Optimal Control of Linear Systems :
A Multiresolution Approach.

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***Abstract*— This paper deals with the optimal control of continuous-time linear systems with state-space constraints. Recently, for linear systems, with discrete-time constraints, it has been shown that the optimal control can be computed by solving a finite dimensional convex quadratic problem. The optimal trajectory belongs to a class of curves called control theoretic splines. In this paper, we consider the optimal control problem with continuous-time constraints. The method proposed here combines the notion of control theoretic splines with multiresolution approximation theory used for some years in the area of signal processing. This work results in a multiresolution approach to linear control which allows the computation of an approximation of the optimal input with great accuracy. Moreover, the associated output satisfies the state-space constraints at all time.**

I. INTRODUCTION

Recently, the relationship between splines and linear control theory has been analyzed [12], resulting in a new class of spline functions : the control theoretic splines. This class of functions has good properties to solve a wide range of continuous-time optimal control problems [4], [5]. Particularly, the optimal trajectory of a continuous-time linear systems with discrete-time state-space constraints (the interval interpolation problem [4]) is a control theoretic spline. In this paper, we consider continuous-time state-space constraints and therefore the optimal trajectory should be the limit of a control theoretic spline.

A conveniant framework to study functions defined by finer and finer grids of samples (given here by the discretization of the state-space constraints) is the theory of multiresolution approximation [2], [9]. The multiscale approach in functional analysis was introduced at the beginning of the 20th century by Haar [7] in order to solve problems that could not be handled by the Fourier transform : analysis of the smoothness and of the local properties of a function. The achievement of this approach came with the introduction of multiresolution approximation by Meyer and Mallat [8], [10].

The basic idea is the following, the approximation of a function f of $L^2(0,1)$ at a resolution 2^N (or at a scale N) is given by a grid of samples which provides local averages of f on intervals of size 2^{-N}. Thus, a multiresolution approximation consists of embedded grids of approximation. More formally, a multiresolution approximation is composed of a sequence $\{V_N\}$ of embedded subspaces of $L^2(0,1)$. The space V_N regroups all the possible approximations at the resolution 2^N. The approximation of the function f at the resolution 2^N is defined as the orthogonal projection on the space V_N.

In this paper, we propose an approach for solving an optimal control problem of continuous-time linear system with state-space constraints using control theoretic splines. This work leads to a generalization of multiresolution approximation to linear control theory.

II. OVERVIEW OF THE MULTIRESOLUTION APPROACH TO OPTIMAL CONTROL

Let us consider the following optimal control problem :

$$\text{Minimize} \int_0^1 u^T(t)u(t)dt$$

$$\text{under} \left\{ \begin{array}{l} x'(t) = Ax(t) + Bu(t),\ x(0) = x_0 \\ c_i^T x(t) - d_i \geq 0,\ i \in \{1 \dots p\} \end{array} \right. \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, $c_i \in \mathbb{R}^{n_x}$, $d_i \in \mathbb{R}$, and A, B are matrices of compatible dimensions. We will assume that the pair (A, B) is controllable. For simplicity, let us assume that there is only one state-space constraint (i.e. $c^T x(t) - d \geq 0$).

The complexity of the problem lies in its continuous-time nature. A reasonnable way to compute an approximate solution is to discretize the constraint.

$$\text{Minimize} \int_0^1 v_N^T(t)v_N(t)dt$$

$$\text{under} \left\{ \begin{array}{l} y_N'(t) = Ay_N(t) + Bv_N(t),\ y_N(0) = x_0 \\ c^T y_N(\frac{q+1}{2^N}) - d \geq 0,\ q \in \{0 \dots 2^N - 1\} \end{array} \right. . \tag{2}$$

A similar approach has been studied in [4]. The solution of (2) is a piecewise continuous function of the $2^N n_x$ dimensional subspace of $L^2(0,1)^{n_u}$:

$$V_N = \left\{ B^T e^{-A^T t} f_N(t) \,\middle|\, \begin{array}{l} f_N : [0,1] \to \mathbb{R}^{n_x},\ \text{constant} \\ \text{on each } [\frac{q}{2^N}, \frac{q+1}{2^N}) \end{array} \right\} \tag{3}$$

Moreover, it can be computed by solving a finite dimensional convex quadratic programming problem. The resulting output belongs to the class of control theoretic splines associated to the pair (A, B). The main drawback of this approach lies in the fact that, in general, the resulting output does not satisfy the constraint for all t in $[0, 1]$.

