

# Approximately Bisimilar Finite Abstractions of Stable Linear Systems<sup>\*</sup>

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**Abstract.** The use of bisimilar finite abstractions of continuous and hybrid systems, greatly simplifies complex computational tasks such as verification or control synthesis. Unfortunately, because of the strong requirements of bisimulation relations, such abstractions exist only for quite restrictive classes of systems. Recently, the notion of approximate bisimulation relations has been introduced, allowing the definition of less rigid relationships between systems. This relaxed notion should certainly allow us to build approximately bisimilar finite abstractions for more general classes of continuous and hybrid systems. In this paper, we show that for the class of stable discrete-time linear systems with constrained inputs, there exists an approximately bisimilar finite state system of any desired precision. We describe an effective procedure for the construction of this abstraction, based on compositional reasoning and samples of the set of initial states and inputs. Finally, we briefly show how our finite abstractions can be used for verification or control synthesis.

## 1 Introduction

Equivalence relationships for systems such as *bisimulation relations* [1, 2] have been very useful to reduce the complexity of computational tasks such as verification or control synthesis for finite state systems. Early research on hybrid systems has focused on the characterization of continuous and hybrid dynamics with *bisimilar* finite abstractions. The first positive results on timed automata [3] were later extended to multirate hybrid automata [4] and hybrid systems with linear dynamics with a particular eigenstructure [5] (see [6] for a survey). More recently, the existence of bisimilar finite state systems has been shown for controllable discrete-time linear systems with unconstrained inputs [7]. The existence of such abstractions provides decidability results as well as computational procedures for verification or control synthesis for these classes of continuous and hybrid systems. Unfortunately, the class of hybrid dynamics admitting bisimilar finite abstractions is quite restrictive since even for very simple systems (*i.e.* three dimensional piecewise constant differential equations [8]), reachability verification is known to be undecidable.

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Recently, the notion of *approximate bisimulation relations* has been introduced in [9], allowing the definition of less rigid relationships between systems. As *exact* bisimulation relations require the observations of two systems to be (and remain) identical, approximate bisimulation relations allow the observations to be different provided the distance between them is (and remains) bounded by some parameter called *precision*. This relaxed assumption should certainly allow us to build approximately bisimilar finite abstractions for more general classes of continuous and hybrid systems.

In this paper, we show that for the class of stable discrete-time linear systems with constrained inputs, there exists an approximately bisimilar finite state system of any desired precision. We show that the linear system can be seen as the composition of two other linear systems, one autonomous and one with inputs but with an initial state set to zero. For each of these systems, we define an approximately bisimilar finite abstraction using a two step procedure. Firstly, by sampling the set of initial states or the set of inputs, we define discrete but infinite abstractions of the linear systems. Secondly, we show that this infinite state systems are approximately bisimilar to a finite state system. The composition of these systems provides us with the finite abstraction of original linear system. Our approach provides an effective way to compute this discrete abstraction. Finally, we briefly show how our finite abstractions can be used for verification or control synthesis.

Let us remark that the idea of sampling the sets of initial states and inputs to compute discrete abstractions for verification has already been proposed [10, 11], though without further reduction to a finite state system. Contrary to these approaches, the abstractions we build are valid for an infinite time-horizon. In [12], a similar technique is proposed to build finite abstractions for stabilizable linear systems, however only a one-sided approximation result is provided making these abstractions suitable for control synthesis but not for verification.

## 2 Approximately bisimilar transition systems

The notion of approximate bisimulation relation has been introduced in [9], in the framework of transition systems.

### 2.1 Transition systems

Essentially, a transition system can be seen as an automaton, possibly with an infinite number of nodes and edges.

**Definition 1.** A *transition system (with observations)* is a tuple  $T = (Q, \rightarrow, Q^0, \Pi, h)$  that consists of:

- a (possibly infinite) set  $Q$  of states,
- a transition relation  $\rightarrow \subseteq Q \times Q$ ,
- a (possibly infinite) set  $Q^0 \subseteq Q$  of initial states,
- a (possibly infinite) set  $\Pi$  of observations,
- an observation map  $h : Q \rightarrow \Pi$ .

If the set of states has a finite number of elements, we say that the transition system is finite. The transition  $(q, q') \in \rightarrow$  is denoted  $q \rightarrow q'$ . A state trajectory of  $T$  is a finite sequence of transitions,  $q^0 \rightarrow \dots \rightarrow q^N$ , where  $q^0 \in Q^0$ . Note that a transition system is possibly non-deterministic: for a given initial state there may be several state trajectories. An external or observed trajectory of  $T$  is a finite sequence of observations,  $\pi^0 \dots \pi^N$  such that there exists a state trajectory of  $T$  which satisfies  $h(q^i) = \pi^i$ , for all  $i \in \{0, \dots, N\}$ . The language of  $T$  is denoted by  $L(T)$  and consists of all external trajectories of  $T$ . The reachable set of  $T$  is the subset of  $\Pi$  defined by:

$$\text{Reach}(T) = \{\pi \in \Pi \mid \exists \pi^0 \dots \pi^N \in L(T), i \in \{0, \dots, N\}, \text{ such that } \pi^i = \pi\}.$$

In this paper, the transition systems we consider are observed over  $\Pi = \mathbb{R}^p$ .

We define a composition operator for transition systems that will be useful in the development of the paper.

**Definition 2.** Let  $T_1 = (Q_1, \rightarrow_1, Q_1^0, \Pi, h_1)$  and  $T_2 = (Q_2, \rightarrow_2, Q_2^0, \Pi, h_2)$  be transition systems. The composition of  $T_1$  and  $T_2$ , denoted by  $T_1 \parallel T_2$ , is the transition system  $T = (Q, \rightarrow, Q^0, \Pi, h)$  defined by:

- the set of states  $Q = Q_1 \times Q_2$ ,
- the transition relation  $\rightarrow$  given by,

$$(q_1, q_2) \rightarrow (q'_1, q'_2) \text{ if } q_1 \rightarrow_1 q'_1 \text{ and } q_2 \rightarrow_2 q'_2,$$

- the set of initial states  $Q^0 = Q_1^0 \times Q_2^0$ ,
- the set of observations  $\Pi = \mathbb{R}^p$ ,
- the observation map  $h(q_1, q_2) = h_1(q_1) + h_2(q_2)$ .

