Symbolic models for nonlinear control systems using approximate bisimulation

Giordano Pola, Antoine Girard and Paulo Tabuada

Abstract— **Control systems are usually modeled by differential equations describing how physical phenomena can be influenced by certain control parameters or inputs. Although these models are very powerful when dealing with physical phenomena, they are less suitable to describe software and hardware interfacing the physical world. This has spurred a recent interest in describing control systems through *symbolic models* that are abstract descriptions of the continuous dynamics, where each "symbol" corresponds to an "aggregate" of continuous states in the continuous model. Since these symbolic models are of the same nature of the models used in computer science to describe software and hardware, they provided a unified language to study problems of control in which software and hardware interact with the physical world. In this paper we show that every incrementally globally asymptotically stable nonlinear control system is approximately equivalent (bisimilar) to symbolic model with a precision that can be chosen a–priori. We also show that for digital controlled systems, in which inputs are piecewise–constant, and under the stronger assumption of incremental input–to–state stability, the symbolic models can be obtained, based on a suitable quantization of the inputs.**

keywords: symbolic models, approximate bisimulation, digital control systems, incremental stability, nonlinear systems.

I. Introduction

The idea of using models at different levels of abstraction has been very successfully used in the formal methods community, with the purpose of mitigating the complexity of software verification. A central notion when dealing with complexity reduction, is the one of bisimulation equivalence, introduced by Milner [1] and Park [2] in the 80s'. The key idea in the notion of bisimulation is to find and compute a symbolic model which shares with the original model most of the properties of interest. In fact, the use of symbolic models provides a unified framework for describing physical processes as well as software and hardware interacting with the physical world. Furthermore, dealing with symbolic models enables one to leverage the rich literature on supervisory control [3] and game theory [4], [5], that can be of help when synthesizing controllers. The problem of constructing symbolic models of control systems is thus quite challenging from a conceptual and technical point of view. After several successful results on the existence of finite bisimulations for discrete–time control systems [6], [7], a new twist in this research line has been recently given by the so–called *approximate bisimulation*, introduced in [8], that captures equivalence of systems in an approximate way. By relaxing the usual notion of bisimulation to approximate bisimulation, a larger class of control systems can be expected to admit symbolic models. In fact in [9], [10], a symbolic model is proposed, which is based on an approximate notion of simulation (one–sided version of the notion of approximate bisimulation). More precisely [10] shows that, if a nonlinear control system is asymptotically stabilizable it is possible to construct a symbolic model that approximates the control system, up to a given precision that is chosen a–priori, as a design parameter. However, if a controller fails to exist for the symbolic model, nothing can be concluded regarding the existence of a controller for the original model. This drawback is a direct consequence of the one–sided approximation notion used in [10]. This motivates the need to extend the results in [10] to bisimulation. The aim of this paper is to identify a class of control systems admitting symbolic models, that are approximately bisimilar to the given control system. The key idea in the results that we propose is to replace the assumption of asymptotic stabilizability of [10] with the stronger notion of asymptotic stability. By doing so, we show that every incrementally globally asymptotically stable [11] nonlinear control system admits a symbolic model that is an approximate bisimulation of the control system, with a precision that is defined a–priori, as a design parameter. Furthermore if the state space of the control system is bounded, which is the case in many realistic situations, the symbolic model is finite. Moreover, by focusing on digital control systems, i.e. systems where control signals are piecewise–constant, a symbolic model can be obtained by quantizing the space of inputs. As a by–product, our results also shed some light into the construction of finite abstractions in the context of quantized control systems [12], [13]. Indeed, by performing a quantization on the input space, we can guarantee that the resulting symbolic model admits a lattice structure. However, while in this paper this lattice structure is exploited for obtaining a finite abstraction of the control system, in the context of quantized control systems literature, it has been used to obtain efficient motion planning algorithms. Similar results have been recently reported in [14]. This paper extends the work in [14] in two directions: (i) by enlarging the class of control systems from linear to nonlinear and (ii) by enlarging the class of input signals from piecewise–constant to measurable. A full version of this paper can be found in

This work has been partially supported by the National Science Foundation CAREER award 0717188.

G. Pola and P. Tabuada are with the Department of Electrical Engineering, University of California at Los Angeles, Los Angeles, CA 90095-1594, USA pola,tabuada@ee.ucla.edu.

A. Girard is with Université Joseph Fourier, Laboratoire Jean Kuntzmann, B.P. 53, 38041, Grenoble, France, Antoine.Girard@imag.fr

[15].

II. Control systems and stability notions

A. Notation

The identity map on a set A is denoted by 1_A. If $A \subseteq B$ we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. Given a function $f : A \to B$, and a set $C \subseteq A$, the symbol $f|_C : C \to B$ denotes the restriction of f to C, so that $f|_C(c) = f(c)$ for any $c \in C$. Given $a, b \in \mathbb{R}$, we denote the closed interval by $[a, b]$ and the open interval by $]a, b[$, i.e. $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ and $]a, b[= \{x \in \mathbb{R} : a < x < b\}$. Given a vector $x \in \mathbb{R}^n$ we denote by x_i the i–th element of x and by $\|x\|$ the infinity norm of x; we recall that $\|x\| = max\{|x_1|, |x_2|, ..., |x_n|\}$, where $|x_i|$ denotes the absolute value of x_i. Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}$, the (essential) supremum of f is denoted by $\|f\|_\infty$. The closed ball centered at $x \in \mathbb{R}^n$ with radius ε is defined by $\mathcal{B}_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq \varepsilon\}$. For any $A \subseteq \mathbb{R}^n$ and $\mu \in \mathbb{R}$ set $[A]_\mu = \{a \in A \mid a_i = k_i\mu, k_i \in \mathbb{Z}, i = 1, ..., n\}$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ belongs to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ belongs to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ iff $(a, b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.

