Towards a Multiresolution Approach to Linear Control

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***Abstract*—We develop a multiresolution approximation framework for linear control. We construct a multiresolution analysis of the set of input functions of a linear system. The approximation of an input u at a scale j is defined as the input u_j of minimal energy such that the trajectories of the system associated with u and u_j coincide on a grid of step length 2^{-j}. We propose a set of wavelet functions which generate this multiresolution analysis. These functions, called control theoretic wavelets, satisfy useful properties for the representation of control inputs of a linear system. As an example of application of our multiresolution approximation framework, we propose a method for efficient encoding of control inputs with regard to several criteria.**

***Index Terms*—Encoding, linear systems, multiresolution approximation, optimal control, wavelet transforms.**

I. Introduction

During the past decade, the relationship between splines and linear control theory has been analyzed in [4]–[6], [11], and [13], resulting in a new class of spline functions: the control theoretic splines. This class of functions has good properties to solve a wide range of continuous-time optimal control problems.

For instance, the optimal trajectory of the interpolation problem, which aims to drive the trajectory of a linear system through specific points at specific times; is a control theoretic spline [5], [13]. Furthermore, the solutions of the smoothing problem [11] and of an optimal control problem with discrete-time state–space constraints [4], are control theoretic splines. More generally, every continuous-time optimal control problem, which can be formulated using samples of the trajectory, can be solved within the control theoretic spline framework.

These problems can be viewed as discrete approximations of optimal control problems involving the value of the trajectory on the whole interval of time, which are, in general, much more difficult to solve. As the discretization step of the time becomes smaller, the solution of the discretized problem provides a finer approximation of the solution of the original one. More generally, any input function can be approximated with desired accuracy by the input associated with a control theoretic spline.

An effective approximation framework involving finer and finer grids of samples is given by the theory of multiresolution approximation [1], [8]–[10], [12]. The basic idea is the following. The approximation of a function f of $L^2(0,1)$ at a resolution 2^j (or at a scale j) is given by a grid of samples which provide local averages of f on intervals of length 2^{-j}. More formally, a multiresolution analysis is composed of a sequence $\{V_j\}$ of embedded subspaces of $L^2(0,1)$. The space V_j regroups all the possible approximations at the resolution 2^j. The approximation of the function f at the resolution 2^j is defined as the orthogonal projection on the space V_j.

In this paper, we develop a multiresolution analysis of the set of input functions of a linear system. The approximation of an input u at a scale j is defined as the input u_j of minimal energy such that the trajectories of the system associated with u and u_j coincide on a grid of step length 2^{-j}. Then, the set of control theoretic splines associated with the uniform grid of step length 2^{-j} is generated by the input functions in the set V_j. A set of wavelet functions generating this multiresolution analysis is derived. We call them control theoretic wavelets. These satisfy several properties which make them interesting basis functions for control inputs of a linear system. As an example of application of our multiresolution approximation framework, we propose a method based on control theoretic wavelets for efficient encoding of input functions with regard to several criteria.

Let us consider the linear control system given by a pair of matrices (A, B)

$$x'(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \quad u \in L^2(0,1)^{n_u} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, A and B are constant matrices of compatible dimensions. Equivalently

$$\forall t \in [0,1] \qquad x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds.$$

We assume without loss of generality that $x_0 = 0$. Let $\mathcal{F}_{(A,B)}$ be the linear map such that $x = \mathcal{F}_{(A,B)}(u)$ is the trajectory of (1) associated with the input u. We assume that $\mathrm{rank}(B) = n_u$ and that the pair (A, B) is controllable. Then, the t-reachability Gramian

$$M(t) = \int_0^t e^{A(t-s)}BB^T e^{A^T(t-s)}ds$$

is invertible for all $t > 0$.

II. Multiresolution Approximation for Control

In this section, we develop a multiresolution approximation framework for linear control. First, we introduce the notion of second generation multiresolution analysis as it was presented in [12]. Then, within this theoretical framework, we develop a

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multiresolution analysis suitable for the approximation of the inputs of system (1). We derive a set of wavelets for this multiresolution analysis that we call control theoretic wavelets. This set of wavelets generates a set of basis functions for the trajectories of system (1) which exhibit interesting properties.

A. Multiresolution Analysis and Wavelets

Definition 1: [12] An orthonormal multiresolution analysis of $L^2(0,1)^n$ is a sequence of closed subspaces $\{V_j \subseteq L^2(0,1)^n | j \in \mathbb{N}\}$ such that

- $V_j \subseteq V_{j+1}$;
- $\bigcup_{j\in\mathbb{N}} V_j$ is dense in $L^2(0,1)^n$;
- for each $j \in \mathbb{N}$, V_j has an orthonormal basis given by scaling functions $\{\varphi_{j,k} | k \in \mathcal{K}(j)\}$ where $\mathcal{K}(j)$ is a set of indexes.

Thus, a multiresolution analysis consists of a sequence of finer and finer approximation subspaces of $L^2(0,1)^n$. The approximation u_j of $u \in L^2(0,1)^n$ at a scale j is defined as the orthogonal projection of u on the subspace V_j. The notion of multiresolution analysis is closely related to the concept of wavelet functions. Wavelets are defined as basis functions of the orthogonal complement of V_j in V_{j+1}.

Definition 2: [12] A set of functions $\{\psi_{j,m} | j \in \mathbb{N}, m \in \mathcal{M}(j)\}$, where $\mathcal{M}(j) = \mathcal{K}(j+1) \setminus \mathcal{K}(j)$ is a set of orthonormal wavelet functions if the following hold.

- The space $W_j = \mathrm{span}(\{\psi_{j,m} | m \in \mathcal{M}(j)\})$ is the orthogonal complement of V_j in V_{j+1}.
- The set $\{\psi_{j,m} | j \in \mathbb{N}, m \in \mathcal{M}(j)\} \cup \{\varphi_{0,k} | k \in \mathcal{K}(0)\}$ is an orthonormal basis of $L^2(0,1)^n$.

Hence, any function $u \in L^2(0,1)^n$ can be written as an infinite linear combination of wavelet functions.

$$u = \sum_{k\in\mathcal{K}(0)} \alpha_{0,k}\varphi_{0,k} + \sum_{j\in\mathbb{N}} \sum_{m\in\mathcal{M}(j)} \beta_{j,m}\psi_{j,m}.$$

Then, its approximation at the scale J is given by the truncated sum

$$u_J = \sum_{k\in\mathcal{K}(0)} \alpha_{0,k}\varphi_{0,k} + \sum_{j=0}^{j=J-1} \sum_{m\in\mathcal{M}(j)} \beta_{j,m}\psi_{j,m}.$$

B. Control Theoretic Multiresolution Analysis

Let us construct a multiresolution analysis of $L^2(0,1)^{n_u}$ suitable for the approximation of the inputs of system (1). In our multiresolution approximation framework, the approximation u_j at scale j of an input u is defined as the solution of the following optimal control problem:

$$\begin{aligned} &\text{Minimize} \quad \int_0^1 v(s)^T v(s) ds \\ &\text{under} \quad x(k/2^j) = y(k/2^j), \qquad k \in \{1 \dots 2^j\} \\ &\text{where} \quad \begin{cases} x = \mathcal{F}_{(A,B)}(u) \\ y = \mathcal{F}_{(A,B)}(v). \end{cases} \end{aligned} \tag{2}$$

Hence, the approximation of a function u at scale j is defined as the input u_j of minimal energy such that the trajectory $\mathcal{F}_{(A,B)}(u_j)$ interpolates $\mathcal{F}_{(A,B)}(u)$ on the grid of [0, 1] of step size 2^{-j}. As shown in [13], u_j is the unique element of the set

$$V_j = \left\{ B^T e^{-A^T t} f_j(t) \,\middle|\, \begin{array}{l} f_j : [0,1] \to \mathbb{R}^{n_x}, \text{ constant} \\ \text{on each } [(k-1)/2^j, k/2^j) \end{array} \right\}$$

satisfying the interpolation constraints of problem (2).