Let us remark that the composition of two transition systems observed on  $\mathbb{R}^p$  is also observed on  $\mathbb{R}^p$ . We can now introduce the notion of approximate bisimulation relation for transition systems as presented in [9].

## 2.2 Approximate bisimulation relations

The notion of exact bisimulation relation allows to characterize the observational equivalence of two transition systems [1, 2]. The notion of approximate bisimulation relation is obtained from the exact one by relaxing the observational equivalence constraint. Instead of requiring that the observations of two systems are and remain the same we require that the distance between them is and remains bounded by some parameter called precision.

**Definition 3.** Let  $T_1 = (Q_1, \rightarrow_1, Q_1^0, \Pi, h_1)$  and  $T_2 = (Q_2, \rightarrow_2, Q_2^0, \Pi, h_2)$  be transition systems. A relation  $R \subseteq Q_1 \times Q_2$  is a  $\delta$ -approximate bisimulation relation between  $T_1$  and  $T_2$ , if for all  $(q_1, q_2) \in R$ :

1.  $\|h_1(q_1) - h_2(q_2)\| \leq \delta$ ,
2. for all  $q_1 \rightarrow_1 q'_1$ , there exists  $q_2 \rightarrow_2 q'_2$ , such that  $(q'_1, q'_2) \in R$ ,
3. for all  $q_2 \rightarrow_2 q'_2$ , there exists  $q_1 \rightarrow_1 q'_1$ , such that  $(q'_1, q'_2) \in R$ .

**Definition 4.**  $T_1$  and  $T_2$  are said to be approximately bisimilar with precision  $\delta$  (denoted  $T_1 \sim_\delta T_2$ ), if there exists  $R$ , a  $\delta$ -approximate bisimulation relation between  $T_1$  and  $T_2$  such that:

1. for all  $q_1 \in Q_1^0$ , there exists  $q_2 \in Q_2^0$ , such that  $(q_1, q_2) \in R$ ,
2. for all  $q_2 \in Q_2^0$ , there exists  $q_1 \in Q_1^0$ , such that  $(q_1, q_2) \in R$ .

*Remark 1.* For  $\delta = 0$ , we recover the usual notion of exact bisimulation relation and of *exactly* bisimilar transition systems. Thus,  $T_1 \sim_0 T_2$  will be denoted  $T_1 \sim T_2$ .

The following proposition states fundamental properties of approximate bisimulation relations. The proof is omitted here but can be found in [9].

**Proposition 1.** Let  $T_1, T_2$  and  $T_3$  be transition systems, then:

1. for all  $\delta \geq 0$ ,  $T_1 \sim_\delta T_1$ ,
2. for all  $\delta \geq 0$ ,  $T_1 \sim_\delta T_2 \iff T_2 \sim_\delta T_1$ ,
3. for all  $\delta \geq 0$ ,  $\delta' \geq 0$ ,  $T_1 \sim_\delta T_2$  and  $T_2 \sim_{\delta'} T_3 \implies T_1 \sim_{\delta+\delta'} T_3$ .

*Remark 2.* Contrarily to the relation  $\sim$ , for  $\delta > 0$ , the relation  $\sim_\delta$  is not an equivalence relation on the set of transition systems. However, the relation defined by  $T_1 \equiv T_2 \iff \exists \delta \geq 0, T_1 \sim_\delta T_2$  is an equivalence relation.

The following proposition shows that approximate bisimulation relations allow compositional reasoning.

**Proposition 2.** Let  $T_1, T_2, S_1$  and  $S_2$  be transition systems, then:

$$T_1 \sim_{\delta_1} S_1 \text{ and } T_2 \sim_{\delta_2} S_2 \implies T_1 || T_2 \sim_{\delta_1 + \delta_2} S_1 || S_2.$$

*Proof.* Let  $T_1 = (Q_1, \rightarrow_1, Q_1^0, \Pi, h_1)$ ,  $T_2 = (Q_2, \rightarrow_2, Q_2^0, \Pi, h_2)$ ,  $S_1 = (P_1, \rightsquigarrow_1, P_1^0, \Pi, g_1)$  and  $S_2 = (P_2, \rightsquigarrow_2, P_2^0, \Pi, g_2)$ .  $T_1 \sim_{\delta_1} S_1$  and  $T_2 \sim_{\delta_2} S_2$ , let  $R_1$  and  $R_2$  be the associated approximate bisimulation relations. Let  $T_1 || T_2 = (Q, \rightarrow, Q^0, \Pi, h)$  and  $S_1 || S_2 = (P, \rightsquigarrow, P^0, \Pi, g)$ . Let us define the following relation  $R \subseteq Q \times P$ :

$$R = \{(q_1, q_2, p_1, p_2) \in Q \times P \mid (q_1, p_1) \in R_1 \text{ and } (q_2, p_2) \in R_2\}.$$

Let  $(q_1, q_2, p_1, p_2) \in R$ ,

$$\begin{aligned} \|h(q_1, q_2) - g(p_1, p_2)\| &= \|h_1(q_1) + h_2(q_2) - g_1(p_1) - g_2(p_2)\| \\ &\leq \|h_1(q_1) - g_1(p_1)\| + \|h_2(q_2) - g_2(p_2)\| \leq \delta_1 + \delta_2. \end{aligned}$$