B. Control Systems

In this paper we consider the following class of nonlinear control systems:

$$\Sigma : \begin{cases} \dot{x} = f(x, u) \\ x \in \mathbb{R}^n, u \in U \subseteq \mathbb{R}^m, \end{cases} \tag{1}$$

where x is the state and u is the control input. We suppose that control input signals are the set, denoted by $\mathcal{U}$, of all measurable functions from intervals of the form $]a, b[\subseteq \mathbb{R}$ to U with $a < 0$ and $b > 0$. Moreover we suppose that the function $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a continuous map satisfying the following Lipschitz assumption: for every compact set $K \subset \mathbb{R}^n$, there exists a constant $\kappa > 0$ such that $\|f(x, u) - f(y, u)\| \leq \kappa\|x - y\|$ for all $x, y \in K$ and all $u \in U$. An absolutely continuous curve $\mathbf{x} :]a, b[\to \mathbb{R}^n$ is said to be a *trajectory* of Σ if there exists $\mathbf{u} \in \mathcal{U}$ satisfying $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$, for almost all $t \in]a, b[$. Although we have defined trajectories over open domains, we shall refer to trajectories $\mathbf{x} :[0, \tau] \to \mathbb{R}^n$ defined on closed domains $[0, \tau]$, $\tau \in \mathbb{R}^+$ with the understanding of the existence of a trajectory $\mathbf{x}' :]a, b[\to \mathbb{R}^n$ such that $\mathbf{x} = \mathbf{x}'|_{[0,\tau]}$. We will also write $\mathbf{x}(\tau, x, \mathbf{u})$ to denote the point reached at time τ under the input $\mathbf{u}$ from initial condition x; this point is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories.

C. Stability notions

The results presented in this paper will assume certain stability assumptions on the control systems. We briefly recall those notions in the following definitions:

Definition 1: [11] A control system Σ is said to be *incrementally globally asymptotically stable (*δ–GAS) if there exist a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}_0^+$, any $x_1, x_2 \in \mathbb{R}^n$ and any input signal $\mathbf{u} \in \mathcal{U}$ the following condition is satisfied:

$$\|\mathbf{x}(t, x_1, \mathbf{u}) - \mathbf{x}(t, x_2, \mathbf{u})\| \leq \beta(\|x_1 - x_2\|, t). \tag{2}$$

Definition 2: [11] A control system Σ is said to be *incrementally input–to–state stable* (δ–ISS) if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function γ such that for any $t \in \mathbb{R}_0^+$, any $x_1, x_2 \in \mathbb{R}^n$ and any pair of input signals $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$ the following condition is satisfied:

$$\begin{aligned} \|\mathbf{x}(t, x_1, \mathbf{u}_1) - \mathbf{x}(t, x_2, \mathbf{u}_2)\| \quad \leq \quad & \beta(\|x_1 - x_2\|, t) \\ & +\gamma(\|\mathbf{u}_1 - \mathbf{u}_2\|_\infty). \end{aligned} \tag{3}$$

By observing (2) and (3), it is not difficult to see that δ–ISS implies δ–GAS, while the converse is not true in general. In general, inequalities (2) and (3) are difficult to check directly. An approach based on Lyapunov functions, can be of help into checking these stability properties.

Definition 3: Consider a control system Σ and a smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$. V is called a δ*–GAS Lyapunov function* for Σ if there exist $\mathcal{K}_\infty$ functions α_1, α_2 and ρ such that:

(i) for any $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\alpha_1(\|x_1 - x_2\|) \leq V(x_1, x_2) \leq \alpha_2(\|x_1 - x_2\|);$$

(ii) for any $u \in U$ and any $x_1, x_2 \in \mathbb{R}^n$

$$\frac{\partial V}{\partial x_1} f(x_1, u) + \frac{\partial V}{\partial x_2} f(x_2, u) < -\rho(\|x_1 - x_2\|).$$

V is called a δ*–ISS Lyapunov function* for Σ, if there exist $\mathcal{K}_\infty$ functions α_1, α_2, ρ and σ such that condition (i) is satisfied and the following holds:

(iii) for any $u_1, u_2 \in U$ and any $x_1, x_2 \in \mathbb{R}^n$

$$\begin{aligned} \frac{\partial V}{\partial x_1} f(x_1, u_1) + \frac{\partial V}{\partial x_2} f(x_2, u_2) \quad < \quad & -\rho(\|x_1 - x_2\|) \\ & +\sigma(\|u_1 - u_2\|). \end{aligned}$$

The following result completely characterizes δ–GAS and δ–ISS in terms of existence of Lyapunov functions.

Theorem 1: [11] Consider a control system Σ. Then:

- If U is compact then Σ is δ–GAS if and only if it admits a δ–GAS Lyapunov function;
- If U is closed, convex, contains the origin and $f(0, 0) = 0$, then Σ is δ–ISS if it admits a δ–ISS Lyapunov function. Moreover if U is compact, existence of a δ–ISS Lyapunov function is equivalent to δ–ISS.