In this paper, we present an approach that allows to compute an approximate solution of the problem (1) such that the associated output satisfies the state-space constraint

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on the entire interval $[0,1]$. The main idea is the following. Two steps are needed for the synthesis of the approximate optimal input u_N. The first step consists in solving the problem (2), that is to compute the input v_N. The associated output y_N is a good approximation of the optimal trajectory x though it does not satisfy the constraint at all time. Therefore, as a second step, we add an other input w_N, *as small as possible*, such that, the solution of the differential equation

$$z'_N(t) = Az_N(t) + Bw_N(t),\ z_N(0) = 0 \qquad (4)$$

satisfies

$$\begin{cases} z_N(\frac{q+1}{2^N}) = 0, & q \in \{0 \dots 2^N - 1\} \\ c^T z_N(t) \geq d - c^T y_N(t). & \end{cases}$$

Thus, the trajectory $x_N = y_N + z_N$ associated to the input $u_N = v_N + w_N$ has the same global behaviour than y_N (it has the same value at the discretization time), and satisfies the state-space constraint on the entire interval $[0,1]$.

For computational tractability, we need to define a family of $L^2(0,1)^{n_u}$:

$$\mathcal{B} = \left\{\bar{u}^l, u^l_{j,k}, j \in \mathbb{N}, k \in \{0 \dots 2^j - 1\}, l \in \{1 \dots n_x\}\right\} ,$$

with the associated output functions :

$$\begin{array}{rcl} \bar{x}^l(t) & = & e^{At}\int_0^t e^{-As}B\bar{u}^l(s)ds \\ x^l_{j,k}(t) & = & e^{At}\int_0^t e^{-As}Bu^l_{j,k}(s)ds. \end{array}$$

If an input can be written as a linear combination of elements of $\mathcal{B}$,

$$u(t) = \sum_{l=1}^{l=n_x} \bar{\alpha}^l \bar{u}^l(t) + \sum_{j=0}^{j=\infty} \sum_{k=0}^{k=2^j-1} \sum_{l=1}^{l=n_x} \alpha^l_{j,k} u^l_{j,k}(t)$$

the associated output is

$$x(t) = e^{At}x_0 + \sum_{l=1}^{l=n_x} \bar{\alpha}^l \bar{x}^l(t) + \sum_{j=0}^{j=\infty} \sum_{k=0}^{k=2^j-1} \sum_{l=1}^{l=n_x} \alpha^l_{j,k} x^l_{j,k}(t) .$$

We want to compute our approximate optimal input as a linear combination of the elements of the set $\mathcal{B}$. Therefore, these functions have to satisfy some specific properties.

First of all, we must be able to write the solution of the discretized optimal control problem (2) as a linear combination of the first $2^N n_x$ elements of $\mathcal{B}$:

Property 1 (Multiresolution): Let V_N be defined as in equation (3), for all $N \in \mathbb{N}$, the set of functions

$$\left\{\bar{u}^l, u^l_{j,k}, j \leq N-1, k \in \{0 \dots 2^j - 1\}, l \in \{1 \dots n_x\}\right\}$$

is a basis of V_N.

Particularly [4], this means that the set of control theoretic splines on the subdivision of $[0,1]$ of size 2^{-N} associated to the pair (A,B) is generated by the set of output functions :

$$\left\{\bar{x}^l, x^l_{j,k}, j \leq N-1, k \in \{0 \dots 2^j - 1\}, l \in \{1 \dots n_x\}\right\} .$$

This property will be useful for the first step of our method, allowing to compute v_N by solving a convex quadratic programming problem of dimension $2^N n_x$. The first step of our method consists in computing the coefficients associated to the first $2^N n_x$ functions of $\mathcal{B}$, which give the global behaviour of the optimal trajectory.

During the second step, we compute the input w_N which allows the approximate optimal trajectory x_N to satisfy the state-space constraint on $[0,1]$. In order to keep what was already computed unchanged, we impose that w_N is a linear combination of the set of functions

$$\left\{u^l_{j,k}, j \geq N, k \in \{0 \dots 2^j - 1\}, l \in \{1 \dots n_x\}\right\} .$$

Particularly, this means that we must have

Property 2 (Interpolation): $\forall N \in \mathbb{N}$, $j \geq N$,

$$x^l_{j,k}(\tfrac{q+1}{2^N}) = 0,$$
$$k \in \{0 \dots 2^N - 1\},\ l \in \{1 \dots n_x\},\ q \in \{0 \dots 2^N - 1\}.$$

Moreover, for computational tractability, it would be conveniant if the synthesis of w_N on the different intervals $[\frac{q}{2^N}, \frac{q+1}{2^N}]$ could be made independently. Indeed, y_N may satisfy the state-space constraint on many intervals of the subdivision, and therefore on these intervals, $w_N(t) = 0$ is enough to satisfy the constraint; therefore, we impose that at any scale j and for any index l the output functions $x^l_{j,k}(t)$ have disjoint supports.