Remark 1: The set of control theoretic splines associated with the uniform grid of [0, 1] of step size 2^{-j} is generated by the input functions of the set V_j.

Theorem 1: The sequence of subspaces $\{V_j | j \in \mathbb{N}\}$ is an orthonormal multiresolution analysis of $L^2(0,1)^{n_u}$.

Proof: Let us remark that the property $V_j \subseteq V_{j+1}$ is obvious. For each $j \in \mathbb{N}$, V_j is a finite-dimensional subspace (of dimension $2^j n_x$) of $L^2(0,1)^{n_u}$, therefore, it has an orthonormal basis. Let u be an element of $L^2(0,1)^{n_u}$. Since $\mathrm{rank}(B) = n_u$, there exists a function f in $L^2(0,1)^{n_x}$ such that

$$u(t) = B^T e^{-A^T t} f(t).$$

From [9], the set of piecewise constant functions over intervals of a dyadic subdivision of [0, 1] is dense in $L^2(0,1)^{n_x}$. Let $\varepsilon > 0$, there exist $j \in \mathbb{N}$ and f_j piecewise constant with respect to the intervals of the uniform subdivision of [0, 1] of step size 2^{-j}, such that

$$\int_0^1 \|f(s) - f_j(s)\|^2 \, ds \leq \varepsilon.$$

Let $u_j(t) = B^T e^{-A^T t} f_j(t)$, it is clear that it is an element of V_j. Moreover, it is easy to show that

$$\int_0^1 \|u(s) - u_j(s)\|^2 \, ds \leq \left(\|B^T\| e^{\|A^T\|} \right)^2 \varepsilon.$$

Hence, $\bigcup_{j\in\mathbb{N}} V_j$ is dense in $L^2(0,1)^{n_u}$. ■

C. Control Theoretic Wavelets

Let W_j be the orthogonal complement of V_j in V_{j+1}.

Lemma 1: W_j is the subspace of V_{j+1} consisting of the inputs u such that the associated trajectory $\mathcal{F}_{(A,B)}(u)$ vanishes at $\{k/2^j | k \in \{1 \dots 2^j\}\}$.

Proof: Let v be an element of V_j

$$v(t) = B^T e^{-A^T t} f_k, \qquad \text{if } t \in \left[(k-1)/2^j, k/2^j\right).$$

Let w be an element of V_{j+1}, such that $\mathcal{F}_{(A,B)}(w)$ vanishes at $\{k/2^j | k \in \{1 \dots 2^j\}\}$. Then, it is easy to show that

$$\forall k \in \{1 \dots 2^j\} \qquad \int_{(k-1)/2^j}^{k/2^j} e^{-As} B w(s) ds = 0.$$

Therefore

$$\int_0^1 v^T(s)w(s)ds = \sum_{k=1}^{k=2^j} f_k^T \int_{(k-1)/2^j}^{k/2^j} e^{-As}Bw(s)ds = 0.$$

Thus

$$\left\{ w \in V_{j+1} \middle| \begin{array}{l} \mathcal{F}_{(A,B)}(w) \text{ vanishes at} \\ \{k/2^j | k \in \{1 \dots 2^j\}\} \end{array} \right\} \subseteq W_j.$$

Moreover, it is easy to show that these subspaces have equal dimensions (i.e. $2^j n_x$), therefore, they are equal. ■

In the following theorem, we introduce a set of wavelet functions for our multiresolution analysis. In the next section, we will show that these functions have useful properties for the representation of the inputs of system (1).

Theorem 2: Let

$$\varphi^l(t) = B^T e^{A^T(1-t)} f^l, \qquad l \in \{1 \dots n_x\}$$

with

$$f^l = \frac{1}{\sqrt{\lambda_0^l}} e_0^l$$

where $\lambda_0^1 \geq \dots \geq \lambda_0^{n_x} > 0$ are the eigenvalues of the matrix $M(1)$, and $\{e_0^1, \dots, e_0^l\}$ is the orthonormal basis of $\mathbb{R}^{n_x}$ composed of associated eigenvectors

$$\psi_{j,m}^l(t) = \begin{cases} 0, & \text{on } \left[0, \frac{m}{2^j}\right) \\ B^T e^{A^T\left(\frac{2m+1}{2^{j+1}} - t\right)} g_j^l, & \text{on } \left[\frac{m}{2^j}, \frac{2m+1}{2^{j+1}}\right) \\ B^T e^{A^T\left(\frac{m+1}{2^j} - t\right)} h_j^l, & \text{on } \left[\frac{2m+1}{2^{j+1}}, \frac{m+1}{2^j}\right) \\ 0, & \text{on } \left[\frac{m+1}{2^j}, 1\right) \end{cases}$$
$$j \in \mathbb{N},\ m \in \{0 \dots 2^j - 1\},\ l \in \{1 \dots n_x\}$$

with

$$g_j^l = \frac{1}{\sqrt{\lambda_{j+1}^l}} M^{-1}\left(\frac{1}{2^{j+1}}\right) e_{j+1}^l$$
$$h_j^l = \frac{-1}{\sqrt{\lambda_{j+1}^l}} M^{-1}\left(\frac{1}{2^{j+1}}\right) e^{A/2^{j+1}} e_{j+1}^l$$

where $0 < \lambda_{j+1}^1 \leq \dots \leq \lambda_{j+1}^{n_x}$ are the eigenvalues of the matrix

$$e^{A^T/2^{j+1}} M^{-1}\left(\frac{1}{2^{j+1}}\right) e^{A/2^{j+1}} + M^{-1}\left(\frac{1}{2^{j+1}}\right)$$

and $\{e_{j+1}^1, \dots, e_{j+1}^l\}$ is the orthonormal basis of $\mathbb{R}^{n_x}$ composed of associated eigenvectors. Then, the set of functions $\{\psi_{j,m}^l | j \in \mathbb{N}, m \in \{0 \dots 2^j - 1\}, l \in \{1 \dots n_x\}\}$ is a set of orthonormal wavelet functions for the multiresolution analysis $\{V_j | j \in \mathbb{N}\}$. We call these functions control theoretic wavelets.