Let  $(q_1, q_2) \rightarrow (q'_1, q'_2)$ , then  $q_1 \rightarrow_1 q'_1$  and  $q_2 \rightarrow_2 q'_2$ . Since  $(q_1, p_1) \in R_1$  and  $(q_2, p_2) \in R_2$ , there exist  $p_1 \rightsquigarrow_1 p'_1$  and  $p_2 \rightsquigarrow_2 p'_2$  such that  $(q'_1, p'_1) \in R_1$  and  $(q'_2, p'_2) \in R_2$ . Then,  $(p_1, p_2) \rightsquigarrow (p'_1, p'_2)$  and  $(q'_1, q'_2, p'_1, p'_2) \in R$ . Similarly, we can show that for all  $(p_1, p_2) \rightsquigarrow (p'_1, p'_2)$  there exists  $(q_1, q_2) \rightarrow (q'_1, q'_2)$  such that  $(q'_1, q'_2, p'_1, p'_2) \in R$ . Hence,  $R$  is a  $\delta_1 + \delta_2$ -approximate bisimulation relation between  $T_1 || T_2$  and  $S_1 || S_2$ . Let  $(q_1, q_2) \in Q^0$ , then  $q_1 \in Q_1^0$  and  $q_2 \in Q_2^0$ . There exist  $p_1 \in P_1^0$  and  $p_2 \in P_2^0$  such that  $(q_1, p_1) \in R_1$  and  $(q_2, p_2) \in R_2$ . Then,  $(p_1, p_2) \in P^0$  and  $(q_1, q_2, p_1, p_2) \in R$ . Similarly, we can show that for all  $(p_1, p_2) \in P^0$  there exists  $(q_1, q_2) \in Q^0$  such that  $(q_1, q_2, p_1, p_2) \in R$ . Therefore, we can conclude that  $T_1 || T_2 \sim_{\delta_1 + \delta_2} S_1 || S_2$ .  $\blacksquare$

### 2.3 Approximation results

The precision of an approximate bisimulation relations allows to quantify how well two systems approximate each other. The following result shows that the distance between the external trajectories of approximately bisimilar systems ( $T_1 \sim_\delta T_2$ ) is bounded by the precision  $\delta$ .

**Theorem 1.** *Let  $T_1$  and  $T_2$  be transition systems such that  $T_1 \sim_\delta T_2$ , then for all  $\pi_1^0 \dots \pi_1^N \in L(T_1)$ , there exists  $\pi_2^0 \dots \pi_2^N \in L(T_2)$ , such that*

$$\text{for all } i \in \{0, \dots, N\}, \|\pi_1^i - \pi_2^i\| \leq \delta$$

and conversely.

*Proof.*  $T_1 \sim_\delta T_2$ , let  $R$  be the associated  $\delta$ -approximate bisimulation relation between  $T_1$  and  $T_2$ . Let  $\pi_1^0 \dots \pi_1^N \in L(T_1)$ , let  $q_1^0 \rightarrow_1 \dots \rightarrow_1 q_1^N$  be the associated state trajectory of  $T_1$ . Since  $q_1^0 \in Q_1^0$ , there exists  $q_2^0 \in Q_2^0$  such that  $(q_1^0, q_2^0) \in R$ . Using the second property of Definition 3, we can show by induction that there exists  $q_2^0 \rightarrow_2 \dots \rightarrow_2 q_2^N$  a state trajectory of  $T_2$  such that for all  $i \in \{0, \dots, N\}$ ,  $(q_1^i, q_2^i) \in R$ . Then, for all  $i \in \{0, \dots, N\}$ ,  $\|\pi_1^i - \pi_2^i\| = \|h_1(q_1^i) - h_2(q_2^i)\| \leq \delta$ . ■

The previous result extends naturally to reachable sets.

**Corollary 1.** *Let  $T_1$  and  $T_2$  be transition systems such that  $T_1 \sim_\delta T_2$ , then*

$$d_H(\text{Reach}(T_1), \text{Reach}(T_2)) \leq \delta$$

where  $d_H$  is the Hausdorff distance<sup>1</sup>.

*Proof.* Let  $\pi_1 \in \text{Reach}(T_1)$ , there exists  $\pi_1^0 \dots \pi_1^N \in L(T_1)$  and  $j \in \{0, \dots, N\}$ , such that  $\pi_1^j = \pi_1$ . From Theorem 1, there exists  $\pi_2^0 \dots \pi_2^N \in L(T_2)$  such that for all  $i \in \{0, \dots, N\}$ ,  $\|\pi_1^i - \pi_2^i\| \leq \delta$ . Particularly,  $\pi_2 = \pi_2^j \in \text{Reach}(T_2)$  and  $\|\pi_2 - \pi_1\| \leq \delta$ . Similarly, we can show that for all  $\pi_2 \in \text{Reach}(T_2)$ , there exists  $\pi_1 \in \text{Reach}(T_1)$ , such that  $\|\pi_2 - \pi_1\| \leq \delta$ . ■

## 3 Finite abstractions of stable linear systems

Let us consider the following discrete-time linear system:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k, & x_0 \in I, u_k \in U \\ y_k = Cx_k \end{cases} \quad (1)$$

where  $A$  is a  $n \times n$  matrix,  $B$  is a  $n \times m$  matrix and  $C$  is a  $p \times n$  matrix. The set of initial states  $I$  is a compact subset of  $\mathbb{R}^n$  and the set of inputs  $U$  is a compact subset of  $\mathbb{R}^m$  containing 0. We assume that the system is asymptotically stable (*i.e.* all the eigenvalues of  $A$  are strictly inside the unit circle in the complex plane).

<sup>1</sup> The Hausdorff distance between two subsets  $A, B \subseteq \mathbb{R}^p$  is defined by  $d_H(A, B) = \max(\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|)$ .

**Proposition 3.** *There exists  $\lambda \in (0, 1)$  and a positive semi-definite symmetric  $n \times n$  matrix  $M$  such that the following linear matrix inequalities hold:*

$$C^T C \leq M \quad (2)$$

$$A^T M A \leq \lambda^2 M \quad (3)$$

*Proof.* Let us search  $M$  under the form  $M = N + C^T C$  where  $N$  is a positive semi-definite symmetric matrix. Then, equation (3) is equivalent to

$$\frac{1}{\lambda^2} A^T N A - N \leq C^T C - \frac{1}{\lambda^2} A^T C^T C A = Q.$$

$Q$  is symmetric and thus can be written as  $Q = Q^+ - Q^-$  where  $Q^+$  and  $Q^-$  are positive semi-definite symmetric matrices. Since all the eigenvalues of  $A$  are strictly inside the unit circle, for  $\lambda$  sufficiently close to 1, all the eigenvalues of  $A/\lambda$  are also strictly inside the unit circle. Then, the discrete-time Lyapunov equation  $A^T N A/\lambda^2 - N = -Q^-$  has a unique solution which is positive semi-definite symmetric. Then, it is easy to see that  $M = N + C^T C$  satisfies equation (2) and (3). ■

