# COMPUTATION AND STABILITY ANALYSIS OF LIMIT CYCLES IN PIECEWISE LINEAR HYBRID SYSTEMS

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Abstract: Hybrid systems often exhibit periodic behaviours. Therefore, the computation and the stability analysis of hybrid limit cycles is an important problem. Recently, several techniques have been proposed for the stability analysis of periodic solutions. However, all of these assume that the cycle has previously been computed, which is not the easiest part of the work. In this paper, we present a method for the numerical computation of limit cycles in piecewise linear hybrid systems. It uses the concept of hybrid Poincaré map and is an extension of the algorithms existing for smooth dynamical systems. An example is treated in order to show the results that can be obtained in practice.

Keywords: hybrid systems, limit cycles, stability, Poincaré map

## 1. INTRODUCTION

Hybrid systems have become the standard modelisation tool for the systems involving interactions between continuous processes and a discrete automata. The applications are numerous in fields such as avionics, automotive industry or biology. The subclass of piecewise linear hybrid systems is doubtless one of the most used. Indeed, they have been used for a long time by engineers to approximate non-linear systems. Recent work has been realized on global geometric analysis (Pettit and Wellstead, 2000), reachability analysis (Asarin *et al.*, 2000), optimal control and Lyapunov stability (Johansson and Rantzer, 1998) ...

Many of these systems exhibit periodic behaviour (see e.g. (Rubensson *et al.*, 1998), (Hiskens, 2001*a*)). Therefore, mathematical tools for the computation and the stability analysis of these solutions need to be developped. Previous work mainly focused on localization (Matveev and Savkin, 2000) or stability of hybrid limit cycles. Methods using discrete time Lyapunov stability (Rubensson and Lennartson, 2000), trajectory sensitivity analysis (Hiskens and Pai, 2000), (Hiskens, 2001*b*) or some similar techniques (Simic *et al.*, 2002) have been proposed.

However, only a few work have been realized on the computation of limit cycles. In (Llibre *et al.*, 2002), non trivial symbolic equations were given for the computation of limit cycles in a planar piecewise linear dynamical systems with two zones. Even in this simple case, the use of numerical methods is needed in order to solve the equations. Consequently, it seems useful to propose some numerical methods that can hold more complicated systems. In (Guckenheimer and Meloon, 2000) and (Viswanath, 2001), efficient techniques were proposed for an accurate computation of limit cycles in continuous dynamical systems.

In this paper, we propose to extend these techniques to the computation of periodic solutions in piecewise linear hybrid systems. Our method lies on a generalization of the Poincaré map to hybrid systems. In a first part, we will present briefly this concept . Then, we will detail our method and finally we will apply it to the classical example of the two tank system.

## 2. POINCARÉ MAP

The computation of limit cycles is one the most important problem of the theory of dynamical systems and one of the most tractable tool for this is the Poincaré map. The principle is illustrated on figure 1.

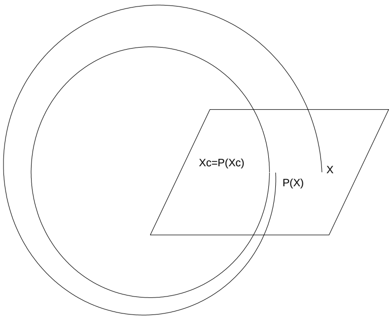

Fig. 1. Poincaré map

Let us consider a dynamical system with a limit cycle $\Gamma$. Let $\mathcal{S}$ be a hyperplane transversal to the cycle $\Gamma$ at a point $x_c$. We now define the Poincaré map $P$ on a neighborhood of $x_c$, to each point $x$ of $\mathcal{S}$ we associate the point $P(x)$ such that the trajectory of the dynamical system emanating from $x$ will cross $\mathcal{S}$ at $P(x)$ after approximatively one period of $\Gamma$.

Then, computing the limit cycle of the dynamical system is equivalent to find the fixed point of the Poincaré map. Moreover, if we are able to compute its differential $dP(x)$, Newton's method applied to $P$ provides an efficient algorithm for the computation of the limit cycle (see (Guckenheimer and Meloon, 2000), (Viswanath, 2001)).