Proof: It is clear that for all $l, \varphi^l \in V_0$. Let l, l' be elements of $\{1 \dots n_x\}$

$$\int_0^1 \varphi^l(s)^T \varphi^{l'}(s) ds = f^{lT} M(1) f^{l'} = \frac{e_0^{lT}}{\sqrt{\lambda_0^l}} M(1) \frac{e_0^{l'}}{\sqrt{\lambda_0^{l'}}}$$
$$= \sqrt{\frac{\lambda_0^{l'}}{\lambda_0^l}} e_0^{lT} e_0^{l'} = \delta_{l,l'}.$$

$\{\varphi^l | l \in \{1 \dots n_x\}\}$ is an orthonormal family of V_0 whose dimension is n_x. Hence, it is an orthonormal basis. Let $j \in \mathbb{N}$, let $m \in \{0 \dots 2^j - 1\}$, $l \in \{1 \dots n_x\}$, it is clear that $\psi_{j,m}^l \in V_{j+1}$. Let us show that $\mathcal{F}_{(A,B)}(\psi_{j,m}^l)$ vanishes at $\{k/2^j | k \in \{1 \dots 2^j\}\}$. For $k \leq m$, $\psi_{j,m}^l = 0$ on the interval $[0, k/2^j)$, therefore, $\mathcal{F}_{(A,B)}(\psi_{j,m}^l) = 0$ on $[0, k/2^j]$. For $k = m+1$

$$\int_0^{(m+1)/2^j} e^{A\left(\frac{m+1}{2^j} - s\right)} B \psi_{j,m}^l(s) ds$$
$$= \int_{m/2^j}^{(2m+1)/2^{j+1}} e^{A\left(\frac{m+1}{2^j} - s\right)} BB^T e^{A^T\left(\frac{2m+1}{2^{j+1}} - s\right)} g_j^l ds$$
$$+ \int_{(2m+1)/2^{j+1}}^{(m+1)/2^j} e^{A\left(\frac{m+1}{2^j} - s\right)} BB^T e^{A^T\left(\frac{m+1}{2^j} - s\right)} h_j^l ds$$
$$= e^{A/2^{j+1}} M\left(\frac{1}{2^{j+1}}\right) g_j^l + M\left(\frac{1}{2^{j+1}}\right) h_j^l = 0.$$

For $k > m+1$, $\psi_{j,m}^l = 0$ on the interval $[(m+1)/2^j, k/2^j)$, therefore, $\mathcal{F}_{(A,B)}(\psi_{j,m}^l) = 0$ on $[(m+1)/2^j, k/2^j]$. Consequently, $\psi_{j,m}^l \in W_j$. Let $j \in \mathbb{N}$, let $m, m' \in \{0 \dots 2^j - 1\}$, $l, l' \in \{1 \dots n_x\}$. If $m \neq m'$, then $\psi_{j,m}^l$ and $\psi_{j,m'}^{l'}$ have disjoint supports; hence, they are orthogonal. If $m = m'$

$$\int_0^1 \psi_{j,m}^{lT}(s) \psi_{j,m}^{l'}(s) ds$$
$$= \int_{m/2^j}^{(2m+1)/2^{j+1}} g_j^{lT} e^{A\left(\frac{2m+1}{2^{j+1}} - s\right)} BB^T e^{A^T\left(\frac{2m+1}{2^{j+1}} - s\right)} g_j^{l'} ds$$
$$+ \int_{(2m+1)/2^{j+1}}^{(m+1)/2^j} h_j^{lT} e^{A\left(\frac{m+1}{2^j} - s\right)} BB^T e^{A^T\left(\frac{m+1}{2^j} - s\right)} h_j^{l'} ds$$
$$= g_j^{lT} M\left(\frac{1}{2^{j+1}}\right) g_j^{l'} + h_j^{lT} M\left(\frac{1}{2^{j+1}}\right) h_j^{l'}$$
$$= \frac{1}{\sqrt{\lambda_{j+1}^l}} e_{j+1}^{lT}$$
$$\times \left[M^{-1}\left(\frac{1}{2^{j+1}}\right) + e^{A^T/2^{j+1}} M^{-1}\left(\frac{1}{2^{j+1}}\right) e^{A/2^{j+1}} \right]$$
$$\times e_{j+1}^{l'} \frac{1}{\sqrt{\lambda_{j+1}^{l'}}}$$
$$= \sqrt{\frac{\lambda_{j+1}^{l'}}{\lambda_{j+1}^l}} e_{j+1}^{lT} e_{j+1}^{l'} = \delta_{l,l'}.$$

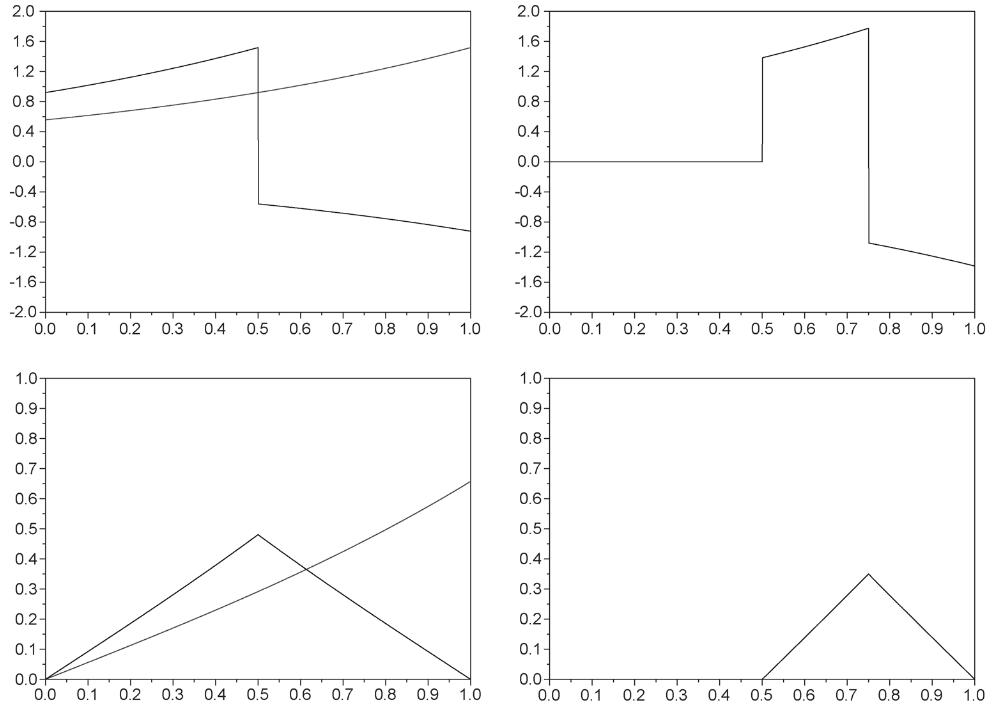

Fig. 1. Functions φ^1, $\psi^1_{0,0}$ (top left), $\psi^1_{1,1}$ (top right), and the associated trajectories χ^1, $\xi^1_{0,0}$ (bottom left), $\xi^1_{1,1}$ (bottom right) for a scalar system $(x' = -x+u)$.

Thus, $\{\psi^l_{j,m} | m \in \{0\dots 2^j-1\}, l \in \{1\dots n_x\}\}$ is an orthonormal family of W_j, whose dimension is $2^j n_x$. Therefore, it is an orthonormal basis. ■

Remark 2: For the scalar system $x' = u$, the set of wavelet functions given by Theorem 2 is composed of dilatations and translations of the well known Haar wavelet which generates the piecewise constant multiresolution analysis of $L^2(0,1)$.