We denote by  $\|\cdot\|_M$  the norm on  $\mathbb{R}^n$  associated to  $M$ :  $\|x\|_M = \sqrt{x^T M x}$ . We define the radii of the sets of initial states and inputs for this norm.

$$r_I = \max_{x \in I} \|x\|_M \text{ and } r_U = \max_{u \in U} \|Bu\|_M.$$

The discrete-time linear system (1) can be seen as a transition system  $\Sigma = (\mathbb{R}^n, \rightarrow, I, \mathbb{R}^p, h)$  where the transition relation  $\rightarrow$  is given by

$$x \rightarrow x' \iff \exists u \in U, x' = Ax + Bu$$

and the observation map by  $h(x) = Cx$ . The dynamics of system (1) can be split into an autonomous dynamics and a controlled dynamics with the initial state set to zero. Thus, we define the following transition systems  $\Sigma_1$  and  $\Sigma_2$ .  $\Sigma_1 = (\mathbb{R}^n, \rightarrow_1, I, \mathbb{R}^p, h)$  where the transition relation is given by

$$x \rightarrow_1 x' \iff x' = Ax.$$

Note that the transition system  $\Sigma_1$ , which captures the autonomous dynamics, is deterministic.  $\Sigma_2 = (\mathbb{R}^n, \rightarrow_2, \{0\}, \mathbb{R}^p, h)$ , where the transition relation  $\rightarrow_2 = \rightarrow$ , holds for the controlled dynamics with zero initial state.

**Proposition 4.**  *$\Sigma$  and  $\Sigma_1 || \Sigma_2$  are exactly bisimilar.*

*Proof.* Let us denote  $\Sigma_1 || \Sigma_2 = (\mathbb{R}^{2n}, \rightarrow, I \times \{0\}, \mathbb{R}^p, g)$ . We define the following relation  $R \subseteq \mathbb{R}^n \times \mathbb{R}^{2n}$ :

$$R = \{(x, x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^{2n} \mid x = x_1 + x_2\}.$$

Let  $(x, x_1, x_2) \in R$ , then

$$\|h(x) - g(x_1, x_2)\| = \|h(x) - h(x_1) - h(x_2)\| = \|C(x - x_1 - x_2)\| = 0.$$

Let  $x \rightarrow x'$ , there exists  $u \in U$  such that  $x' = Ax + Bu$ . Then, we have  $x_2 \rightarrow_2 x'_2$  where  $x'_2 = Ax_2 + Bu$  and  $x_1 \rightarrow_1 x'_1$  where  $x'_1 = Ax_1$ . Thus,  $(x_1, x_2) \rightarrow (x'_1, x'_2)$  and

$$x' = Ax + Bu = A(x_1 + x_2) + Bu = Ax_1 + Ax_2 + Bu = x'_1 + x'_2.$$

Then,  $(x', x'_1, x'_2) \in R$ . Similarly, it is easy to see that for all  $(x_1, x_2) \rightarrow (x'_1, x'_2)$  there exists  $x \rightarrow x'$  such that  $(x', x'_1, x'_2) \in R$ . Hence,  $R$  is an exact bisimulation relation between  $\Sigma$  and  $\Sigma_1 \parallel \Sigma_2$ . For all  $x \in I$ ,  $(x, 0) \in I \times \{0\}$  satisfies  $(x, x, 0) \in R$ . Therefore,  $\Sigma \sim \Sigma_1 \parallel \Sigma_2$ .  $\blacksquare$

*Remark 3.* Proposition 4 is the translation in the framework of transition systems of the fundamental superposition principle in linear systems theory.

Thus, the linear system  $\Sigma$  can be seen as the composition of the autonomous dynamics  $\Sigma_1$  and of the controlled dynamics  $\Sigma_2$ . In the following, we construct a finite state system  $S$  which is approximately bisimilar to  $\Sigma$ . The approach is the following. First, we construct finite state systems  $S_1$  and  $S_2$  that are approximately bisimilar respectively to  $\Sigma_1$  and  $\Sigma_2$ . Then, from Proposition 2, we can obtain the abstraction  $S$  from the composition of  $S_1$  and  $S_2$ .

### 3.1 Abstraction of the autonomous dynamics

The construction of a finite state system that is approximately bisimilar to  $\Sigma_1$  is processed in two steps. First, by using a sample of the set of initial states  $I$ , we compute a system with a discrete but infinite set of states. Then, a finite abstraction is derived from this system by remarking that all its external trajectories converge to 0.

**Lemma 1.** *Let  $\varepsilon_1 > 0$ , there exists a finite set  $I_\sigma = \{x_0, \dots, x_{J_1}\} \subseteq I$  such that*

$$\forall x \in I, \exists x_j \in I_\sigma, \|x - x_j\|_M \leq \varepsilon_1.$$

*Proof.* Let  $\alpha_M$  be the largest eigenvalue of the matrix  $M$ , then for all  $(x, x') \in \mathbb{R}^n \times \mathbb{R}^n$ ,  $\|x - x'\|_M \leq \alpha_M \|x - x'\|$ . Now, let us assume that for all finite set of points  $\{x_0, \dots, x_r\} \subseteq I$ , there exists  $x \in I$ , such that for all  $x_j$ ,  $\|x - x_j\| \geq \varepsilon_1 / \alpha_M$ . Then, starting from a point  $x_0 \in I$ , we can construct a sequence  $\{x_j\}_{j \in \mathbb{N}}$  such that for all  $j, j' \in \mathbb{N}$ ,  $j \neq j'$ , we have  $\|x_j - x_{j'}\| \geq \varepsilon_1 / \alpha_M$ . Therefore, we cannot extract a converging subsequence of  $\{x_j\}_{j \in \mathbb{N}}$  and  $I$  cannot be a compact set. Therefore, we proved by contradiction that there exists a finite set of points  $I_\sigma = \{x_0, \dots, x_{J_1}\} \subseteq I$  such that for all  $x \in I$ , there exists  $x_j \in I_\sigma$  satisfying the inequality  $\|x - x_j\| \leq \varepsilon_1 / \alpha_M$ .  $\blacksquare$

Then, let us define the transition system  $T_1 = (Q_1, \rightarrow_1, Q_1^0, \mathbb{R}^p, h_1)$  where the set of symbolic states is

$$Q_1 = \{q_j^k \mid j \in \{0, \dots, J_1\}, k \in \mathbb{N}\},$$

the transition relation  $\rightarrow_1$  is deterministic, given by  $q_j^k \rightarrow_1 q_j^{k+1}$ , the set of initial states is  $Q_1^0 = \{q_j^0 \mid j \in \{0, \dots, J_1\}\}$ , and the observation map  $h_1(q_j^k) = CA^k x_j$ .