In the next section, we will show that this concept can be generalized to piecewise linear hybrid systems.

## 3. COMPUTATION AND ANALYSIS OF HYBRID LIMIT CYCLES

In this paper, we consider the class of piecewise linear hybrid systems.

$$\begin{array}{l} \dot{x}(t) = A_{q(t)}x(t) + b_{q(t)} \\ q(t) \in T(x(t), q(t^-)) \end{array} \qquad (1)$$

where $x(t) \in \Re^N$ is the continuous variable of the system and $q(t) \in \mathcal{Q} = \{1, \ldots, n\}$ is the discrete state. In each discrete state, the vector $x(t)$ satisfies a linear differential equation. The switching procedure is given by a transition map $T$:

$$\begin{array}{lcl} i \in T(x,i) & \Longleftrightarrow & \forall j \neq i,\ x \notin S_{i,j} \\ \forall j \neq i,\ j \in T(x,i) & \Longleftrightarrow & x \in S_{i,j} \end{array} \qquad (2)$$

where $S_{i,j}$ are the switching planes

$$S_{i,j} = \{x \in \Re^N | k_{i,j}^t x - d_{i,j} = 0\} \qquad (3)$$

### *3.1 Notations and assumptions*

Let us assume that the system 1 has a periodic solution $(x^*(t), q^*(t))$ and that $x^*(0)$ is on a switching plane of the system. There exists a sequence of discrete states $\{q_1, \ldots, q_p\}$ and $p+1$ scalars $0 = t_0^* < t_1^* < \ldots < t_p^*$ such that :

$$\begin{cases} q^*(t) = q_i,\ \forall t \in [t_{i-1}^*, t_i^*) \\ x^*(t_p) = x^*(0) \end{cases} \qquad (4)$$

$\{q_i\}$ is the sequence of the successive discrete states of the periodic solution and $\{t_i^*\}$ is the associated sequence of switching times. We note,

$$x_i^* = x^*(t_i) \text{ and } s_i^* = t_i^* - t_{i-1}^* \ . \qquad (5)$$

We now have to make two assumptions.

*Assumption 1.* For all $i \in \{1, \ldots, p\}$, there exists a unique $q \in \mathcal{Q}$ such that $x_i^* \in S_{q_i,q}$.

*Assumption 2.* $x^*(t)$ does not reach any switching plane tangentially, i.e.

$$\begin{cases} k_{q_i,q_{i+1}}^t \ \ (A_{q_i}x_i^* + b_{q_i}) \neq 0,\ i \in \{1, \ldots, p-1\} \\ k_{q_p,q_1}^t \ \ (A_{q_p}x_p^* + b_{q_p}) \neq 0 \ . \end{cases} \qquad (6)$$

These two assumptions are directly related to the stability of the hybrid limit cycles. Indeed, if the first one does not hold, then at the time $t_i^*$ there are at least two possible transitions. We easily imagine that this can have dramatic consequences on the stability of the periodic solution. If the second one does not hold then a small perturbation on the initial condition $x^*(0)$, might involve dramatic changes of behaviour, indeed in this case the perturbated solution might not reach the switching plane and then the switching sequence will be different.

Under these two assumptions and due to the continuity of the flow in each discrete state, we can show that a trajectory of the hybrid system emanating from a point $x_0$ on the switching plane $S_{q_p,q_1}$ in a neighborhood of the point $x_0^*$ with $q_1$ as initial discrete state, will have the same switching sequence than the periodic solution.

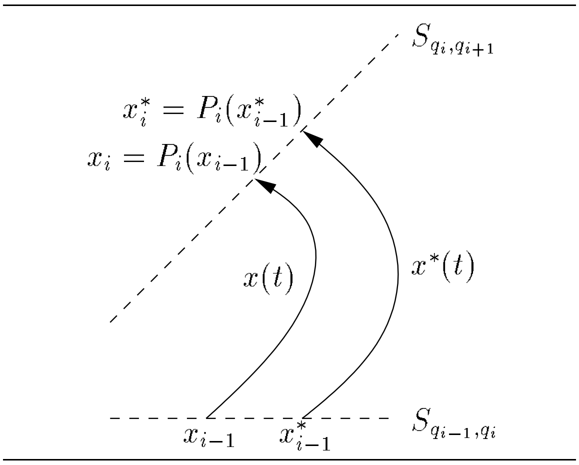

Fig. 2. Definition of the function $P_i$

### 3.2 Hybrid Poincaré map

Let $i$ be an element of $\{1, \ldots, p\}$, let us consider the initial value problem :