D. Basis Functions for the Trajectories

The set of control theoretic wavelets generates a set of trajectories of system (1)

$$\begin{aligned} \chi^l &= \mathcal{F}_{(A,B)}(\varphi^l) \\ \xi^l_{j,m} &= \mathcal{F}_{(A,B)}\left(\psi^l_{j,m}\right) \\ &\quad j \in \mathbb{N}, \qquad m \in \{0\dots 2^j-1\}, \quad l \in \{1\dots n_x\}. \end{aligned}$$

On Fig. 1, some control theoretic wavelets for a scalar system $(x' = -x+u)$ and the associated trajectories are shown. Let u be an element of $L^2(0,1)^{n_u}$, from Theorem 2, there exist coefficients $\{\alpha^l | l \in \{1\dots n_x\}\}$, $\{\beta^l_{j,m} | j \in \mathbb{N}, m \in \{0\dots 2^j - 1\}, l \in \{1\dots n_x\}\}$ such that

$$u = \sum_{l=1}^{l=n_x} \alpha^l \varphi^l + \sum_{j=0}^{j=\infty} \sum_{m=0}^{m=2^j-1} \sum_{l=1}^{l=n_x} \beta^l_{j,m} \psi^l_{j,m}. \qquad (3)$$

Proposition 1: Let $x = \mathcal{F}_{(A,B)}(u)$. For all $t \in [0,1]$

$$x(t) = \sum_{l=1}^{l=n_x} \alpha^l \chi^l(t) + \sum_{j=0}^{j=\infty} \sum_{m=0}^{m=2^j-1} \sum_{l=1}^{l=n_x} \beta^l_{j,m} \xi^l_{j,m}(t). \qquad (4)$$

Proof: Let $J \in \mathbb{N}$, we define

$$u_J = \sum_{l=1}^{l=n_x} \alpha^l \varphi^l + \sum_{j=0}^{j=J-1} \sum_{m=0}^{m=2^j-1} \sum_{l=1}^{l=n_x} \beta^l_{j,m} \psi^l_{j,m}.$$

From Theorem 2, we have that

$$\lim_{J\to\infty} \int_0^1 \|u(s) - u_J(s)\|^2 \, ds = 0.$$

Let $x_J = \mathcal{F}_{(A,B)}(u_J)$, then for all $t \in [0,1]$

$$x_J(t) = \sum_{l=1}^{l=n_x} \alpha^l \chi^l(t) + \sum_{j=0}^{j=J-1} \sum_{m=0}^{m=2^j-1} \sum_{l=1}^{l=n_x} \beta^l_{j,m} \xi^l_{j,m}(t).$$

Let $c \in \mathbb{R}^{n_x}$ such that $\|c\| = 1$. For all $t \in [0,1]$,

$$\begin{aligned} \left|c^T\left(x(t) - x_J(t)\right)\right|^2 &= \left| \int_0^t c^T e^{A(t-s)} B\left(u(s) - u_J(s)\right) ds \right|^2 \\ &\le c^T M(t) c \int_0^1 \|u(s) - u_J(s)\|^2 \, ds \\ &\le c^T M(1) c \int_0^1 \|u(s) - u_J(s)\|^2 \, ds \\ &\le \lambda^1_0 \int_0^1 \|u(s) - u_J(s)\|^2 \, ds. \end{aligned}$$

Since the inequality holds for all unitary $c \in \mathbb{R}^{n_x}$, then

$$\lim_{J \to \infty} \|x(t) - x_J(t)\| = 0.$$

■

We now show that the control theoretic wavelets satisfy properties that make them interesting basis functions for control inputs of system (1). Let $u \in L^2(0,1)^{n_u}$ and $x = \mathcal{F}_{(A,B)}(u)$, then u and x can be written as in (3) and (4).

Let $J \in \mathbb{N}$, for all $j \geq J$, $m \in \{0 \dots 2^j - 1\}$, $l \in \{0 \dots n_x\}$, we have by construction that $\xi^l_{j,m}$ vanishes at $\{k/2^J | k \in \{1 \dots 2^J\}\}$. Thus, the value of a trajectory $x = \mathcal{F}_{(A,B)}(u)$ at these instants is uniquely determined by the coefficients $\{\alpha^l | l \in \{1 \dots n_x\}\}$ and $\{\beta^l_{j,m} | j \in \{0 \dots J-1\}, m \in \{0 \dots 2^j - 1\} l \in \{1 \dots n_x\}\}$ of the decomposition of u in the control theoretic wavelet basis. These coefficients can be computed by solving a set of linear equations involving the value of the trajectory x at $\{k/2^J | k \in \{1 \dots 2^J\}\}$.

Moreover, let us remark that for all $j \geq J, m \in \{0 \dots 2^j - 1\}$, $l \in \{0 \dots n_x\}$, the support of the function $\xi^l_{j,m}$ is included in the open interval $(m/2^j, (m+1)/2^j)$. Particularly, this means that the set of linear equations that need to be solved to determine the coefficients of the decomposition of u is sparse. Thus, it follows that the computation of the decomposition of u in the control theoretic wavelet basis can be done in a very efficient way.

Moreover, the information on the input u is represented in an intrinsically hierarchical and localized manner when u is decomposed in the control theoretic wavelet basis. Indeed, at lower scales, the coefficients of the decomposition of the input u capture the global behavior of the trajectory x. At higher scales, the coefficients capture the details of the trajectory. This is done in a localized way.

In the next section, these properties are used to encode efficiently the control input of a linear system.

III. Efficient Encoding of Control Inputs

Efficient encoding of control inputs is a problem of great importance in several control problems. For instance, for distant control, involving wireless communications, the delay due to the time of transmission is proportional to the number of bytes needed to encode the control input. Thus, for efficiency, the encoding of the control inputs should contain the maximum of information on the minimum of bytes. In the signal processing area, the properties of wavelets have been used successfully for image compression [9]. In this section, we extend some of these techniques to control inputs for linear systems. Let us consider an input $u \in L^2(0,1)^{n_u}$

$$u = \sum_{l=1}^{l=n_x} \alpha^l \varphi^l + \sum_{j=0}^{j=\infty} \sum_{m=0}^{m=2^j-1} \sum_{l=1}^{l=n_x} \beta^l_{j,m} \psi^l_{j,m}.$$

Let $J \in \mathbb{N}$, $N = 2^J n_x$, the principle of compression by wavelets is to choose the N coefficients which contains the more information, among those of the decomposition of u in the wavelet basis (see, e.g., [9]). The compressed control input can be written as

$$u_J = \sum_{l=1}^{l=n_x} \bar{\alpha}^l \varphi^l + \sum_{j=0}^{j=\infty} \sum_{m=0}^{m=2^j-1} \sum_{l=1}^{l=n_x} \bar{\beta}^l_{j,m} \psi^l_{j,m}$$

where $\bar{\alpha}^l$ is either α^l or 0, $\bar{\beta}^l_{j,m}$ is either $\beta^l_{j,m}$ or 0 and such that

$$\#\left\{\bar{\alpha}^l \neq 0, \bar{\beta}^l_{j,m} \neq 0\right\} \leq N.$$

Several strategies can be used. In the following, we analyze three of them. First, let us prove the following result.