Property 3 (Local influence): $\forall j \in \mathbb{N}, l \in \{1 \dots n_x\}$,

$$\bigcap_{k=0}^{k=2^j-1} \left\{t \in [0,1] \mid x^l_{j,k}(t) \neq 0\right\} = \emptyset .$$

In the next section, we give a theorem which shows that such a family exists and that it is an orthonormal basis of $L^2(0,1)^{n_u}$.

III. A MULTIRESOLUTION BASIS FOR LINEAR CONTROL

From now on, we assume that $n_x \geq n_u$ and that $\text{rank}(B) = n_u$. First of all, since the pair (A,B) is controllable, let us remark that the matrix

$$M(t) = \int_0^t e^{-As}BB^Te^{-A^Ts}ds$$

is invertible [12]. We define the sequence $\{M_j\}$ of matrices such that $M_j = M(\frac{1}{2^{j+1}})$.

Lemma 1: The matrix M_{-1} is positive-definite symmetric. For all $j \in \mathbb{N}$, the matrices M_j and $M_j + M_j e^{A^T/2^{j+1}} M_j^{-1} e^{A/2^{j+1}} M_j$ are positive-definite symmetric.

Proof: By its structure, M_j is positive symmetric. Moreover, it is regular. Hence, it is positive-definite symmetric. It is clear that M_j positive-definite symmetric implies $M_j + M_j e^{A^T/2^{j+1}} M_j^{-1} e^{A/2^{j+1}} M_j$ positive-definite symmetric. ■

Thus, there exist orthonormal bases of $\mathbb{R}^{n_x}$ composed of eigenvectors of these matrices. The following theorem states that there exists an orthonormal basis of $L^2(0,1)^{n_u}$ satisfying the three properties. The construction of the basis as well as the proof of the theorem can be found in [6].

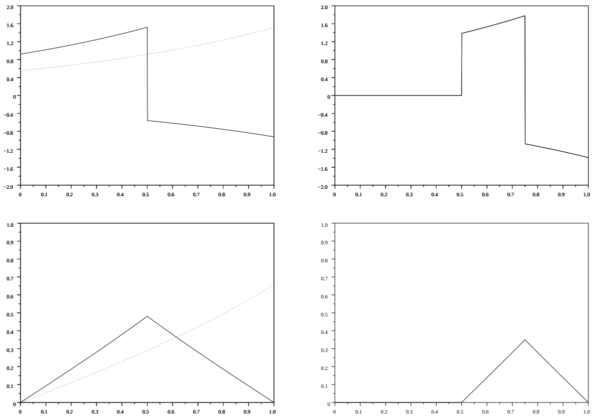

Fig. 1. The functions $\bar{u}$, $u_{0,0}$ (top left) and $u_{1,1}$ (top right) and the associated outputs $\bar{x}$, $x_{0,0}$ (bottom left) and $x_{1,1}$ (bottom right) for a scalar system ($a = -1$, $b = 1$).

Theorem 1: There exists an orthonormal basis $\mathcal{B}$ of $L^2(0,1)^{n_u}$ satisfying properties 1, 2 and 3. It is made up of the following functions :

$$\bar{u}^l(t) = B^T e^{-A^T t}\bar{f}^l \ ,$$

$$u^l_{j,k}(t) = \begin{cases} 0 & \text{on } \left[0, \frac{k}{2^j}\right) \\ B^T e^{A^T(\frac{k}{2^j}-t)} f^l_j & \text{on } \left[\frac{k}{2^j}, \frac{2k+1}{2^{j+1}}\right) \\ B^T e^{A^T(\frac{k}{2^j}-t)} g^l_j & \text{on } \left[\frac{2k+1}{2^{j+1}}, \frac{k+1)}{2^j}\right) \\ 0 & \text{on } \left[\frac{k+1}{2^j}, 1\right) \end{cases}$$

$$j \in \mathbb{N}, k \in \{0 \dots 2^j - 1\}, l \in \{1 \dots n_x\}$$

with

$$\bar{f}^l = \frac{1}{\sqrt{\lambda^l_{-1}}} e^l_{-1}$$

where e^l_{-1} is an element of the orthonormal basis composed of eigenvectors of M_{-1} and λ^l_{-1} is the associated eigenvalue.

$$f^l_j = \frac{1}{\sqrt{\lambda^l_j}} e^l_j \ ,$$

$$g^l_j = \frac{1}{\sqrt{\lambda^l_j}} (e^l_j - \lambda^l_j M_j^{-1} e^l_j) \ ,$$

where e^l_j is an element of the orthonormal basis composed of eigenvectors of $M_j + M_j e^{A^T/2^{j+1}} M_j^{-1} e^{A/2^{j+1}} M_j$ and λ^l_j is the associated eigenvalue.

