0 + tv^*(t) + \int_0^t (e^{(t-s)A}-I)v(s)ds \end{aligned}$$

where $v^*(t) = \frac{1}{t}\int_0^t v(s)ds$. Since V is a convex set, it follows that $v^*(t) \in V$. For $t \in [0,\tau]$, let us consider the point $z(t) \in P$ given by

$$z(t) = \frac{\tau-t}{\tau}x_0 + \frac{t}{\tau}(e^{\tau A}x_0 + \tau v^*(t)) = x_0 + \frac{t}{\tau}(e^{\tau A}-I)x_0 + tv^*(t).$$

Then,

$$x(t)-z(t) = e^{tA}x_0 - x_0 - \frac{t}{\tau}(e^{\tau A}-I)x_0 + \int_0^t (e^{(t-s)A}-I)v(s)ds.$$

Let us remark that

$$e^{tA}x_0 - x_0 - \frac{t}{\tau}(e^{\tau A}-I)x_0 = \frac{t}{\tau}\sum_{k=2}^{+\infty}\frac{\tau(t^{k-1}-\tau^{k-1})}{k!}A^k x_0$$

It follows that

$$\|e^{tA}x_0 - x_0 - \frac{t}{\tau}(e^{\tau A}-I)x_0\| \leq \frac{t}{\tau}(e^{\tau\|A\|}-1-\tau\|A\|)R_{X_0}.$$

We also have

$$\begin{aligned} \|\int_0^t (e^{(t-s)A}-I)v(s)ds\| &\leq \int_0^t \|e^{(t-s)A}-I\|\|v(s)\|ds \\ &\leq R_V\int_0^t (e^{(t-s)\|A\|}-1)ds \\ &\leq (e^{t\|A\|}-\|A\|t-1)\frac{R_V}{\|A\|} \\ &\leq \frac{t}{\tau}(e^{\tau\|A\|}-\|A\|\tau-1)\frac{R_V}{\|A\|} \end{aligned}$$

by convexity of $e^{t\|A\|}-\|A\|t-1$. Then, from the two previous inequalities we have

$$\|x(t)-z(t)\| \leq \frac{t}{\tau}\alpha_\tau.$$

It follows that there exists $y(t)$ in $\alpha_\tau \mathrm{B}$ such that:

$$x(t) = \frac{\tau - t}{\tau} x_0 + \frac{t}{\tau}(e^{\tau A} x_0 + \tau v^*(t) + y(t))$$

thus $\mathcal{R}_{[0,\tau]}(X_0) \subseteq \mathrm{CH}\left(X_0, e^{\tau A} X_0 \oplus \tau V \oplus \alpha_\tau \mathrm{B}\right) = \Omega_0$, which proves the first part of the lemma.

For the second part of the lemma, let us note that

$$\mathcal{R}_{[0,\tau]}(X_0) \subseteq \Omega_0 \subseteq Q \oplus \alpha_\tau \mathrm{B},$$

which implies that:

$$d_H(\Omega_0, \mathcal{R}_{[0,\tau]}(X_0)) \leq d_H(Q \oplus \alpha_\tau \mathrm{B}, \mathcal{R}_{[0,\tau]}(X_0)).$$

Moreover, since $\|x(t) - z(t)\| \leq \frac{t}{\tau}\alpha_\tau \leq \alpha_\tau$, we have $d_H\left(P, \mathcal{R}_{[0,\tau]}(X_0)\right) \leq \alpha_\tau$ and $d_H(P \oplus \alpha_\tau \mathrm{B}, \mathcal{R}_{[0,\tau]}(X_0)) \leq 2\alpha_\tau$. By the triangular inequality:

$$\begin{aligned} d_H(Q \oplus \alpha_\tau \mathrm{B}, \mathcal{R}_{[0,\tau]}(X_0)) &\leq d_H(Q \oplus \alpha_\tau \mathrm{B}, P \oplus \alpha_\tau \mathrm{B}) \\ &\quad + d_H(P \oplus \alpha_\tau \mathrm{B}, \mathcal{R}_{[0,\tau]}(X_0)) \\ &\leq d_H(Q, P) + 2\alpha_\tau \\ &\leq \frac{1}{4}(e^{\tau\|A\|} - 1) D_{X_0} + 2\alpha_\tau. \end{aligned}$$

Thus,

$$d_H(\Omega_0, \mathcal{R}_{[0,\tau]}(X_0)) \leq \frac{1}{4}(e^{\tau\|A\|} - 1) D_{X_0} + 2\alpha_\tau.$$