Lemma 2: For all $j \in \mathbb{N}$, $m \in \{0 \dots 2^j - 1\}$, $l \in \{0 \dots n_x\}$

$$\left\|\xi^l_{j,m}\right\|_\infty \leq \sqrt{\|BB^T\|} e^{\|A\|} 2^{-j/2}.$$

Proof: For $t \notin [m/2^j, (m+1)/2^j]$, $\xi^l_{j,m}(t) = 0$. Let $t \in [m/2^j, (m+1)/2^j]$, $c \in \mathbb{R}^{n_x}$, and $\|c\| = 1$

$$\begin{aligned}
\left|c^T \xi^l_{j,m}(t)\right|^2 &= \left|c^T \int_0^t e^{A(t-s)} B \psi^l_{j,m}(s) ds\right|^2 \\
&\leq \left\|\psi^l_{j,m}\right\|_2^2 c^T \int_{m/2^j}^t e^{A(t-s)} BB^T e^{A^T(t-s)} ds c \\
&\leq \int_{m/2^j}^{(m+1)/2^j} e^{2\|A\|\left((m+1)/2^j - s\right)} \|BB^T\| ds \\
&\leq \frac{\|BB^T\|}{2\|A\|} \left(e^{2\|A\|/2^j} - 1\right) \\
&\leq \|BB^T\| e^{2\|A\|} 2^{-j}.
\end{aligned}$$

Since the inequality holds for all $t \in [m/2^j, (m+1)/2^j]$ and for all $c \in \mathbb{R}^{n_x}$ such that $\|c\| = 1$, then $\|\xi^l_{j,m}\|_\infty \leq \sqrt{\|BB^T\|} e^{\|A\|} 2^{-j/2}$. ■

Two criteria will be used to evaluate the quality of the compression of the control input. The first one is the L^2-norm of the error of approximation of the input:

$$\|u - u_J\|_2^2 = \sum_{l=1}^{l=n_x} (\alpha^l - \bar{\alpha}^l)^2 + \sum_{j=0}^{j=\infty} \sum_{m=0}^{m=2^j-1} \sum_{l=1}^{l=n_x} \left(\beta^l_{j,m} - \bar{\beta}^l_{j,m}\right)^2. \quad (5)$$

The second one is the L^∞-norm of the error of approximation of the trajectory

$$\|x - x_J\|_\infty \leq \sum_{l=1}^{l=n_x} |\alpha^l - \bar{\alpha}^l| \|\chi^l\|_\infty + \sum_{l=1}^{l=n_x} \sum_{j=0}^{j=\infty} \left\| \sum_{m=0}^{m=2^j-1} \left(\beta^l_{j,m} - \bar{\beta}^l_{j,m}\right) \xi^l_{j,m} \right\|_\infty.$$

For given l and j, the functions $\xi^l_{j,m}$ have disjoint supports. Thus

$$\left\| \sum_{m=0}^{m=2^j-1} \left(\beta^l_{j,m} - \bar{\beta}^l_{j,m}\right) \xi^l_{j,m} \right\|_\infty = \max_{m=0}^{m=2^j-1} \left|\beta^l_{j,m} - \bar{\beta}^l_{j,m}\right| \left\|\xi^l_{j,m}\right\|_\infty \le \max_{m=0}^{m=2^j-1} \left|\beta^l_{j,m} - \bar{\beta}^l_{j,m}\right| 2^{-j/2}\sqrt{\|BB^T\|} e^{\|A\|}.$$

Therefore

$$\|x - x_J\|_\infty \le \sqrt{\|BB^T\|} e^{\|A\|} \sum_{l=1}^{l=n_x} |\alpha^l - \bar{\alpha}^l| + \sqrt{\|BB^T\|} e^{\|A\|} \sum_{l=1}^{l=n_x} \sum_{j=0}^{j=\infty} \max_{m=0}^{m=2^j-1} \left|\beta^l_{j,m} - \bar{\beta}^l_{j,m}\right| 2^{-j/2}. \quad (6)$$

It is clear that from a control viewpoint, the second performance criterion is more important than the first one. On that point, our problem differs from traditional signal compression problems where the main objective is to minimize the L^2-norm of the error of approximation of a signal.

In the following, we evaluate these errors for three different compression strategies. First, we have to quantify the decrease of the coefficients of the decomposition of the input u in the control theoretic wavelet basis.

A. Decrease of the Coefficients

Let us examine how the coefficients $\beta^l_{j,m}$ decrease under some assumptions on the input u.

Lemma 3: Let u be bounded on [0, 1], for all $j \in \mathbb{N}$, $m \in \{0 \dots 2^j - 1\}$, $l \in \{1 \dots n_x\}$

$$\left|\beta^l_{j,m}\right| \le \|u\|_\infty 2^{-j/2}.$$

Proof: Let $j \in \mathbb{N}$, $m \in \{0 \dots 2^j - 1\}$, $l \in \{1 \dots n_x\}$

$$\left|\beta^l_{j,m}\right|^2 = \left| \int_0^1 u^T(s)\psi^l_{j,m}(s) ds \right|^2 \le \int_{m/2^j}^{(m+1)/2^j} \|u(s)\|^2 ds \int_{m/2^j}^{(m+1)/2^j} \left\|\psi^l_{j,m}(s)\right\|^2 ds \le \int_{m/2^j}^{(m+1)/2^j} \|u\|^2_\infty ds = \|u\|^2_\infty 2^{-j}.$$

■

Lemma 4: Let u be bounded and L-Lipschitz, for all $j \in \mathbb{N}$, $m \in \{0 \dots 2^j - 1\}$, $l \in \{1 \dots n_x\}$

$$\left|\beta^l_{j,m}\right| \le k 2^{-3j/2}$$

where

$$k = L + \|B^T\| \|B\| \left\|\left(B^T B\right)^{-1}\right\| \|A^T\| e^{\|A^T\|} \|u\|_\infty.$$

Proof: Let $j \in \mathbb{N}$, $m \in \{0 \dots 2^j - 1\}$, $l \in \{1 \dots n_x\}$. From Theorem 2, we have

$$\forall f \in \mathbb{R}^{n_x} \qquad \int_0^1 f^T e^{-As} B \psi^l_{j,m}(s) ds = 0.$$

Then, for all $f \in \mathbb{R}^{n_x}$

$$\left|\beta^l_{j,m}\right| = \left| \int_0^1 u^T(s)\psi^l_{j,m}(s) ds \right| = \left| \int_0^1 \left(B^T e^{-A^T s} f - u(s)\right)^T \psi^l_{j,m}(s) ds \right| \le 2^{-j/2} \sup_{s \in \left[\frac{m}{2^j}, \frac{m+1}{2^j}\right]} \left\|B^T e^{-A^T s} f - u(s)\right\|.$$

Since $\mathrm{rank}(B) = n_u$, the matrix $B^T B$ is invertible. Let $f = e^{A^T m/2^j} B \left(B^T B\right)^{-1} u(m/2^j)$, then

$$\left\|B^T e^{-A^T s} f - u(s)\right\| \le \left\|B^T e^{-A^T s} f - B^T e^{-A^T m/2^j} f\right\| + \left\|u(s) - u(m/2^j)\right\|.$$

Since u is L-Lipschitz, we have

$$\left\|u(s) - u(m/2^j)\right\| \le L|s - m/2^j| \le L 2^{-j}.$$

Furthermore

$$\left\|B^T e^{-A^T s} f - B^T e^{-A^T m/2^j} f\right\| \le \|B^T\| \left\|e^{A^T(m/2^j - s)} - I\right\| \left\|e^{-A^T m/2^j} f\right\| \le \|B^T\| \|A^T\| e^{\|A^T\|} 2^{-j} \|B\| \left\|\left(B^T B\right)^{-1}\right\| \|u\|_\infty$$

■

The assumption that the control input u is bounded is reasonable for practical applications. However, the coefficients of the decomposition in the control theoretic wavelet basis of a function which is bounded but not Lipschitz decrease very slowly compared to those of a bounded and Lipschitz function. Unfortunately, the assumption that control inputs are bounded and Lipschitz may be too strong for many interesting applications. From our perspective, it is reasonable to consider control inputs that are bounded, piecewise continuous with a finite number of discontinuities and Lipschitz on each interval where they are continuous.