On figure 1, some functions of $\mathcal{B}$ and the associated ouputs are represented for the scalar system $x'(t) = -x(t) + u(t)$. In the next section, we show how the basis $\mathcal{B}$ can be used to solve the optimal control problem (1).

IV. SOLVING THE OPTIMAL CONTROL PROBLEM

The synthesis of an approximate solution of the problem (1) consists of two steps. In the next part, we describe the synthesis of the solution of the discretized problem (2).

A. Computing the Solution of the Discretized Problem

In [4], it is shown that the solution v_N of problem (2) is an element of V_N. Hence, we can write :

$$v_N(t) = \sum_{l=1}^{l=n_x} \bar{\alpha}^l \bar{u}^l(t) + \sum_{j=0}^{j=N-1} \sum_{k=0}^{k=2^j-1} \sum_{l=1}^{l=n_x} \alpha^l_{j,k} u^l_{j,k}(t)$$

and the associated output

$$y_N(t) = e^{At}x_0 + \sum_{l=1}^{l=n_x} \bar{\alpha}^l \bar{x}^l(t) + \sum_{j=0}^{j=N-1} \sum_{k=0}^{k=2^j-1} \sum_{l=1}^{l=n_x} \alpha^l_{j,k} x^l_{j,k}(t) \ .$$

For more tractability, let us number the discretization points in a multiresolution manner :

$$t_{i,p} = \frac{1}{2^{i+1}} + \frac{p}{2^i}, \ i \in \{0 \dots N-1\}, \ p \in \{0 \dots 2^i - 1\}.$$

Lemma 2: For all $i \in \{0 \dots N-1\}$, $p \in \{0 \dots 2^i - 1\}$,

$$\begin{array}{rcl} y_N(t_{i,p}) & = & e^{At_{i,p}}x_0 + \sum_{l=1}^{l=n_x} \bar{\alpha}^l \bar{x}^l(t_{i,p}) \\ & & + \sum_{j=0}^{j=i} \sum_{l=1}^{l=n_x} \alpha^l_{j,\lfloor 2^j t_{i,p} \rfloor} x^l_{j,\lfloor 2^j t_{i,p} \rfloor}(t_{i,p}), \\ y_N(1) & = & e^A x_0 + \sum_{l=1}^{l=n_x} \bar{\alpha}^l \bar{x}^l(1) \ . \end{array}$$

Proof: From property 2, it is clear that the only output functions which do not vanish at $t = 1$ are $\bar{x}^l$, for all $l \in \{1 \dots n_x\}$. Consequently, the second equality is proved. Let $i \in \{0 \dots N-1\}$, $p \in \{0 \dots 2^i - 1\}$, from property 2, for all $j \geq i+1$, we have $x^l_{j,k}(t_{i,p}) = 0$. Thus,

$$\begin{array}{rcl} y_N(t_{i,p}) & = & e^{At_{i,p}}x_0 + \sum_{l=1}^{l=n_x} \bar{\alpha}^l \bar{x}^l(t_{i,p}) \\ & & + \sum_{j=0}^{j=i} \sum_{k=0}^{2^j-1} \sum_{l=1}^{l=n_x} \alpha^l_{j,k} x^l_{j,k}(t_{i,p}). \end{array}$$

Moreover, from theorem 1, it is clear that the support of $x^l_{j,k}$ is $(\frac{k}{2^j}, \frac{k+1}{2^j})$. Therefore, $x^l_{j,k}(t_{i,p}) \neq 0$ only if $\frac{k}{2^j} < t_{i,p} < \frac{k+1}{2^j}$, which implies $k = \lfloor 2^j t_{i,p} \rfloor$. ■