III. Approximate bisimulations

In this section we introduce a notion of approximate equivalence upon which all the results in this paper rely. We start by introducing the class of transition systems that will be used as abstract models for control systems.

Definition 4: A transition system is quintuple:

$$T = (Q, L, \longrightarrow, O, H),$$

consisting of:

- A set of states Q;
- A set of labels L;
- A transition relation $\longrightarrow \subseteq Q \times L \times Q$;
- An output set O;
- An output function $H : Q \to O$.

A transition system $(Q, L, \longrightarrow, O, H)$ is said to be:

- *metric*, if the output set O is equipped with a metric $d : O \times O \to \mathbb{R}_0^+$;
- *countable*, if Q and L are countable sets;
- *finite*, if Q and L are finite sets.

We will follow standard practice and denote an element $(p, l, q) \in \longrightarrow$ by $p \overset{l}{\longrightarrow} q$. Transition systems capture dynamics through the transition relation. For any states $p, q \in Q$, $p \overset{l}{\longrightarrow} q$ simply means that it is possible to evolve or jump from state p to state q under the action labeled by l. We will use transition systems as an abstract representation of control systems. There are several different ways in which we can transform control systems into transition systems. We now describe one of these which has the property of capturing all the information contained in a control system Σ. Given a control system Σ define the transition system $T(\Sigma) := (Q, L, \longrightarrow, O, H)$, where:

- $Q = \mathbb{R}^n$;
- $L = \mathcal{U}$;
- $q \overset{\mathbf{u}}{\longrightarrow} p$ if there exists a trajectory $\mathbf{x} : [0, \tau] \to \mathbb{R}^n$ of Σ satisfying $\mathbf{x}(\tau, q, \mathbf{u}) = p$ for some $\tau \in \mathbb{R}^+$;
- $O = \mathbb{R}^n$;
- $H = 1_{\mathbb{R}^n}$.

Note that $T(\Sigma)$ is a metric transition system when we regard $O = \mathbb{R}^n$ as being equipped with the metric $d(p, q) = \|p - q\|$. We now introduce a notion of approximate equivalence. The notion of equivalence that we consider, is the one of bisimulation equivalence [1], [2]. Bisimulation relations are standard mechanisms to relate the properties of transition systems. Intuitively, a bisimulation relation between a pair of transition systems T_1 and T_2 is a relation between the corresponding state sets explaining how a sequence of transitions r_1 of T_1 can be transformed into a sequence of transitions r_2 of T_2 and vice versa. While typical bisimulation relations require that r_1 and r_2 are observationally indistinguishable, that is, $H_1(r_1) = H_2(r_2)$, we shall relax this by requiring $H_1(r_1)$ to simply be close to $H_2(r_2)$ where closeness is measured with respect to the metric on the output set. The following notion has been introduced in [8] and in a slightly different formulation in [10].

Definition 5: Let $T_1 = (Q_1, L_1, \longrightarrow_1, O, H_1)$ and $T_2 = (Q_2, L_2, \longrightarrow_2, O, H_2)$ be metric transition systems with the same output set O and the same metric d and let $\varepsilon \in \mathbb{R}_0^+$. A relation $R \subseteq Q_1 \times Q_2$ is said to be an ε*–approximate bisimulation* relation between T_1 and T_2, if for any $(q_1, q_2) \in R$:

(i) $d(H_1(q_1), H_2(q_2)) \leq \varepsilon$;
(ii) $q_1 \overset{l_1}{\longrightarrow}_1 p_1$ implies the existence of $q_2 \overset{l_2}{\longrightarrow}_2 p_2$ such that $(p_1, p_2) \in R$;
(iii) $q_2 \overset{l_2}{\longrightarrow}_2 p_2$ implies the existence of $q_1 \overset{l_1}{\longrightarrow}_1 p_1$ such that $(p_1, p_2) \in R$.

Moreover T_1 is ε*–bisimilar* to T_2 if there exists a ε–approximate bisimulation R between T_1 and T_2 such that $R(Q_1) = Q_2$ and $R^{-1}(Q_2) = Q_1$.

IV. Approximately bisimilar symbolic models

In the following we will work with a sub–transition system of $T(\Sigma)$ obtained by selecting those transitions from $T(\Sigma)$ describing trajectories of duration τ for some chosen $\tau \in \mathbb{R}^+$. This can be seen as a time discretization or sampling process.

Definition 6: Given a control system Σ and a parameter $\tau \in \mathbb{R}^+$, define the transition system:

$$T_\tau(\Sigma) := (Q_1, L_1, \longrightarrow_1, O_1, H_1),$$

where:

- $Q_1 = \mathbb{R}^n$;
- $L_1 = \{\mathbf{u} \in \mathcal{U} \mid \mathbf{x}(\tau, x, \mathbf{u}) \text{ is defined for all } x \in \mathbb{R}^n\}$;
- $q \overset{\mathbf{u}}{\longrightarrow}_1 p$ if there exists a trajectory $\mathbf{x} : [0, \tau] \to \mathbb{R}^n$ of Σ satisfying $\mathbf{x}(\tau, q, \mathbf{u}) = p$;
- $O_1 = \mathbb{R}^n$;
- $H_1 = 1_{\mathbb{R}^n}$.