Let u be such an input, we note $\{t_1 \dots t_d\}$ the times at which u is not continuous. At each scale $j \in \mathbb{N}$, we define the following subsets of $\{0 \dots 2^j - 1\}$:

$$\mathcal{B}_j = \left\{m | \exists t_i \in \left[m/2^j, (m+1)/2^j\right)\right\}$$
$$\mathcal{C}_j = \{0 \dots 2^j - 1\} \setminus \mathcal{B}_j.$$

At each scale, the elements of $\mathcal{B}_j$ are the indices of the intervals containing the instant at which u is discontinuous. Note that the set $\mathcal{B}_j$ has at most d elements.

Proposition 2: For all $j \in \mathbb{N}$, $m \in \{0 \dots 2^j - 1\}$, $l \in \{1 \dots n_x\}$

$$\left|\beta_{j,m}^l\right| \leq \begin{cases} \|u\|_\infty 2^{-j/2}, & \text{if } m \in \mathcal{B}_j \\ k2^{-3j/2}, & \text{if } m \in \mathcal{C}_j. \end{cases}$$

Proof: Obvious and relies on Lemma 3 and 4. ■

In the following, we use this result to analyze the performance of some compression strategies for control inputs.

B. Uniform Interpolation of the Trajectory

The first strategy we consider is the most classical one. It consists in choosing the N first coefficients of the decomposition. Thus, the compressed input u_J is defined by the coefficients

$$\begin{aligned} \bar{\alpha}_l &= \alpha_l \\ \bar{\beta}_{j,m}^l &= \begin{cases} \beta_{j,m}^l, & j \leq J-1 \\ 0, & j \geq J. \end{cases} \end{aligned}$$

In that case, the trajectory x_J associated with the compressed input u_J interpolates the original trajectory x at $\{q/2^J | q \in \{1 \dots 2^J\}\}$.

Proposition 3: Let u_J and x_J be the input and trajectory obtained by uniform interpolation of the trajectory x, then

$$\begin{aligned} \|u - u_J\|_2 &= O(2^{-J/2}) \\ \|x - x_J\|_\infty &= O(2^{-J}). \end{aligned}$$

Proof: Equation (5) becomes

$$\|u_J - u\|_2^2 = \sum_{l=1}^{l=n_x} \sum_{j=J}^{j=\infty} \sum_{m=0}^{m=2^j-1} \beta_{j,m}^{l2}.$$

For any $j \in \mathbb{N}$, there exist at most d elements in $\mathcal{B}_j$ and 2^j elements in $\mathcal{C}_j$. Hence

$$\begin{aligned} \|u_J - u\|_2^2 &\leq \sum_{l=1}^{l=n_x} \sum_{j=J}^{j=\infty} \left(d\|u\|_\infty^2 2^{-j} + 2^j k^2 2^{-3j}\right) \\ &\leq n_x 2^{-J} \left(2d\|u\|_\infty^2 + \frac{4}{3} k^2 2^{-J}\right). \end{aligned}$$

Equation (6) becomes

$$\|x - x_J\|_\infty \leq \sqrt{\|BB^T\| e^{\|A\|}} \sum_{l=1}^{l=n_x} \sum_{j=J}^{j=\infty} \max_{m=0}^{m=2^j-1} \left|\beta_{j,m}^l\right| 2^{-j/2}.$$

Let us assume that J is large enough so that $k2^{-3J/2}$ is smaller than $\|u\|_\infty 2^{-J/2}$. Therefore

$$\begin{aligned} \|x - x_J\|_\infty &\leq \sqrt{\|BB^T\| e^{\|A\|}} \sum_{l=1}^{l=n_x} \sum_{j=J}^{j=\infty} \|u\|_\infty 2^{-j} \\ &\leq 2^{-J+1} n_x \|u\|_\infty \sqrt{\|BB^T\| e^{\|A\|}}. \end{aligned}$$

■

C. Best L^2-Approximation of the Input

We analyze a second strategy which consists in choosing the N largest coefficients (in absolute value) of the decomposition of u in the wavelet basis. Thus, the compressed input u_J is the best L^2-approximation of u in the wavelet basis with N nonzero coefficients. Let θ_J be the absolute value of the N^{th} largest coefficient (in absolute value). Therefore

$$\begin{aligned} \bar{\alpha}_l &= \begin{cases} \alpha_l, & \text{if } |\alpha_l| \geq \theta_J \\ 0, & \text{if } |\alpha_l| < \theta_J \end{cases} \\ \bar{\beta}_{j,m}^l &= \begin{cases} \beta_{j,m}^l, & \text{if } \left|\beta_{j,m}^l\right| \geq \theta_J \\ 0, & \text{if } \left|\beta_{j,m}^l\right| < \theta_J. \end{cases} \end{aligned}$$

Lemma 5: For J sufficiently large, θ_J is smaller than $\max(k, \|u\|_\infty) 2^{-3(J-1)/2}$.

Proof: From Proposition 2, for all $j \geq J-1$, $m \in \mathcal{C}_j$, $l \in \{1 \dots n_x\}$

$$\left|\beta_{j,m}^l\right| \leq k2^{-3(J-1)/2}.$$

and, for all $j \geq 3(J-1)$, $m \in \mathcal{B}_j$, $l \in \{1 \dots n_x\}$

$$\left|\beta_{j,m}^l\right| \leq \|u\|_\infty 2^{-3(J-1)/2}.$$

Hence, there are at most $n_x(2^{J-1} + 2d(J-1))$ coefficients with absolute value greater than $\max(k, \|u\|_\infty) 2^{-3(J-1)/2}$. For J sufficiently large, $N = n_x 2^J$ is greater than $n_x(2^{J-1} + 2d(J-1))$. This allows to conclude. ■

Proposition 4: Let u_J and x_J be the input and trajectory obtained by the best L^2-approximation of the input u, then

$$\begin{aligned} \|u - u_J\|_2 &= O(2^{-J}) \\ \|x - x_J\|_\infty &= O(2^{-3J/2}). \end{aligned}$$

Proof: Clearly, for all $l \in \{1 \dots n_x\}$, $|\bar{\alpha}^l - \alpha^l| \leq \theta_J$. Similarly, for all $j \in \mathbb{N}$, $m \in \{1 \dots 2^j - 1\}$, $l \in \{1 \dots n_x\}$, $|\bar{\beta}_{j,m}^l - \beta_{j,m}^l|$ is upper-bounded by θ_J and $|\beta_{j,m}^l|$. Equation (5) becomes

$$\begin{aligned} &\|u - u_J\|_2^2 \\ &\leq n_x \left[\theta_J^2 + \sum_{j=0}^{j=J-2} \sum_{m=0}^{m=2^j-1} \theta_J^2 \right. \\ &\quad + \sum_{j=J-1}^{j=2J-3} \left(\sum_{m \in \mathcal{B}_j} \theta_J^2 + \sum_{m \in \mathcal{C}_j} k^2 2^{-3j}\right) \\ &\quad \left. + \sum_{j=2J-2}^{j=\infty} \left(\sum_{m \in \mathcal{B}_j} \|u\|_\infty^2 2^{-j} + \sum_{m \in \mathcal{C}_j} k^2 2^{-3j}\right)\right] \end{aligned}$$

which leads to

$$\begin{aligned} \|u - u_J\|_2^2 \leq\ & n_x \theta_J^2 \left(2^{J-1} + d(J-1)\right) \\ & + n_x 2^{-2(J-1)} \left(4k^2/3 + 2d\|u\|_\infty^2\right). \end{aligned}$$

Using the upper bound of θ_J given by Lemma 5, the latest inequality leads to $\|u - u_J\|_2 = O(2^{-J})$.