Using lemma 2, and the fact that the basis is orthonormal, the problem (2) is equivalent to

$$\begin{array}{ll} \underset{\bar{\alpha}^l, \alpha^l_{j,k}}{\text{Minimize}} & \sum_{l=1}^{l=n_x} (\bar{\alpha}^l)^2 + \sum_{j=0}^{j=N-1} \sum_{k=0}^{k=2^j-1} \sum_{l=1}^{l=n_x} (\alpha^l_{j,k})^2 \end{array}$$

$$\text{under} \begin{cases} \sum_{l=1}^{l=n_x} \bar{\alpha}^l c^T \bar{x}^l(1) - \left(d - c^T e^A x_0\right) \geq 0 \\ \sum_{l=1}^{l=n_x} \bar{\alpha}^l c^T \bar{x}^l(t_{i,p}) \\ + \sum_{j=0}^{j=i} \sum_{l=1}^{l=n_x} \alpha^l_{j,\lfloor 2^j t_{i,p} \rfloor} c^T x^l_{j,\lfloor 2^j t_{i,p} \rfloor}(t_{i,p}) \\ - \left(d - c^T e^{At_{i,p}} x_0\right) \geq 0 \\ i = \in \{0 \dots N-1\}, \ p \in \{0 \dots 2^i - 1\} \ . \end{cases} \tag{5}$$

Thus, the input v_N can be computed by solving a $2^N n_x$ dimensional quadratic programming problem. Note that the constraints have a sparse structure. Indeed, we have 2^N constraints with $2^N n_x$ variables. However, due to the appropriate structure of the basis $\mathcal{B}$, the constraint matrix has only $(N2^N + 1)n_x$ non-zero coefficients. Therefore, problem (5) can be efficiently solved by using special solvers [11].

B. From Discretized Constraints to Continuous-Time Constraints

In the previous part, we synthesized an input v_N such that the associated output y_N satisfies the state-space constraint at given discretization points. For N sufficiently large, it is generally a good approximation of the solution of the continuous time problem. Unfortunately, it does not satisfy the constraint at all time.

In this part, we explain the synthesis of the second input w_N such that the output x_N associated to $v_N + w_N$ has the same global behaviour than y_N (it interpolates y_N at the discretization points) and such that x_N satisfies the constraint on the entire interval $[0, 1]$.

As we explained before, we do not want to modify the coefficients that were already computed during the first step of our method. Consequently, we search w_N in the orthogonal complementary of V_N, which is an infinite dimensional vector space. For obvious computational reasons, w_N can only be expressed within a finite basis. Hence, we first search w_N has a linear combination of the functions $u^l_{N,k}$:

$$w_N(t) = \sum_{k=0}^{k=2^N-1} \sum_{l=1}^{l=n_x} \alpha^l_{N,k} u^l_{N,k}(t)\ . \tag{6}$$

Since the outputs $x^l_{N,k}$ have disjoint supports, the synthesis of w_N can be made independently on each interval $[\frac{q}{2^N}, \frac{q+1}{2^N})$. For $q \in \{0 \dots 2^N - 1\}$, we define the open intervals :

$$I_{N,q} = \left(\tfrac{q}{2^N}, \tfrac{2q+1}{2^{N+1}}\right),\ J_{N,q} = \left(\tfrac{2q+1}{2^{N+1}}, \tfrac{q+1}{2^N}\right).$$

In the following, we need to evaluate the limit of the derivative of some output functions. For simplicity, for any output function o, $o'(\frac{q}{2^N})$ will denote the right-limit of $o'(t)$ at $t = \frac{q}{2^N}$ and $o'(\frac{q+1}{2^N})$ will denote the left-limit of $o'(t)$ at $t = \frac{q+1}{2^N}$. Thus, on the interval $I_{N,q}$,

$$\begin{array}{rcl} c^T x_N(t) & \geq & c^T x_N(\frac{q}{2^N}) + c^T x'_N(\frac{q}{2^N})(t - \frac{q}{2^N}) \\ & & - \sup_{I_{N,q}} |c^T x''_N(t)| \frac{(t-\frac{q}{2^N})^2}{2}\ . \end{array}$$

Using $x_N = y_N + z_N$, the fact that $z_N(\frac{q}{2^N}) = 0$ and

$$z_N(t) = \sum_{l=1}^{l=n_x} \alpha^l_{N,q} x^l_{N,q}(t) \text{ on } I_{N,q},$$

we obtain

$$\begin{array}{l} c^T x_N(t) \geq c^T y_N(\frac{q}{2^N}) + \\ \left[c^T y'_N(\frac{q}{2^N}) + \sum_{l=1}^{l=n_x} \alpha^l_{N,q} c^T x'^l_{N,q}(\frac{q}{2^N})\right] (t - \frac{q}{2^N}) - \\ \left[\sup_{I_{N,q}} |c^T y''_N(t)| + \sum_{l=1}^{l=n_x} |\alpha^l_{N,q}| \sup_{I_{N,q}} |c^T x''^l_{N,q}(t)|\right] \frac{(t-\frac{q}{2^N})^2}{2} \end{array}$$