**Lemma 2.** *The transition systems  $\Sigma_1$  and  $T_1$  are approximately bisimilar with precision  $\varepsilon_1$ .*

*Proof.* Let us define the relation  $R \subseteq \mathbb{R}^n \times Q_1$ :

$$R = \{(x, q_j^k) \mid \|x - A^k x_j\|_M \leq \varepsilon_1\}.$$

From equation (2), for all  $(x, q_j^k) \in R$ ,

$$\|h(x) - h_1(q_j^k)\| = \|C(x - A^k x_j)\| \leq \|x - A^k x_j\|_M \leq \varepsilon_1.$$

Further, from equation (3),

$$\|Ax - A^{k+1} x_j\|_M = \|A(x - A^k x_j)\|_M \leq \lambda \|x - A^k x_j\|_M \leq \varepsilon_1.$$

Therefore,  $(Ax, q_j^{k+1}) \in R$ . Hence,  $R$  is an  $\varepsilon_1$ -approximate bisimulation relation between  $\Sigma_1$  and  $T_1$ . Moreover, for all  $x \in I$ , there exists  $x_j \in I_\sigma \subseteq I$  such that  $\|x - x_j\|_M \leq \varepsilon_1$ , (i.e.  $(x, q_j^0) \in R$ ) and conversely. Thus,  $\Sigma_1 \sim_{\varepsilon_1} T_1$ .  $\blacksquare$

Let us remark that as  $k$  goes to infinity,  $h_1(q_j^k)$  converges to 0. Then, by replacing the states for large values of  $k$  (typically for  $k$  greater than a given parameter  $K_1$ ) by an invariant state associated with the observation 0, we define the finite state transition system  $S_1 = (P_1, \rightsquigarrow_1, P_1^0, \mathbb{R}^p, g_1)$  where the finite set of symbolic states is

$$P_1 = \{p_j^k \mid j \in \{0, \dots, J_1\}, k \in \{0, \dots, K_1\}\} \cup \{p^\infty\},$$

and the transition relation  $\rightsquigarrow_1$  is deterministic and given by

$$\forall j \in \{0, \dots, J_1\}, k \in \{0, \dots, K_1 - 1\}, p_j^k \rightsquigarrow_1 p_j^{k+1}, p_j^{K_1} \rightsquigarrow_1 p^\infty \text{ and } p^\infty \rightsquigarrow_1 p^\infty.$$

The set of initial states is  $P_1^0 = \{p_j^0 \mid j \in \{0, \dots, J_1\}\}$ , and the observation map is given by

$$\forall j \in \{0, \dots, J_1\}, k \in \{0, \dots, K_1\}, g_1(p_j^k) = CA^k x_j \text{ and } g_1(p^\infty) = 0.$$

**Lemma 3.** *The transition systems  $T_1$  and  $S_1$  are approximately bisimilar with precision  $r_I \lambda^{K_1+1}$ .*

*Proof.* We define the following relation  $R \subseteq Q_1 \times P_1$ :

$$R = \{(q_j^k, p_j^k) \mid j \in \{0, \dots, J_1\}, k \in \{0, \dots, K_1\}\} \cup \{(q_j^k, p^\infty) \mid j \in \{0, \dots, J_1\}, k > K_1\}.$$

Let  $(q_j^k, p_j^k) \in R$ , for  $j \in \{0, \dots, J_1\}$ ,  $k \in \{0, \dots, K_1\}$ , then  $\|h_1(q_j^k) - g_1(p_j^k)\| = 0$ . If  $k < K_1$ , then  $(q_j^{k+1}, p_j^{k+1}) \in R$ . If  $k = K_1$ , then  $(q_j^{K_1+1}, p^\infty) \in R$ . Let  $(q_j^k, p^\infty) \in R$ , for  $j \in \{0, \dots, J_1\}$ ,  $k > K_1$ , then from equations (2) and (3)

$$\|h_1(q_j^k) - g_1(p^\infty)\| = \|CA^k x_j\| \leq \|A^k x_j\|_M \leq \lambda^k \|x_j\|_M \leq \lambda^{K_1+1} r_I.$$

Further  $(q_j^{k+1}, p^\infty) \in R$ . Thus,  $R$  is a  $r_I \lambda^{K_1+1}$ -approximate bismulation relation between  $T_1$  and  $S_1$ . Moreover, for all  $q_j^0 \in Q_1^0$ , there exists  $p_j^0 \in P_1^0$  such that  $(q_j^0, p_j^0) \in R$ , and conversely. Therefore,  $T_1 \sim_{r_I \lambda^{K_1+1}} S_1$ .  $\blacksquare$

From Lemmas 2 and 3 and from Proposition 1, the following result is straightforward.

**Theorem 2.** *The transition systems  $\Sigma_1$  and  $S_1$  are approximately bisimilar with precision  $\varepsilon_1 + r_I \lambda^{K_1+1}$ .*

Let us remark that by choosing appropriately the parameters  $\varepsilon_1$  and  $K_1$ , any desired precision can be achieved by the finite abstraction  $S_1$ . The number of states of  $S_1$  is  $(J_1 + 1)(K_1 + 1) + 1$ .

### 3.2 Abstraction of the controlled dynamics

The construction of a finite abstraction that is approximately bisimilar to  $\Sigma_2$  is also processed in two steps. First by using a sample of the set of inputs  $U$ , we compute an abstraction with an infinite number of states. Then, we show that this system is approximately bisimilar to a finite state system.