We now consider the L_∞-norm of the error of approximation of the trajectory; from (6)

$$\|x - x_J\|_\infty \leq \sqrt{\|BB^T\| e^{\|A\|}} n_x \theta_J \left(1 + \sum_{j=0}^{j=\infty} 2^{-j/2}\right).$$

Hence, $\|x - x_J\|_\infty = O(\theta_J) = O(2^{-3J/2})$. ■

D. Best L^∞-Approximation of the Trajectory

While the second strategy tried to minimize the error of approximation of the input, the third strategy aims to minimize the error of approximation of the trajectory. According to Lemma 2, for $j \in \mathbb{N}$, $m \in \{1 \dots 2^j - 1\}$, $l \in \{1 \dots n_x\}$, the contribution of the coefficient $\beta^l_{j,m}$ to the trajectory is bounded by

$$\left|\beta^l_{j,m}\right| 2^{-j/2} \sqrt{\|BB^T\| e^{\|A\|}}.$$

The third strategy consists in keeping the largest contributions to the trajectory. Let θ_J be the value of the N^{th} largest element of the set of normalized coefficients $\{|\alpha^l|, |\beta^l_{j,m}| 2^{-j/2} | j \in \mathbb{N}, m \in \{1 \dots 2^j - 1\}, l \in \{1 \dots n_x\}\}$, the compressed input u_J is defined by the coefficients

$$\bar{\alpha}_l = \begin{cases} \alpha_l & \text{if } |\alpha_l| \geq \theta_J \\ 0 & \text{if } |\alpha_l| < \theta_J \end{cases}$$

$$\bar{\beta}^l_{j,m} = \begin{cases} \beta^l_{j,m} & \text{if } \left|\beta^l_{j,m}\right| 2^{-j/2} \geq \theta_J \\ 0 & \text{if } \left|\beta^l_{j,m}\right| 2^{-j/2} < \theta_J. \end{cases}$$

Lemma 6: For J sufficiently large, θ_J is smaller than $\max(k, \|u\|_\infty) 2^{-2(J-1)}$.

Proof: From Proposition 2, for all $j \geq J - 1$, $m \in \mathcal{C}_j$, $l \in \{1 \dots n_x\}$

$$\left|\beta^l_{j,m}\right| 2^{-j/2} \leq k 2^{-2(J-1)}$$

and, for all $j \geq 2(J-1)$, $m \in \mathcal{B}_j$, $l \in \{1 \dots n_x\}$

$$\left|\beta^l_{j,m}\right| 2^{-j/2} \leq \|u\|_\infty 2^{-2(J-1)}.$$

Hence, there are at most $n_x(2^{J-1} + d(J-1))$ normalized coefficients strictly greater than $\max(k, \|u\|_\infty) 2^{-2(J-1)}$. For J large enough, $N = 2^J n_x$ is greater than $n_x(2^{J-1} + d(J-1))$. This leads to the conclusion. ■

Proposition 5: Let u_J and x_J be the input and trajectory obtained by the best L^∞-approximation of the trajectory x, then

$$\|u - u_J\|_2 = O(2^{-J})$$
$$\|x - x_J\|_\infty = O(J 2^{-2J}).$$

TABLE I
L^2-Norm of the Approximation Error of the Input for the Different Strategies and Several Compression Rates

	Strategy 1	Strategy 2	Strategy 3
$J=4$	1.9578358	1.0680063	1.2174699
$J=5$	1.2462848	0.4734766	0.5814066
$J=6$	0.8116987	0.2277683	0.2336332
$J=7$	0.5343714	0.1101325	0.1120497
$J=8$	0.3430447	0.0512162	0.0521266

TABLE II
L^∞-Norm of the Error of Approximation of the Trajectory for the Different Strategies and Several Compression Rates

	Strategy 1	Strategy 2	Strategy 3
$J=4$	0.1702431	0.2197090	0.1115303
$J=5$	0.0842527	0.1091285	0.0257247
$J=6$	0.0405219	0.0864505	0.0067362
$J=7$	0.0201837	0.0025790	0.0014312
$J=8$	0.0080961	0.0006317	0.0003677

Proof: For all $l \in \{1 \dots n_x\}$, $|\bar{\alpha}^l - \alpha^l| \leq \theta_J$. For all $j \in \mathbb{N}$, $m \in \{1 \dots 2^j - 1\}$, $l \in \{1 \dots n_x\}$, $|\bar{\beta}^l_{j,m} - \beta^l_{j,m}|$ is bounded by $\theta_J 2^{j/2}$ and $|\beta^l_{j,m}|$. Equation (5) becomes

$$\begin{aligned} &\|u_J - u\|_2^2 \\ &\quad \leq n_x \Bigg[\theta_J^2 + \sum_{j=0}^{j=J-2} \sum_{m=0}^{2^j-1} \theta_J^2 2^j \\ &\qquad + \sum_{j=J-1}^{j=2J-3} \left(\sum_{m \in \mathcal{B}_j} \theta_J^2 2^j + \sum_{m \in \mathcal{C}_j} k^2 2^{-3j} \right) \\ &\qquad + \sum_{j=2J-2}^{j=\infty} \left(\sum_{m \in \mathcal{B}_j} \|u\|_\infty^2 2^{-j} + \sum_{m \in \mathcal{C}_j} k^2 2^{-3j} \right) \Bigg] \end{aligned}$$

which leads to

$$\begin{aligned} \|u - u_J\|_2^2 \leq n_x \theta_J^2 &\left(\frac{2^{2(J-1)} + 2}{3} + d 2^{J-1}(2^{J-1} - 1) \right) \\ &+ n_x 2^{-2(J-1)} \left(4k^2/3 + 2d\|u\|_\infty^2 \right). \end{aligned}$$

Since $\theta_J = O(2^{-2J})$, $\|u_J - u\|_2 = O(2^{-J})$.

We now consider the error of approximation of the trajectory. We assume that J is large enough so that $k 2^{-2J} \leq \|u\|_\infty 2^{-J}$. Equation (6) becomes

$$\begin{aligned} \|x - x_J\|_\infty &\leq \sqrt{\|BB^T\| e^{\|A\|}} n_x \left(\theta_J + \sum_{j=0}^{j=2J-3} \theta_J \right) \\ &\quad + \sqrt{\|BB^T\| e^{\|A\|}} n_x \sum_{j=2J-2}^{j=\infty} \|u\|_\infty 2^{-j}. \\ &\leq \sqrt{\|BB^T\| e^{\|A\|}} n_x \theta_J (2J-1) \\ &\quad + \sqrt{\|BB^T\| e^{\|A\|}} n_x 2 \|u\|_\infty 2^{-2(J-1)}. \end{aligned}$$

Using the bound of θ_J given by Lemma 6, this inequality leads to the expected result. ■