Note that the set of labels L_1 is composed by those control inputs $u \in \mathcal{U}$ for which $\mathbf{x}(\tau, x, \mathbf{u})$ is defined for all initial condition $x \in \mathbb{R}^n$. In the following we show existence of a countable transition system that is approximately bisimilar to $T_\tau(\Sigma)$, provided that Σ satisfies some stability properties. For doing so we will extract a countable set of states and a countable set of labels from $T_\tau(\Sigma)$ in a way that the obtained transition system satisfies the required approximation. By simple considerations on the infinity norm, for any given precision $\eta \in \mathbb{R}^+$ we can approximate the state space $Q_1 = \mathbb{R}^n$ of $T_\tau(\Sigma)$ by means of the countable set $Q_2 := [\mathbb{R}^n]_\eta$ so that for any $x \in \mathbb{R}^n$ there exists $q \in [\mathbb{R}^n]_\eta$ such that $\|x - q\| \leq \eta/2$. The approximation of the set of labels L_1 of $T_\tau(\Sigma)$ is more involved and it requires the notion of reachable sets. Given a control system Σ, any time $\tau \in \mathbb{R}^+$ and any state $x \in \mathbb{R}^n$ consider the set:

$$\mathcal{R}(\tau, x) = \{z \in \mathbb{R}^n : z = \mathbf{x}(\tau, x, \mathbf{u}), \mathbf{u} \in \mathcal{U}\},$$

of reachable states at time τ from initial state x. Given a precision $\mu \in \mathbb{R}^+$, we approximate the reachable set $\mathcal{R}(\tau, x)$ by:

$$\mathcal{P}_\mu(\tau, x) = \{y \in [\mathbb{R}^n]_\mu : \exists z \in \mathcal{R}(\tau, x) \text{ s.t. } \|y - z\| \leq \mu\}.$$

Since $\mathcal{P}_\mu(\tau, x) \subseteq [R^n]_\mu$, it is countable. Notice that for any $y \in \mathcal{P}_\mu(\tau, x)$ there exists a (possibly infinite) set of labels $\mathbf{l} \in L_1$ so that $\mathbf{d}(y, \mathbf{x}(\tau, q, \mathbf{l})) \leq \mu$. In order to approximate the set of labels L_1 we consider for any $y \in \mathcal{P}_\mu(\tau, x)$ only *one* label $\mathbf{l} \in L_1$, as "representative" of all labels associated with p. The choice of representatives is defined by the function:

$$\psi^{\mu}_{\tau,x} : \mathcal{P}_{\mu}(\tau, x) \rightarrow \mathcal{U},$$

that associates to any $y \in \mathcal{P}_{\mu}(\tau, x)$ one control input $\mathbf{u} = \psi^{\mu}_{\tau,x}(y) \in \mathcal{U}$ such that $\|y - \mathbf{x}(\tau, x, \mathbf{u})\| \leq \mu$. Notice that the function $\psi^{\mu}_{\tau,x}$ is not unique. We can now propose the following symbolic model. Given a control system Σ, any $\tau \in \mathbb{R}^{+}$, $\eta \in \mathbb{R}^{+}$ and $\mu \in \mathbb{R}^{+}$ define the following transition system:

$$T_{\tau,\eta,\mu}(\Sigma) := (Q_2, L_2, \longrightarrow_2, O_2, H_2), \tag{4}$$

where:

- $Q_2 = [\mathbb{R}^n]_{\eta}$;
- $L_2 = \bigcup_{q \in [\mathbb{R}^n]_{\eta}} L_2(q)$, where:
$$L_2(q) := \{\mathbf{l} \in \mathcal{U} : \mathbf{l} = \psi^{\mu}_{\tau,q}(p), p \in \mathcal{P}_{\mu}(\tau, q)\}; \tag{5}$$
- $q \stackrel{\mathbf{l}}{\longrightarrow}_2 p$, if $\mathbf{l} \in L_2(q)$ and $\|p - \mathbf{x}(\tau, q, \mathbf{l})\| \leq \eta$;
- $O_2 = \mathbb{R}^n$;
- $H_2 = \imath : Q_2 \longrightarrow O_2$.

Parameters τ, η and μ in the transition system $T_{\tau,\eta,\mu}(\Sigma)$ can be thought of, respectively, as a sampling time, a state space and an input space quantization. Notice that the quantization μ is given on the space of reachable states $\mathcal{R}(\tau, q)$ rather than on the infinite dimensional space $\mathcal{U}$. The set appearing in (5) is countable since it is the image through the function $\psi^{\mu}_{\tau,q}$ of the countable set $\mathcal{P}_{\mu}(\tau, q)$. Hence, the set of labels L_2 is the union of a countable sequence of countable sets and therefore it is countable as well. Finally since also the set of states Q_2 is countable, the transition system $T_{\tau,\eta,\mu}(\Sigma)$ is countable. Furthermore if the state space of the control system Σ is bounded, the corresponding transition system $T_{\tau,\eta,\mu}(\Sigma)$ is finite.

We now have all the ingredients to state the main result relating δ–GAS to the existence of symbolic models.