**Lemma 4.** *Let  $\varepsilon_2 > 0$ , there exists a finite list  $U_\sigma = \{u_0, \dots, u_{J_2}\} \subseteq U$  such that  $u_0 = 0$  and*

$$\forall u \in U, \exists u_j \in U_\sigma, \|Bu - Bu_j\|_M \leq \varepsilon_2.$$

The proof of this result is similar to that of Lemma 1. We define the alphabet  $V = \{v_0, \dots, v_{J_2}\}$ . We define the transition system  $T_2 = (Q_2, \rightarrow_2, Q_2^0, \mathbb{R}^p, h_2)$  where the set of symbolic states is the set of infinite words on the alphabet  $V$ :

$$Q_2 = \{v_{j_0} v_{j_1} v_{j_2} \dots \mid \forall k \in \mathbb{N}, j_k \in \{0, \dots, J_2\}\}.$$

The transition relation  $\rightarrow_2$  is given by

$$\forall v_{j_0} v_{j_1} v_{j_2} \dots \in Q_2, \forall v_j \in V, v_{j_0} v_{j_1} v_{j_2} \dots \rightarrow_2 v_j v_{j_0} v_{j_1} v_{j_2} \dots$$

The set of initial states consists of a single infinite word  $Q_2^0 = \{v_0 v_0 v_0 \dots\}$ . The observation map is given by

$$\forall v_{j_0} v_{j_1} v_{j_2} \dots \in Q_2, h_2(v_{j_0} v_{j_1} v_{j_2} \dots) = \sum_{k=0}^{k=\infty} CA^k B u_{j_k}.$$

Let us remark that since the linear system (1) is stable, the observation map is well defined for any infinite word.

**Lemma 5.** *The transition systems  $\Sigma_2$  and  $T_2$  are approximately bisimilar with precision  $\varepsilon_2/(1-\lambda)$ .*

*Proof.* Let us define the relation  $R \subseteq \mathbb{R}^n \times Q_2$ ,

$$R = \left\{ (x, v_{j_0} v_{j_1} v_{j_2} \dots) \mid \left\| x - \sum_{k=0}^{k=\infty} A^k B u_{j_k} \right\|_M \leq \varepsilon_2 / (1 - \lambda) \right\}.$$

From equation (2), for all  $(x, v_{j_0} v_{j_1} v_{j_2} \dots) \in R$ ,

$$\begin{aligned} \|h(x) - h_2(v_{j_0} v_{j_1} v_{j_2} \dots)\| &= \|C(x - \sum_{k=0}^{k=\infty} A^k B u_{j_k})\| \\ &\leq \|x - \sum_{k=0}^{k=\infty} A^k B u_{j_k}\|_M \leq \varepsilon_2 / (1 - \lambda). \end{aligned}$$

Let  $x \rightarrow_2 x'$ , then there exists  $u \in U$  such that  $x' = Ax + Bu$ . There exists  $u_j \in U_\sigma$ , such that  $\|Bu - Bu_j\|_M \leq \varepsilon_2$ . Then,  $v_{j_0} v_{j_1} v_{j_2} \dots \rightarrow_2 v_j v_{j_0} v_{j_1} v_{j_2} \dots$  and from equation (3)

$$\begin{aligned} \|x' - Bu_j - \sum_{k=0}^{k=\infty} A^{k+1} B u_{j_k}\|_M &\leq \|A(x - \sum_{k=0}^{k=\infty} A^k B u_{j_k})\|_M + \|Bu - Bu_j\|_M \\ &\leq \lambda \|x - \sum_{k=0}^{k=\infty} A^k B u_{j_k}\|_M + \varepsilon_2 \leq \varepsilon_2 / (1 - \lambda). \end{aligned}$$

Thus, we have  $(x', v_j v_{j_0} v_{j_1} v_{j_2} \dots) \in R$ . Similarly, it is easy to show that for all  $v_{j_0} v_{j_1} v_{j_2} \dots \rightarrow_2 v_j v_{j_0} v_{j_1} v_{j_2} \dots$ , there exists  $x \rightarrow_2 x'$  such that we have  $(x', v_j v_{j_0} v_{j_1} v_{j_2} \dots) \in R$ . Thus,  $R$  is an  $\varepsilon_2/(1-\lambda)$ -approximate bisimulation relation between  $\Sigma_2$  and  $T_2$ . Moreover, it is clear that  $(0, v_0 v_0 v_0 \dots) \in R$ , then  $\Sigma_2 \sim_{\varepsilon_2/(1-\lambda)} T_2$ .  $\blacksquare$

We now show that the system  $T_2$  is approximately bisimilar to a finite system. Let us remark that for an infinite word  $v_{j_0} v_{j_1} v_{j_2} \dots \in Q_2$ , the value of  $h_2(v_{j_0} v_{j_1} v_{j_2} \dots)$  does not depend much on  $v_{j_k}$  for large values of  $k$  (since the matrices  $A^k$  converge to 0). Then, by replacing an infinite word by its finite prefix of a given length, we can define a finite state transition system  $S_2 = (P_2, \rightsquigarrow_2, P_2^0, \mathbb{R}^p, g_2)$  where the set of symbolic states is the set of words of length  $K_2 + 1$  on the alphabet  $V$ :

$$P_2 = \{v_{i_0} \dots v_{i_{K_2}} \mid \forall k \in \{0, \dots, K_2\}, i_k \in \{0, \dots, J_2\}\}.$$

The transition relation  $\rightsquigarrow_2$  is given by

$$\forall v_{i_0} \dots v_{i_{K_2}} \in P_2, \forall v_i \in V, v_{i_0} \dots v_{i_{K_2}} \rightsquigarrow_2 v_i v_{i_0} \dots v_{i_{K_2-1}}.$$

The set of initial states consists of a single word of length  $K_2 + 1$ ,  $P_2^0 = \{v_0 \dots v_0\}$ .