The main idea of the second step of our method is the following : if the right-hand side of the inequality is greater than d on $I_{N,q}$, then x_N satisfies the state-space constraint on $I_{N,q}$. Since the right-hand side is a second order polynomial with a negative quadratic term, it is concave. Moreover, for $t = \frac{q}{2^N}$, its value is $c^T y_N(\frac{q}{2^N})$ which is greater than d, therefore the right-hand side of the inequality is greater than d on $I_{N,q}$ if and only if its value at the point $\frac{2q+1}{2^{N+1}}$ is greater than d. Hence,

$$\begin{array}{l} d \leq c^T y_N(\frac{q}{2^N}) + \\ \left[c^T y'_N(\frac{q}{2^N}) + \sum_{l=1}^{l=n_x} \alpha^l_{N,q} c^T x'^l_{N,q}(\frac{q}{2^N})\right] 2^{-N-1} - \\ \left[\sup_{I_{N,q}} |c^T y''_N(t)| + \sum_{l=1}^{l=n_x} |\alpha^l_{N,q}| \sup_{I_{N,q}} |c^T x''^l_{N,q}(t)|\right] 2^{-2N-3} \end{array} \tag{7}$$

is enough to insure that $c^T x_N(t) \geq d$ on the entire interval $I_{N,q}$. Using similar considerations, we can show that

$$\begin{array}{l} d \leq c^T y_N(\frac{q+1}{2^N}) - \\ \left[c^T y'_N(\frac{q+1}{2^N}) + \sum_{l=1}^{l=n_x} \alpha^l_{N,q} c^T x'^l_{N,q}(\frac{q+1}{2^N})\right] 2^{-N-1} - \\ \left[\sup_{J_{N,k}} |c^T y''_N(t)| + \sum_{l=1}^{l=n_x} |\alpha^l_{N,q}| \sup_{J_{N,q}} |c^T x''^l_{N,q}(t)|\right] 2^{-2N-3} \end{array} \tag{8}$$

guarantees that $c^T x_N(t) \geq d$ on $J_{N,q}$.

Thus, we choose w_N as small as possible such that it satisfies those two inequalities. Therefore, for q in $\{0 \dots 2^N - 1\}$, the coefficients $\alpha^l_{N,q}$ have to be the solution of the optimization problem :

$$\underset{\alpha^l_{N,q}}{\text{Minimize}} \quad \sum_{l=1}^{l=n_x} (\alpha^l_{N,q})^2$$

$$\text{under} \left\{ \begin{array}{l} \sum_{l=1}^{l=n_x} \alpha^l_{N,q} c^T B B^T f^l_N \\ -|\alpha^l_{N,q}| 2^{-N-2} \sup_{I_{N,q}} |c^T x''^l_{N,q}(t)| \geq m^1_{n,q} \\ \sum_{l=1}^{l=n_x} -\alpha^l_{N,q} c^T B B^T e^{A^T/2^N} g^l_N \\ -|\alpha^l_{N,q}| 2^{-N-2} \sup_{J_{N,q}} |c^T x''^l_{N,q}(t)| \geq m^2_{n,q} \end{array} \right.$$

where

$$m^1_{n,q} = 2^{N+1}(d - c^T y_N(\frac{q}{2^N})) - c^T y'_N(\frac{q}{2^N}) + \frac{\sup_{I_{N,q}} |c^T y''_N(t)|}{2^{N+2}}$$

$$m^2_{n,q} = 2^{N+1}(d - c^T y_N(\frac{q+1}{2^N})) + c^T y'_N(\frac{q+1}{2^N}) + \frac{\sup_{J_{N,q}} |c^T y''_N(t)|}{2^{N+2}}$$

Using a classical trick of optimization theory, we introduce the auxiliary variables $\beta^l_{N,q} = |\alpha^l_{N,q}|$. Finally, we arrive to the following result.

Proposition 1: Let us consider the 2^N following quadratic programming problems indexed by q :

$$\underset{\alpha^l_{N,q},\, \beta^l_{N,q}}{\text{Minimize}} \quad \sum_{l=1}^{l=n_x} (\beta^l_{N,q})^2$$

$$\text{under} \left\{ \begin{array}{l} \sum_{l=1}^{l=n_x} \alpha^l_{N,q} c^T B B^T f^l_N \\ -\beta^l_{N,q} 2^{-N-2} \sup_{I_{N,q}} |c^T x''^l_{N,q}(t)| \geq m^1_{n,q} \\ \sum_{l=1}^{l=n_x} -\alpha^l_{N,q} c^T B B^T e^{A^T/2^N} g^l_N \\ -\beta^l_{N,q} 2^{-N-2} \sup_{J_{N,q}} |c^T x''^l_{N,q}(t)| \geq m^2_{n,q} \\ \beta^l_{N,q} - \alpha^l_{N,q} \geq 0\ , \quad l \in \{1 \dots n_x\} \\ \beta^l_{N,q} + \alpha^l_{N,q} \geq 0\ , \quad l \in \{1 \dots n_x\} \end{array} \right. \tag{9}$$

Then,

$$w_N(t) = \sum_{k=0}^{k=2^N-1} \sum_{l=1}^{l=n_x} \alpha^l_{N,k} u^l_{N,k}(t)$$

is the smallest element of $L^2(0,1)^{n_u}$ of the form (6) satisfying inequalities (7) and (8).