Theorem 2: Consider a control system Σ and any desired precision $\varepsilon \in \mathbb{R}^{+}$. If Σ is δ–GAS then for any $\tau \in \mathbb{R}^{+}$, $\eta \in \mathbb{R}^{+}$ and $\mu \in \mathbb{R}^{+}$ satisfying the following condition:

$$\beta(\varepsilon, \tau) + 2\mu + \eta \leq \varepsilon, \tag{6}$$

where β is a $\mathcal{KL}$ function satifying inequality (2), the transition system $T_{\tau}(\Sigma)$ is ε–bisimilar to $T_{\tau,\eta,\mu}(\Sigma)$.

Before giving the proof of this result we point out that if Σ is δ–GAS, there always exist parameters τ, η and μ satisfying condition (6). In fact, if Σ is δ–GAS then there exists a sufficiently large τ so that $\beta(\varepsilon, \tau) < \varepsilon$; then by choosing sufficiently small values of μ and η, condition (6) is fulfilled. As pointed out in Section II-C it is difficult in general to find a $\mathcal{KL}$ function β satisfying inequality (2). However, once a δ–GAS Lyapunov function V for Σ has been found, an expression for the function β can be derived on the basis of V.

Proof: Consider the relation $R \subseteq Q_1 \times Q_2$ defined by $(x, q) \in R$ if and only if $||x - q|| \leq \varepsilon$. By construction $R(Q_1) = Q_2$; by geometrical considerations on the infinity norm, $Q_1 \subseteq \bigcup_{q_2 \in Q_2} \mathcal{B}_{\eta}(q_2)$ and therefore, since by (6), $\eta < \varepsilon$, we have that $R^{-1}(Q_2) = Q_1$. We now show that R is an ε–approximate bisimulation relation between $T_{\tau}(\Sigma)$ and $T_{\tau,\eta,\mu}(\Sigma)$. Consider any $(x, q) \in R$. Condition (i) in Definition 5 is satisfied by the definition of R. Let us now show that condition (ii) in Definition 5 holds. Consider any $\mathbf{u}_1 \in L_1$ and the transition $x \stackrel{\mathbf{u}_1}{\longrightarrow}_1 y$ in $T_{\tau}(\Sigma)$. Let $v = \mathbf{x}(\tau, q, \mathbf{u}_1)$; since $\mathbb{R}^n \subseteq \bigcup_{w \in [\mathbb{R}^n]_{\mu}} \mathcal{B}_{\mu}(w)$, there exists $w \in [\mathbb{R}^n]_{\mu}$ such that:

$$\|v - w\| \leq \mu. \tag{7}$$

Since $v \in \mathcal{R}(\tau, q)$, it is clear that $w \in \mathcal{P}_{\mu}(\tau, q)$ by definition of $\mathcal{P}_{\mu}(\tau, q)$. Then, let $\mathbf{l}_2 \in L_2(q)$ be given by $\mathbf{l}_2 = \psi^{\mu}_{\tau,q}(w)$. By setting $z = \mathbf{x}(\tau, q, \mathbf{l}_2)$, it follows that:

$$\|w - z\| \leq \mu. \tag{8}$$

Since $\mathbb{R}^n \subseteq \bigcup_{q_2 \in Q_2} \mathcal{B}_{\eta}(q_2)$, there exists $p \in Q_2$ such that:

$$\|z - p\| \leq \eta. \tag{9}$$

Thus, $q \stackrel{\mathbf{l}_2}{\rightarrow}_2 p$ and since Σ is δ–GAS and by (7), (8), (9) and (6), the following chain of inequalities holds:

$$\begin{aligned} \|y - p\| &= \|y - v + v - w + w - z + z - p\| \\ &\leq \|y - v\| + \|v - w\| + \|w - z\| + \|z - p\| \\ &\leq \beta(||x - q||, \tau) + \|v - w\| + \|w - z\| + \|z - p\| \\ &\leq \beta(\varepsilon, \tau) + \mu + \mu + \eta \leq \varepsilon. \end{aligned}$$

We now show that condition (iii) holds as well. Consider any $\mathbf{l}_2 \in L_2$ and the transition $q \stackrel{\mathbf{l}_2}{\longrightarrow}_2 p$ in $T_{\tau,\eta,\mu}(\Sigma)$. By definition of $T_{\tau,\eta,\mu}(\Sigma)$ there exists $z \in Q_1$ such that $z = \mathbf{x}(\tau, q, \mathbf{l}_2)$ and $\|z - p\| \leq \eta$. Choose $\mathbf{u}_1 = \mathbf{l}_2$ and consider now the transition $x \stackrel{\mathbf{u}_1}{\longrightarrow}_1 y$ in $T_{\tau}(\Sigma)$. Since Σ is δ–GAS and by condition (6), the following chain of inequalities holds:

$$\begin{aligned} \|y - p\| &= \|y - z + z - p\| \leq \|y - z\| + \|z - p\| \\ &\leq \beta(\|x - q\|, \tau) + \|z - p\| \leq \beta(\varepsilon, \tau) + \eta \leq \varepsilon. \end{aligned}$$

Thus $(y, p) \in R$, which completes the proof. ■

This result represents a substantial improvement over previously known classes of control systems admitting symbolic models, which included output controllable linear systems in discrete–time [7] and stable linear systems in discrete–time [14]. Despite its conceptual importance, highlighting stability as a sufficient condition for the existence of symbolic models, Theorem 2 does not suggest how to construct such models. In the next section we address this issue by identifying input quantizations, leading to the desired symbolic models.