The observation map is given by

$$\forall v_{i_0} \dots v_{i_{K_2}} \dots \in P_2, g_2(v_{i_0} \dots v_{i_{K_2}}) = \sum_{k=0}^{k=K_2} C A^k B u_{i_k}.$$

**Lemma 6.** *The transition systems  $T_2$  and  $S_2$  are approximately bisimilar with precision  $r_U \lambda^{K_2+1} / (1 - \lambda)$ .*

*Proof.* We define the relation  $R \subseteq Q_2 \times P_2$ :

$$R = \{(v_{j_0} v_{j_1} v_{j_2} \dots, v_{i_0} \dots v_{i_{K_2}}) \mid \forall k \in \{0, \dots, K_2\}, j_k = i_k\}.$$

Let  $(v_{j_0} v_{j_1} v_{j_2} \dots, v_{i_0} \dots v_{i_{K_2}}) \in R$ , from equations (2) and (3)

$$\begin{aligned} \|h_2(v_{j_0} v_{j_1} v_{j_2} \dots) - g_2(v_{i_0} \dots v_{i_{K_2}})\| &= \left\| \sum_{k=K_2+1}^{k=\infty} CA^k Bu_{j_k} \right\| \\ &\leq \left\| \sum_{k=K_2+1}^{k=\infty} A^k Bu_{j_k} \right\|_M \\ &\leq \lambda^{K_2+1} \left\| \sum_{k=0}^{k=\infty} A^k Bu_{j_{k-K_2-1}} \right\|_M \\ &\leq \lambda^{K_2+1} \sum_{k=0}^{k=\infty} \lambda^k r_U = r_U \lambda^{K_2+1} / (1 - \lambda). \end{aligned}$$

For all  $v_{j_0} v_{j_1} v_{j_2} \dots \rightarrow_2 v_j v_{j_0} v_{j_1} v_{j_2} \dots$ , we have  $v_{i_0} \dots v_{i_{K_2}} \rightsquigarrow_2 v_j v_{i_0} \dots v_{i_{K_2-1}}$  and it is easy to see that  $(v_j v_{j_0} v_{j_1} v_{j_2} \dots, v_j v_{i_0} \dots v_{i_{K_2-1}}) \in R$ . Similarly, for all  $v_{i_0} \dots v_{i_{K_2}} \rightsquigarrow_2 v_i v_{i_0} \dots v_{i_{K_2-1}}$ , we have  $v_{j_0} v_{j_1} v_{j_2} \dots \rightarrow_2 v_i v_{j_0} v_{j_1} v_{j_2} \dots$  and  $(v_i v_{j_0} v_{j_1} v_{j_2} \dots, v_i v_{i_0} \dots v_{i_{K_2-1}}) \in R$ . Hence,  $R$  is a  $r_U \lambda^{K_2+1} / (1 - \lambda)$ -approximate bisimulation relation between  $T_2$  and  $S_2$ . Moreover,  $(v_0 v_0 v_0 \dots, v_0 \dots v_0) \in R$ , then  $T_2 \sim_{r_U \lambda^{K_2+1} / (1 - \lambda)} S_2$ .  $\blacksquare$

Intuitively,  $S_2$  can be seen as an abstraction which keeps track of the last  $K_2 + 1$  values of the input of  $\Sigma_2$  (the larger  $K_2$  the better the precision). From Lemmas 5 and 6 and from Proposition 1, the following result is straightforward.

**Theorem 3.** *The transition systems  $\Sigma_2$  and  $S_2$  are approximately bisimilar with precision  $(\varepsilon_2 + r_U \lambda^{K_2+1}) / (1 - \lambda)$ .*

Let us remark that by choosing appropriately the parameters  $\varepsilon_2$  and  $K_2$ , any desired precision can be achieved by the finite abstraction  $S_2$ . The number of states of  $S_2$  is  $(J_2 + 1)^{(K_2+1)}$ .

### 3.3 Abstraction of the linear system

We now define the finite abstraction  $S$  of  $\Sigma$  by composition of  $S_1$  and  $S_2$ :  $S = S_1 || S_2$ . Then, we can give the main result of the paper.

**Theorem 4.** *The transition system  $\Sigma$  associated with the stable linear system (1) and the finite state transition system  $S$  are approximately bisimilar with precision  $(\varepsilon_1 + r_I \lambda^{K_1+1}) + (\varepsilon_2 + r_U \lambda^{K_2+1}) / (1 - \lambda)$ .*

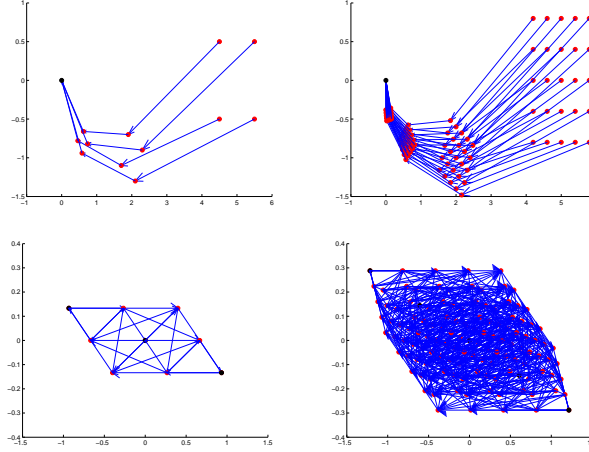
*Proof.* From Proposition 2 and Theorems 2 and 3,  $\Sigma_1 || \Sigma_2$  and  $S_1 || S_2$  are approximately bisimilar with precision  $(\varepsilon_1 + r_I \lambda^{K_1+1}) + (\varepsilon_2 + r_U \lambda^{K_2+1}) / (1 - \lambda)$ . Then, Theorem 4 follows from Propositions 1 and 4.  $\blacksquare$

Hence, for any stable linear systems of the form (1), there exists a finite abstraction that is approximately bisimilar. Moreover, by choosing appropriately  $\varepsilon_1$ ,  $K_1$ ,  $\varepsilon_2$  and  $K_2$ , any desired precision can be achieved. Let us remark that the paper provides an effective way to compute these finite abstractions using samples of the sets of initial states and inputs.

*Example 1.* We consider the linear system of the form (1) given by the matrices

$$A = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

the set of initial states  $I = [4, 6] \times [-1, 1]$  and the set of inputs  $U = [-1, 1]$ . We computed finite abstractions  $S_1$  and  $S_2$  of the autonomous and controlled dynamics. Figure 1 shows simple discrete abstractions of  $S_1$  for  $\varepsilon_1 = 1/2$  and  $K_1 = 2$  (precision 0.85) and for  $\varepsilon_1 = 1/5$  and  $K_1 = 3$  (precision 0.44) and of  $S_2$  for  $\varepsilon_2 = 1/3$  and  $K_2 = 1$  (precision 0.97) and for  $\varepsilon_2 = 1/5$  and  $K_2 = 2$  (precision 0.52). More precise abstractions for smaller  $\varepsilon_1$  and  $\varepsilon_2$  and larger  $K_1$  and  $K_2$  can be computed; however, the visualization becomes problematic.