The output x_N associated to the input $u_N = v_N + w_N$ satisfies the state-space constraints on the entire interval $[0, 1]$ and interpolates the optimal output y_N of problem (2) at the points of discretization.

Thus, w_N can be computed by solving 2^N small quadratic programming problems. It might happen that one of these has no solution. In this case, we choose the values $\alpha^l_{N,q}$ such that $c^T x_N(\frac{2q+1}{2^{N+1}}) \geq d$. Afterwards, we add to w_N a linear combination of the functions $u^l_{N+1,2q}$, $u^l_{N+1,2q+1}$ in order to satisfy the state-space constraints on the whole interval. This can be done by solving 2 quadratic programming problems similar to (9).

However, in some cases, we can not find any input function w_N such that the output associated to $v_N + w_N$ satisfies the state-space constraints. For example, this happens when $c^T B = 0$. Indeed, in that case, if $c^T y_N(t) - d$ vanishes at a discretization point and if $c^T y'_N(t)$ is negative at this point, then, it is impossible to find a function w_N such that $x_N(t)$ satisfies the state-space constraints. To solve this kind of problems, we have to regroup both optimization problems (5) and (9) in one big problem which is much more complicated to solve.

V. EXAMPLES

We implemented our method in the free scientific software package scilab. We experimented it on various examples. We show here two of them which seem illustrative.

A. Numerical Experiments

We applied our method to the scalar system

$$x'(t) = -x(t) + u(t),\ x(0) = 1$$

with the state-space constraints $x(t) \geq 0.8$. Let u be the optimal input of the control problem. Since u satisfies the constraints of the problem (2) and u_N satisfies those of the problem (1), it is clear that

$$\forall N \in \mathbb{N},\ ||v_N||_{L^2} \leq ||u||_{L^2} \leq ||u_N||_{L^2}\ .$$

Thus, the value of $||u_N||_{L^2} - ||v_N||_{L^2}$ gives an idea of the quality of approximation of the optimal input. The results are shown in table 1.

We can see that computational time increases rapidly as N becomes bigger. However, for N relatively small, we obtain a very good approximation of the optimal input. Experimentally, the convergence is of order $O(2^{-2N})$, that is proportional to the square of the step of discretization.

TABLE I

QUALITY OF THE APPROXIMATION, COMPUTATION TIME IN SECONDS (PENTIUM III 1 GHZ, SCILAB).

N	3	4	5	6	7	8
Error	1.6e-3	3.5e-4	8.5e-5	2.1e-5	5.2e-6	1.3e-6
Cputime	0	0.01	0.02	0.04	0.24	3.58

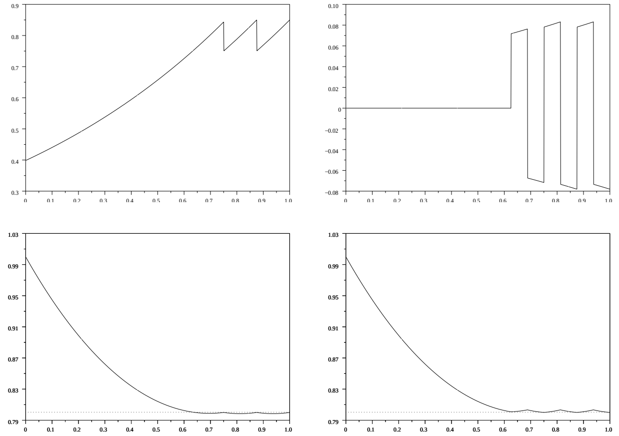

Fig. 2. The inputs v_3 (top left) and w_3 (top right) and the outputs y_3 (bottom left) and x_3 (bottom right) for a scalar system.

On figure 2, we represented some curves which we think representative. For $N = 3$, we show the optimal input v_3 of the discretized problem and the input w_3 added in order to satisfy the state-space constraints. Please note that the two plots are made at different scales, and that $w_3(t)$ is actually about ten times smaller than $v_3(t)$. We also show the outputs y_3 and x_3, we can see that they have the same values at the discretization points $k/8$, $k \in \{1 \ldots 8\}$. We also see that x_3 satisfies the state-space constraint at all time whereas y_3 does not.