V. Digital Control Systems

In this section we specialize the results of the previous section to the case of digital control systems, i.e. control systems where control signals are piecewise–constant. In many man made systems, input signals are physically implemented as piecewise–constant signals. Our assumptions are then in consonance with real physical constraints. Moreover, input quantization can be seen as a very powerful complexity reduction mechanism, simplifying several control synthesis problems [12], [13]. In the following we suppose that the input space U of the control systems involved, contains the origin and it is a hyper rectangle, i.e. $U = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_m, b_m]$, for some $a_i < b_i, i = 1, 2, ..., m$. Given $\tau \in \mathbb{R}^+$, we now consider the class of constant inputs of duration τ, that is $\mathcal{U}_\tau = \{\mathbf{u} \in \mathcal{U} : \mathbf{u}(t) = \mathbf{u}(0), t \in [0, \tau]\}$. We denote by u the control input $\mathbf{u} \in \mathcal{U}_\tau$ for which $\mathbf{u}(t) = u$, $t \in [0, \tau]$. Let us denote by $T_{\mathcal{U}_\tau}(\Sigma)$ the sub–transition system of $T_\tau(\Sigma)$ where only control inputs in $\mathcal{U}_\tau$ are considered. More formally define:

$$T_{\mathcal{U}_\tau}(\Sigma) := (Q_1, L_1, \longrightarrow_1, O_1, H_1),$$

where:

- $Q_1 = \mathbb{R}^n$;
- $L_1 = U$;
- $q \xrightarrow{l}_1 p$ if there exists a trajectory $\mathbf{x}$ of Σ satisfying $\mathbf{x}(\tau, q, l) = p$;
- $O_1 = \mathbb{R}^n$;
- $H_1 = 1_{\mathbb{R}^n}$.

We now define a suitable symbolic model associated with transition system $T_{\mathcal{U}_\tau}(\Sigma)$. Given a control system Σ, any $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^+$, define the following transition system:

$$T_{\tau,\eta,\mu}(\Sigma) := (Q_2, L_2, \longrightarrow_2, O_2, H_2), \tag{10}$$

where:

- $Q_2 = [\mathbb{R}^n]_\eta$;
- $L_2 = [U]_\mu$;
- $q \xrightarrow{l}_2 p$ if $\|p - \mathbf{x}(\tau, q, l)\| \leq \eta$;
- $O_2 = \mathbb{R}^n$;
- $H_2 = \imath : Q_2 \longrightarrow O_2$.

Notice that transition system in (10) differs from transition system in (4), (only) in the way that we use for approximating control inputs. In particular, the choice of labels in transition system (10) is done without assuming the knowledge of reachable set associated with the control system, and therefore the construction of $T_{\tau,\eta,\mu}(\Sigma)$ is effective. We refer to [15] for a discussion on the computational issues related to the construction of the proposed symbolic model.

We are now able to give the following result that relates δ–ISS to the existence of symbolic models.

Theorem 3: Consider a control system Σ and any desired precision $\varepsilon \in \mathbb{R}^+$. If Σ is δ–ISS then for any $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$, and $\mu \in \mathbb{R}^+$ satisfying the following condition:

$$\beta(\varepsilon, \tau) + \gamma(\mu) + \eta \leq \varepsilon, \tag{11}$$

where β is a $\mathcal{KL}$ function and γ is a $\mathcal{K}_\infty$ function satisfying inequality (3), the transition system $T_{\mathcal{U}_\tau}(\Sigma)$ is ε–bisimilar to $T_{\tau,\eta,\mu}(\Sigma)$.

Before giving the proof of this result we point out that, analogously to conditions of Theorem 2, there always exist parameters τ, η, and μ satisfying condition (11). As pointed out in Section II-C it is difficult to find in general, a $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function γ satisfying inequality (3). However, once a δ–ISS Lyapunov function V for Σ has been found, an expression for the functions β and γ can be derived on the basis of V; we show this in the next section by means of an illustrative example.

Proof: Consider the relation $R \subseteq Q_1 \times Q_2$ defined by $(x, q) \in R$ if and only if $||x - q|| \leq \varepsilon$. By construction $R(Q_1) = Q_2$; by geometrical considerations on the infinity norm, $Q_1 \subseteq \bigcup_{q_2 \in [\mathbb{R}^n]_\eta} \mathcal{B}_\eta(q_2)$ and therefore, since by (11), $\eta < \varepsilon$, we have that $R^{-1}(Q_2) = Q_1$. We now show that R is an ε–approximate bisimulation relation between $T_\tau(\Sigma)$ and $T_{\tau,\eta,\mu}(\Sigma)$. Consider any $(x, q) \in R$. Condition (i) in Definition 5 is satisfied by the definition of R. Let us now show that condition (ii) in Definition 5 holds. Consider any $u_1 \in U$ and the transition $x \xrightarrow{u_1}_1 y$. Consider an input $l_2 \in L_2$ such that:

$$\|u_1 - l_2\| \leq \mu, \tag{12}$$

and set $z = \mathbf{x}(\tau, q, l_2)$. (Notice that such input $l_2 \in L_2$ exists because the assumptions on U make $[U]_\mu$ non–empty.) Since $Q_1 \subseteq \bigcup_{q_2 \in [\mathbb{R}^n]_\eta} \mathcal{B}_\eta(q_2)$, there exists $p \in Q_2$ such that:

$$\|z - p\| \leq \eta, \tag{13}$$

and $q \xrightarrow{l_2}_2 p$. Since Σ is δ–ISS and by (11), (12) and (13), the following chain of inequalities holds:

$$\begin{aligned} \|y - p\| &= \|y - z + z - p\| \leq \|y - z\| + \|z - p\| \\ &\leq \beta(\|x - q\|, \tau) + \gamma(\|u_1 - l_2\|_\infty) + \eta \\ &\leq \beta(\varepsilon, \tau) + \gamma(\mu) + \eta \leq \varepsilon. \end{aligned}$$