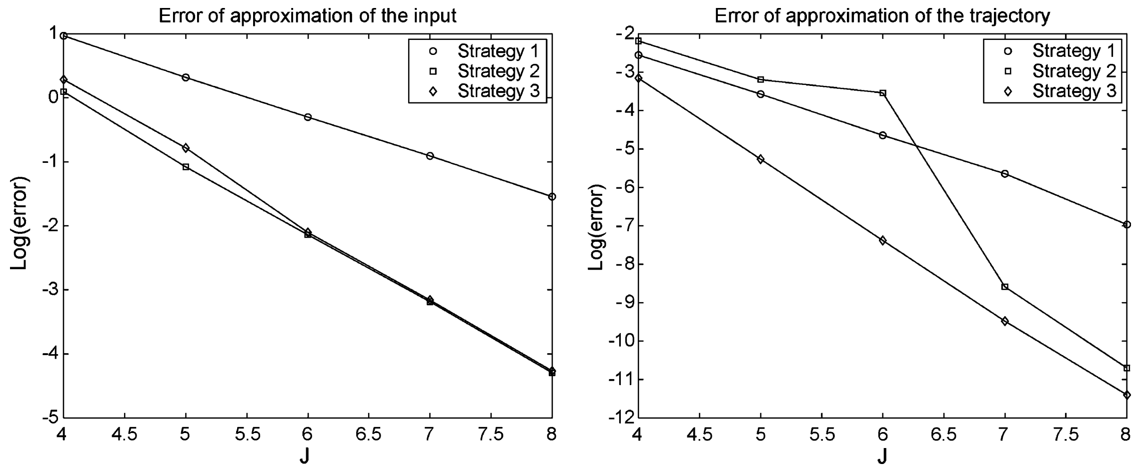

Fig. 2. Approximation errors against the compression parameter J.

E. Numerical Experiments

In this part, we check the validity of the theoretical results from the previous part. We consider the following control system:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 3\pi & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \quad (7)$$

The desired trajectory is

$$x(t) = \begin{cases} \sin(3\pi t), & \text{if } t \in [0, 2/3] \\ -1 + e^{2-3t}, & \text{if } t \in [2/3, 1] \end{cases}$$

$$y(t) = \begin{cases} e^{-t} \left(\cos(3\pi t) - 1 \right), & \text{if } t \in [0, 2/3] \\ 0, & \text{if } t \in [2/3, 1] \end{cases}$$

The corresponding input $(u, v)^T$ is computed in the control theoretic wavelet basis by solving a sparse system of linear equations. We computed the 2048 first coefficients given by the value of the trajectory on the grid of step size 1/1024 of the interval [0, 1].

Then, we applied the three compression strategies presented in the previous section. The estimations of the different approximation errors are presented in Tables I and II. Fig. 2 shows the graphs of the approximation errors against the compression parameter J.

The L^2-approximation errors of the input are presented in Table I. The experimental results confirm approximatively the theoretical ones. Indeed, the error seems to evolve as $2^{-0.6J}$ with the first strategy (truncation of the sum) as it evolves roughly as 2^{-J} for the other ones. We can see, that the quality of approximation is quite the same for the second and the third strategies though it is always better with the second strategy (thresholding the coefficients).

In Table II, the L^∞-approximation errors of the trajectory are presented. We can see that they agree with theoretical estimations for the first and the third strategies. The interpretation of the results obtained with the second strategy is much harder since the error of approximation does not decrease in a *regular* way. This may be evidence that the second strategy is not adapted for approximating the trajectory.

In Fig. 3, we represented one of the original inputs and the associated trajectory on the top of the figure. Below, we plotted one of the compressed inputs and the associated trajectory obtained by the third compression strategy (thresholding the normalized coefficients). In that case, the input is encoded using 64 nonzero coefficients. We can see that the trajectory associated with the compressed input restores accurately the original trajectory. Particularly, we can see that irregularities of the trajectories are conserved.

IV. Conclusion

In this paper, we developed a new approximation framework for linear control. It results in a multiresolution analysis of the space of input functions of a linear control system. We computed an associated set of orthonormal wavelet functions which we call control theoretic wavelets. The main advantage over classical wavelet bases is that the hierarchical structure is given in the space of trajectories of the system while the inner product is consistent with the space of input functions.

We presented an application of the multiresolution approximation framework for linear control. Given a desired trajectory, we synthesized an input which can be coded on a given number of bytes and such that the associated trajectory remains close to the desired one. We analyzed three different approaches to this problem. These methods of compression have applications in distant control problems where the time of transmission of input signals has a critical role. The methods of compression using wavelet bases can generally be adapted to denoising problems [9]. In [6], a denoising problem has been handled using the framework of control theoretic splines. It is likely that control theoretic wavelets will be useful to propose an alternative approach to this problem. This will be part of future research.

We think that the hierarchical and localized properties of the control theoretic wavelets can be used advantageously to solve a wide range of control synthesis problems. In [7], for instance, we proposed a method for computing an approximation of the solution of an optimal control problem for linear systems subject to continuous-time state constraints. On theoretical side, it

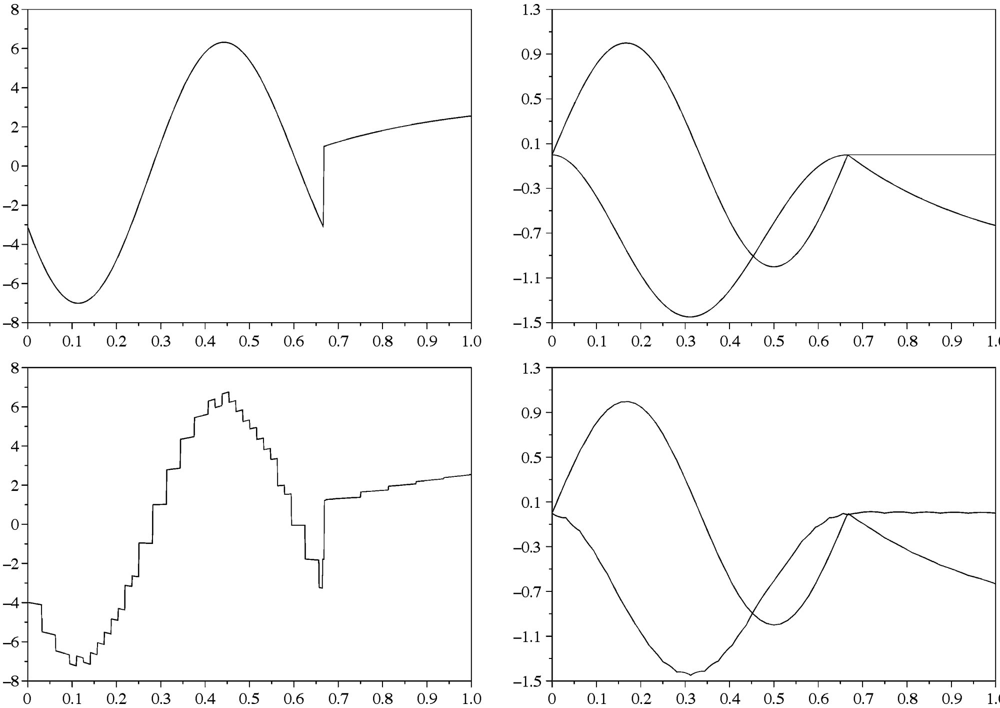

Fig. 3. Second component of the input v (top left), and the components x and y of associated trajectory (top right). The second component of the compressed input v_5 (bottom left), and the components x_5 and y_5 of associated trajectory (bottom right) for the third compression strategy.

would be interesting to analyze the link between control theoretic wavelets and spline wavelets as these seems to be control theoretic wavelets associated with specific matrices A and B.

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