B. Optimal Control of Switched Hybrid Systems

A similar method can be applied to the optimal control of linear switched hybrid systems with a priori known switchings. Here, we considered the following example :

$$x'(t) = \left\{ \begin{array}{l} A_1 x(t) + B_1 u(t),\ \text{for } t \in [0, 1) \\ A_2 x(t) + B_2 u(t),\ \text{for } t \in [1, 2] \end{array} \right.$$

where

$$A_1 = \begin{pmatrix} -0.1 & 0 \\ 0.5 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ -0.1 & 0.2 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} -0.1 & -0.5 \\ 0.5 & -0.1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.8 & 0 \\ 0.2 & 1 \end{pmatrix}.$$

We used our method to synthesize an approximate solution of the switched optimal control problem :

$$\text{Minimize } \int_0^2 u^T(t) u(t) dt$$

$$\text{under} \left\{ \begin{array}{lll} x^T(0) & = - & (1\ \ 1) \\ x^T(2) & = & (1\ \ 1) \\ (1\ \ 0).x(t) + 1.5 & \geq & 0,\ \text{for } t \in [0, 1) \\ (0\ \ 1).x(t) - 0.5 & \geq & 0,\ \text{for } t \in [1, 2]\ . \end{array} \right.$$

The main steps of the extension of our method to switched hybrid systems is the following. First, we define on the interval $[0, 1)$ the basis of $L^2(0, 1)^{n_u}$ associated to the first system and on $[1, 2]$ the one associated to the second system.

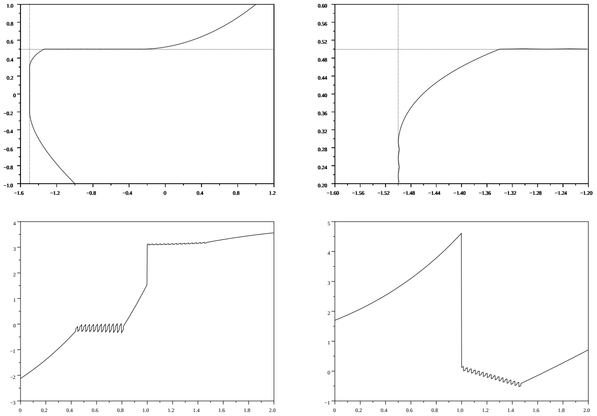

Fig. 3. The approximate optimal output x_5 (top) and both componants of the input u_5 (bottom) of the switched optimal control problem.

The union of these two families constitutes an orthonormal basis of $L^2(0,2)^{n_u}$.

Then, we choose a discretization step 2^{-N}, and solve a discretized version of the switched optimal control problem using the $2^N n_x$ first elements of both bases. Afterwards, we solve 2^{N+1} quadratic programming problems similar to (9), in order to satisfy the constraints at all time.

For our example, we chose $N = 5$, consequently the approximate optimal input u_N is coded by 256 coefficients, moreover, within these coefficients only 122 are non zeros. This allows to code this input very efficiently.

The L^2-norm of u_N is 4.5876 while the one of the discretized problem solution is 4.5864. Therefore, the input u_N costs less than 0.03 percent more than the effective optimal input of the continuous problem.

On figure (3), we represented the approximate optimal input and output of the switched optimal control problem. On the top-right figure we can see that the state-space constraints are satisfied at all time.

VI. CONCLUSION

In this paper, we presented a tractable method for the synthesis of approximate solutions of linear optimal control problems. It is easily implementable and has good computational features. Its main advantage is that the outputs associated to these solutions satisfy the state-space constraints at all time. Therefore, it can be used for the synthesis of inputs of linear systems involving safety purpose.

We are currently working on the extension of this method to hybrid systems where the switching times are not known a priori. Perhaps, this could be done by borrowing techniques of discrete time piecewise affine systems [1] to solve a discretized version of the hybrid optimal control problem. Then, we could locally modify this trajectory in order to satisfy the continuous-time state-space constraints.

The second point of this paper is the introduction of multiresolution approximations for linear control. In that aim, we have used the concept of control theoretic smoothing splines [4], [5], [12]. This results in a basis of $L^2(0,1)^{n_u}$ which has ideal properties for computing, analysing and coding inputs of linear systems.

We are convinced that the multiresolution approach to linear control has many applications in various problems such as reconstruction of the input from noised output samples or optimal coding of input signals.

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