Hence $(y, p) \in R$ and condition (ii) in Definition 5 holds. We now show that condition (iii) holds as well. Consider any $l_2 \in L_2$ and the transition $q \xrightarrow{l_2}_2 p$ in $T_{\tau,\eta,\mu}(\Sigma)$. By definition of $T_{\tau,\eta,\mu}(\Sigma)$:

$$\|z - p\| \leq \eta, \tag{14}$$

where $z = \mathbf{x}(\tau, q, l_2)$. Consider now the transition $x \xrightarrow{l_2}_1 y$ in $T_{\mathcal{U}_\tau}(\Sigma)$. Since Σ is δ–ISS and by (11) and (14), the following chain of inequalities holds:

$$\begin{aligned} \|y - p\| &= \|y - z + z - p\| \leq \|y - z\| + \|z - p\| \\ &\leq \beta(\|x - q\|, \tau) + \gamma(\|l_2 - l_2\|_\infty) + \eta \\ &\leq \beta(\varepsilon, \tau) + \eta \leq \varepsilon. \end{aligned}$$

Thus $(y, p) \in R$, which completes the proof. ■

VI. A SIMPLE EXAMPLE

Consider a control system:

$$\Sigma : \begin{cases} \dot{x} = f(x, u) \\ x \in \mathbb{R}^2, u \in U \subseteq \mathbb{R}, \end{cases} \tag{15}$$

where $U = [-0.1, 0.1]$ and $f : \mathbb{R}^2 \times U \to \mathbb{R}^2$ is defined by:

$$f((x_1, x_2), u) = \begin{bmatrix} -2x_1 + x_2^2 - 7u & -2(1 + u^2)x_2 \end{bmatrix}'.$$

We work in the compact set $X = [-1, 1] \times [-1, 1]$. The set X is invariant for the control system Σ, i.e. $\mathbf{x}(t, x, \mathbf{u}) \in X$, for any $x \in X$, any $\mathbf{u} \in \mathcal{U}$, and any time $t \in \mathbb{R}_0^+$. Given a desired precision $\varepsilon \in \mathbb{R}^+$, the goal is to find suitable parameters $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^+$, so that transition system $T_{\tau,\eta,\mu}(\Sigma)$ as defined in (10) is ε–bisimilar to transition system $T_{\mathcal{U}_\tau}$. In order to find such parameters we can apply Theorem 3. We start by showing that the control system Σ defined by (15) is δ–ISS. Consider the function $V:\mathbb{R}^2\times\mathbb{R}^2\rightarrow\mathbb{R}_0^+$ defined by $V(x,y)=0.5\left((x_1-y_1)^2+(x_2-y_2)^2\right)$, for any $x=(x_1,x_2), y=(y_1,y_2)\in X$. Function V satisfies condition (i) of Definition 3, by choosing $\alpha_1(r)=0.5\,r^2$ and $\alpha_2(r)=r^2$. Moreover it can be shown that:

$$\frac{\partial V}{\partial x}f(x,u)+\frac{\partial V}{\partial y}f(y,v)\leq -\left\|x-y\right\|_2^2+14.8\left|u-v\right|. \quad (16)$$

Thus condition (iii) of Definition 3 is satisfied with $\rho(r)=r^2$ and $\sigma(r)=14.8\,r$ and therefore by Theorem 1, the control system Σ is δ–ISS. In order to apply Theorem 3, we need to find a $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function γ satisfying inequality (3). By inequality (16), the definition of V and the comparison lemma, the following inequalities hold for any $x,y\in X$, any $\mathbf{u},\mathbf{v}\in\mathcal{U}$ and any time $t\in\mathbb{R}_0^+$:

$$\begin{aligned}&\|\mathbf{x}(t,x,\mathbf{u})-\mathbf{x}(t,y,\mathbf{v})\|\leq\sqrt{2V(t)}\leq\\ &\sqrt{2e^{-2t}V(0)}+\sqrt{29.6\left(\int_0^\infty e^{-2\alpha}d\alpha\right)\|u-v\|_\infty}\leq\\ &\sqrt{2}e^{-t}\left\|x-y\right\|+\sqrt{14.8\|u-v\|_\infty}. \quad (17)\end{aligned}$$

Define $\beta(r,s):=\sqrt{2}e^{-s}r$ and $\gamma(r):=\sqrt{14.8\,r}$ for any $r,s\in\mathbb{R}$. Functions β and γ are respectively a $\mathcal{KL}$ function and a $\mathcal{K}_\infty$ function and by (17) they satisfy inequality (3). We can now apply Theorem 3. Condition (11) becomes:

$$\sqrt{2}e^{-\tau}\varepsilon+\sqrt{14.8\,\mu}+\eta\leq\varepsilon. \quad (18)$$

Set the precision $\varepsilon=0.5$ and choose $\eta=1/3$ and $\tau=5$; inequality (18) implies $\mu\leq 0.0017$ and therefore we can choose $\mu=0.001$. The resulting transition system:

$$T_{\tau,\eta,\mu}(\Sigma)=(Q_2,L_2,\underset{2}{\longrightarrow},O_2,H_2)$$

is defined by:

- $Q_2=\{-\eta,0,\eta\}\times\{-\eta,0,\eta\}$;
- $L_2=[U]_{0.001}$;
- $\underset{2}{\longrightarrow}$ is depicted in Figure 1;
- $O_2=X$;
- $H_2=\imath:Q_2\hookrightarrow O_2$.