■

Proof of Lemma 2

Proof. Let us consider $x(.)$ a trajectory of the system, then there exists an initial state $x_0 \in \Omega$ and an input $v(.)$ such that for all t, $v(t) \in V$ and

$$\begin{aligned} x(\tau) &= e^{\tau A} x_0 + \int_0^\tau e^{(\tau - s)A} v(s) ds \\ &= e^{\tau A} x_0 + \int_0^\tau v(s) ds + \int_0^\tau (e^{(\tau - s)A} - I) v(s) ds \\ &= e^{\tau A} x_0 + \tau v^* + \int_0^\tau (e^{(\tau - s)A} - I) v(s) ds \end{aligned}$$

where $v^* = \frac{1}{\tau}\int_0^\tau v(s)ds$. Since V is a convex set, it follows that $v^* \in V$. Let us consider the point $z \in e^{\tau A}\Omega \oplus \tau V$ given by $z = e^{\tau A} x_0 + \tau v^*$. Then, similar to lemma 1 we can show that $\|x(\tau) - z\| \leq \beta_\tau$. It follows that

$$d_H\left(e^{\tau A}\Omega \oplus \tau V, \mathcal{R}_\tau(\Omega)\right) \leq \beta_\tau$$

which implies that $\mathcal{R}_\tau(\Omega) \subseteq \Omega'$ and $d_H(\Omega', \mathcal{R}_\tau(\Omega)) \leq 2\beta_\tau$. ■

Proof of Theorem 1

PROOF. From lemma 1, $\mathcal{R}_{[0,\tau]}(X_0) \subseteq \Omega_0$. Assume $\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0) \subseteq \Omega_i$, then

$$\mathcal{R}_{[(i+1)\tau,(i+2)\tau]}(X_0) = \mathcal{R}_\tau\left(\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0)\right) \subseteq \mathcal{R}_\tau\left(\Omega_i\right) \subseteq e^{\tau A}\Omega_i \oplus \tau V \oplus \beta_\tau \mathrm{B}.$$

Therefore, $\mathcal{R}_{[(i+1)\tau,(i+2)\tau]}(X_0) \subseteq \Omega_{i+1}$ and the first part of the Theorem holds. Let us note $\delta_i = d_H(\Omega_i, \mathcal{R}_{[i\tau,(i+1)\tau]}(X_0))$, then

$$\begin{aligned}\delta_{i+1} &= d_H(\Omega_{i+1}, \mathcal{R}_\tau\left(\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0)\right)) \\ &\leq d_H(\Omega_{i+1}, \mathcal{R}_\tau\left(\Omega_i\right)) + d_H(\mathcal{R}_\tau\left(\Omega_i\right), \mathcal{R}_\tau\left(\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0)\right))\end{aligned}$$

From lemma 2, $d_H(\Omega_{i+1}, \mathcal{R}_\tau\left(\Omega_i\right)) \leq 2\beta_\tau$ and it is easy to show that

$$d_H(\mathcal{R}_\tau\left(\Omega_i\right), \mathcal{R}_\tau\left(\mathcal{R}_{[i\tau,(i+1)\tau]}(X_0)\right)) \leq e^{\tau\|A\|} d_H(\Omega_i, \mathcal{R}_{[i\tau,(i+1)\tau]}(X_0)).$$

Thus, we have that $\delta_{i+1} \leq e^{\tau\|A\|}\delta_i + 2\beta_\tau$. Therefore, for all $i = 0, \ldots, N-1$

$$\delta_i \leq e^{i\tau\|A\|}\delta_0 + 2\beta_\tau \sum_{k=0}^{i-1} e^{k\tau\|A\|}.$$

Then, from lemma 1,

$$\begin{aligned}\delta_i &\leq e^{i\tau\|A\|}\left(\frac{e^{\tau\|A\|}-1}{4} D_{X_0} + 2\alpha_\tau\right) + 2\beta_\tau \sum_{k=0}^{i-1} e^{k\tau\|A\|} \\ &\leq e^{i\tau\|A\|}\left(\frac{e^{\tau\|A\|}-1}{4} D_{X_0} + 2(e^{\tau\|A\|} - 1 - \tau\|A\|)R_{X_0}\right) + \\ &\quad 2(e^{\tau\|A\|} - 1 - \tau\|A\|)\frac{R_V}{\|A\|} \sum_{k=0}^{i} e^{k\tau\|A\|} \\ &\leq e^{i\tau\|A\|}\left(\frac{\tau\|A\|e^{\tau\|A\|}}{4} D_{X_0} + \tau^2\|A\|^2 e^{\tau\|A\|} R_{X_0}\right) + \\ &\quad \tau^2\|A\|^2 e^{\tau\|A\|}\frac{R_V}{\|A\|}\frac{e^{(i+1)\tau\|A\|}-1}{e^{\tau\|A\|}-1} \\ &\leq \tau e^{(i+1)\tau\|A\|}\left(\frac{\|A\|}{4} D_{X_0} + \tau\|A\|^2 R_{X_0}\right) + \\ &\quad \tau^2\|A\|^2 e^{\tau\|A\|}\frac{R_V}{\|A\|}\frac{e^{(i+1)\tau\|A\|}}{\tau\|A\|} \\ &\leq \tau e^{(i+1)\tau\|A\|}\left(\frac{\|A\|}{4} D_{X_0} + \tau\|A\|^2 R_{X_0} + e^{\tau\|A\|} R_V\right).\end{aligned}$$

This leads to the estimate of the theorem since $(i+1)\tau \leq T$. ■