**Fig. 1.** Finite abstractions of the autonomous dynamics (top) and of the controlled dynamics (bottom).

## 4 Application to verification and control synthesis

In this section, we show how the finite abstraction developed in the previous section can be used to solve verification or control synthesis problems. Let us define  $\delta = (\varepsilon_1 + r_I \lambda^{K_1+1}) + (\varepsilon_2 + r_U \lambda^{K_2+1}) / (1 - \lambda)$  and let  $F$  be a subset of  $\mathbb{R}^p$ . We define the following sets:

$$\forall \mu \geq 0, I(F, \mu) = \{\pi \in F \mid \forall \pi' \in \mathbb{R}^p, \|\pi - \pi'\| \leq \mu \implies \pi' \in F\}.$$

$$\forall \mu \geq 0, O(F, \mu) = \{\pi \in \mathbb{R}^p \mid \exists \pi' \in F \text{ such that } \|\pi - \pi'\| \leq \mu\}.$$

$I(F, \mu)$  denotes the set of points of  $F$  whose distance to the boundary is greater than  $\mu$ .  $O(F, \mu)$  denotes the  $\mu$  neighborhood of  $F$ .

### 4.1 Verification

We want to determine whether  $F$  is reachable by the linear system (1) (*i.e.*  $\text{Reach}(\Sigma) \cap F \neq \emptyset$ ). The idea is to do the reachability computation on the finite abstraction  $S$ , which is a simple task since  $S$  is finite, and then to interpret the results for  $\Sigma$  using the following proposition.

**Proposition 5.** *The following assertions hold*

$$\text{Reach}(S) \cap I(F, \delta) \neq \emptyset \implies \text{Reach}(\Sigma) \cap F \neq \emptyset \quad (4)$$

$$\text{Reach}(S) \cap O(F, \delta) = \emptyset \implies \text{Reach}(\Sigma) \cap F = \emptyset \quad (5)$$

$$\text{Reach}(\Sigma) \cap I(F, 2\delta) \neq \emptyset \implies \text{Reach}(S) \cap I(F, \delta) \neq \emptyset \quad (6)$$

$$\text{Reach}(\Sigma) \cap O(F, 2\delta) = \emptyset \implies \text{Reach}(S) \cap O(F, \delta) = \emptyset \quad (7)$$

The proof of this result is not stated here but is straightforward from Corollary 1. Equations (4) and (5) allow us to determine whether  $\text{Reach}(\Sigma)$  intersects  $F$ . The interpretation of equation (6) is that if  $\text{Reach}(\Sigma)$  *sufficiently* intersects  $F$ , then the reachability analysis of the finite abstraction  $S$  will allow to conclude that  $\text{Reach}(\Sigma) \cap F \neq \emptyset$ . Similarly, equation (7) tells that if  $\text{Reach}(\Sigma)$  is *sufficiently* far from  $F$ , then the reachability analysis of  $S$  will allow to conclude that  $\text{Reach}(\Sigma) \cap F = \emptyset$ . This means that the more robust  $\Sigma$  is with respect to the reachability property, the coarser the finite abstraction  $S$  we can use. Let us remark that, contrarily to most approximate reachability techniques, the approach based on approximately bisimilar abstractions allows to verify reachability properties on infinite time-intervals.

## 4.2 Control synthesis

Let us assume that  $\text{Reach}(S) \cap I(F, \delta) \neq \emptyset$ , then from Proposition 5, we know that  $\text{Reach}(\Sigma)$  intersects  $F$ . We now want to determine an initial state and a sequence of inputs which drives the system  $\Sigma$  to an observation in  $F$ . For the sake of simplicity, we assume that  $K_1 = K_2 = K$ .

There exists an element  $(p, w) \in P_1 \times P_2$  reachable by a state trajectory of  $S$  and such that  $g_1(p) + g_2(w) \in I(F, \delta)$ . We can show that there are only two possibilities:

1.  $(p, w) = (p_j^k, v_{i_0} \dots v_{i_K})$  where for all  $r \in \{k, \dots, K\}$ ,  $v_{i_r} = v_0$ . Then, a state trajectory of  $S$  leading to  $(p, w)$  is

$$(p_j^0, v_0 \dots v_0) \rightsquigarrow (p_j^1, v_{i_{k-1}} v_0 \dots v_0) \rightsquigarrow (p_j^2, v_{i_{k-2}} v_{i_{k-1}} v_0 \dots v_0) \rightsquigarrow \dots \quad (8)$$

$$\dots \rightsquigarrow (p_j^k, v_{i_0} \dots v_{i_{k-1}} v_0 \dots v_0).$$

2.  $(p, w) = (p^\infty, v_{i_0} \dots v_{i_K})$ . Then, a state trajectory of  $S$  leading to  $(p, w)$  is

$$(p_j^0, v_0 \dots v_0) \rightsquigarrow (p_j^1, v_{i_K} v_0 \dots v_0) \rightsquigarrow \dots \quad (9)$$

$$\dots \rightsquigarrow (p_j^K, v_{i_1} \dots v_{i_K} v_0) \rightsquigarrow (p^\infty, v_{i_0} v_{i_1} \dots v_{i_K}).$$

Thereof, we can choose an initial state and a sequence of inputs for the system  $\Sigma$  such that its associated external trajectory tracks with precision  $\delta$  the external trajectory of the discrete abstraction  $S$  associated to the state trajectory given by equation (8) or (9).

## 5 Conclusion

In this paper, we showed that stable linear systems with constrained inputs admit approximately bisimilar finite abstractions of arbitrary precision. An effective way of computing these abstractions, based on compositional reasoning and samples of the set of initial states and inputs, is described. We showed how these abstractions can be used to simplify some computational tasks such as verification and control synthesis for reachability specifications.

Future work includes further reductions of these abstractions using approximation techniques for finite systems in order to obtain abstractions achieving a desired precision with a minimal number of states. We also intend to extend these results to continuous-time systems, as well as non-linear dynamical systems and to use our discrete abstractions to solve verification and control synthesis problems for more complex specifications such as those expressed in temporal logics using model checking or supervisory control for purely discrete systems.

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