VII. Discussion

This paper extends the results of [9], [10], from approximate simulation to approximate bisimulation. The key idea was to replace the assumption of asymptotic stabilizability of [9], [10] by the stronger notion of asymptotic stability. Note that for control systems with bounded inputs, which is the case in many realistic situations, even if a feedback control law rendering the closed–loop system δ–GAS were found, there is no guarantee that such feedback satisfies the input constraints. For the class of digital control systems, a symbolic model is proposed and based on a quantization of the control input space. The construction of this symbolic model is effective. We refer to [15], for a discussion on the computational issues arising in the construction of this symbolic model.

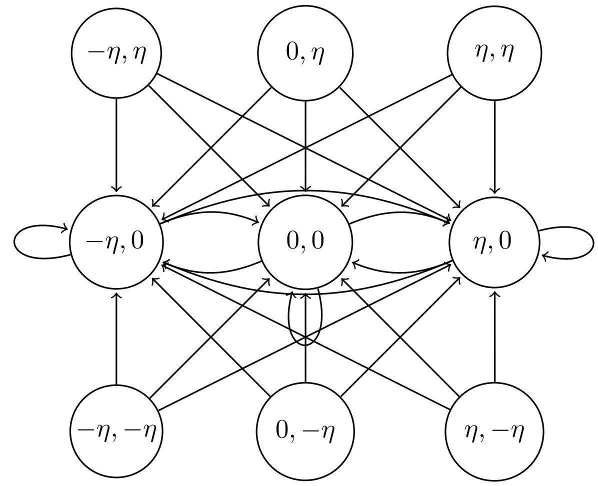

Fig. 1. Symbolic model $T_{5,1/3,10^{-3}}(\Sigma)$ associated with the control system Σ, as defined in (15). An arrow from a state q to a state p means that there exists *at least* one label l in L_2 so that $\mathbf{x}(5,q,l)$ is in $\mathcal{B}_{\eta 2}(p)$.

References

[1] R. Milner, *Communication and Concurrency*. Prentice Hall, 1989.

[2] D. Park, "Concurrency and automata on infinite sequences," ser. Lecture Notes in Computer Science, Springer-Verlag, Ed., vol. 104, 1981, pp. 167–183.

[3] P. Ramadge and W. Wonham, "Supervisory control of a class of discrete event systems," *SIAM Journal on Control and Optimization*, vol. 25, no. 1, pp. 206–230, 1987.

[4] W. Zielonka, "Infinite games on finitely coloured graphs with applications to automata on infinite trees," *Theoretical Computer Science*, vol. 200, pp. 135–183, 1998.

[5] A. Arnold, A. Vincent, and I. Walukiewicz, "Games for synthesis of controllers with partial observation," *Theoretical Computer Science*, vol. 28, no. 1, pp. 7–34, 2003.

[6] P. Tabuada and G. Pappas, "Linear Time Logic control of discrete-time linear systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 12, pp. 1862–1877, 2006.

[7] P. Tabuada, "Symbolic models for control systems," *Acta Informatica*, vol. 43, no. 7, pp. 477–500, February 2007, special Issue on Hybrid Systems.

[8] A. Girard and G. Pappas, "Approximation metrics for discrete and continuous systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 782–798, 2007.

[9] P. Tabuada, "Symbolic control of linear systems based on symbolic subsystems," *IEEE Transactions on Automatic Control, Special issue on symbolic methods for complex control systems*, vol. 51, no. 6, pp. 1003–1013, June 2006.

[10] ——, "Approximate simulation relations and finite abstractions of quantized control systems," in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, A. Bemporad, A. Bicchi, and G. Buttazzo, Eds. Berlin: Springer Verlag, 2007, vol. 4416, pp. 529–542.

[11] D. Angeli, "A Lyapunov approach to incremental stability properties," *IEEE Transactions on Automatic Control*, vol. 47, no. 3, pp. 410–421, 2002.

[12] A. Bicchi, A. Marigo, and B. Piccoli, "On the rechability of quantized control systems," *IEEE Transaction on Automatic Control*, April 2002.

[13] ——, "Feedback encoding for efficient symbolic control of dynamical systems," *IEEE Transaction on Automatic Control*, vol. 51, no. 6, pp. 987– 1002, June 2006.

[14] A. Girard, "Approximately bisimilar finite abstractions of stable linear systems," in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, A. Bemporad, A. Bicchi, and G. Buttazzo, Eds. Berlin: Springer Verlag, 2007, vol. 4416, pp. 231–244.

[15] G. Pola, A. Girard, and P. Tabuada, "Approximately bisimilar symbolic models for nonlinear control systems," 2007, available at arXiv:0706.0246v2 [math.OC].