$$\dot{x}(t) = A_{q_i}x(t) + bq_i,\ x(0) = x_{i-1} \qquad (7)$$

where $x_{i-1}$ is in a neighborhood of $x_{i-1}^*$ on the same switching plane than $x_{i-1}^*$. We call $P_i$ the function that associates to the point $x_{i-1}$ the point $x_i$ which is the first intersection of the solution of 7 with the switching plane $S_{q_i,q_{i+1}}$ (see figure 2). Note that we have $P_i(x_{i-1}^*) = x_i^*$. This function has the following properties :

*Lemma 1.* $P_i$ is defined in a neighborhood of $x_{i-1}^*$. Moreover, it is continuous and differentiable. We note $x_i = P_i(x_{i-1})$, $s_i$ is the time that the solution of 7 takes to reach the point $x_i$. The expression of the derivative of $P_i$ is

$$dP_i(x_{i-1}) = \left(I - \frac{(A_{q_i}x_i + b_{q_i})k^t_{q_i,q_{i+1}}}{k^t_{q_i,q_{i+1}}(A_{q_i}x_i + b_{q_i})}\right) e^{s_i A_{q_i}} \qquad (8)$$

Now, let us consider a point $x_0$ in a neighborhood of $x_0^*$ and on the plane $S_{q_p,q_1}$. The solution $x(t)$ of the hybrid system 1 with initial value $(x_0, q_1)$, reaches the switching plane $S_{q_1,q_2}$ at the point $x_1 = P_1(x_0)$. Then the discrete state of the system becomes $q_2$, since $P_1$ is continuous on a neighborhood of $x_0^*$ then $x_1$ is as close to $x_1^*$ as we want. Consequently, $x(t)$ reaches the switching plane $S_{q_2,q_3}$ at the point $x_2 = P_2(x_1) = P_2oP_1(x_0)$ and so on...
Finally, $x(t)$ comes back on the plane $S_{q_p,q_1}$ at the point $x_p = P_poP_{p-1}o \ldots oP_1(x_0)$. Therefore, the function

$$P = P_poP_{p-1}o \ldots oP_1 \qquad (9)$$

is the Poincaré map associated to the plane $S_{q_p,q_1}$. Moreover we have the following result.

*Theorem 1.* $P$ is defined in a neighborhood of $x_0^*$. Moreover it is continuous, differentiable and its differential at the point $x_0$ is

$$\begin{aligned} dP(x_0) &= \prod_{i=p}^{i=1} dP_i(x_{i-1}) \\ \text{where } x_i &= P_io \ldots P_1(x_0)\ . \end{aligned} \qquad (10)$$

The proof is not given here but is obvious using equation 9 and lemma 1.

### 3.3 Stability of the limit cycle

The result given in this paragraph is quite similar to those of (Simic *et al.*, 2002), (Hiskens, 2001*a*), (Hiskens, 2001*b*). The cycle is said to be stable if for any $x_0$ in a neighborhood of $x_0^*$, the solution of the hybrid system 1 emanating from $x_0$ with the initial discrete state $q_1$ converges to the limit cycle. In the other cases, the cycle is said to be unstable. Let $\lambda_j$ be the eigenvalues of the matrix $dP(x_0^*)$.

*Theorem 2.* (Stability of the limit cycle).

- If for all $j$, $|\lambda_j| < 1$ then the limit cycle is stable.
- If there exists one $j'$, $|\lambda_{j'}| > 1$ then the limit cycle is unstable.
- If there exists one $j'$, $|\lambda_{j'}| = 1$ and that for all $j \neq j'$, $|\lambda_j| \leq 1$ then the method does not yield a conclusion about stability.

This theorem is a classical result of the discrete time dynamical system theory.

### 3.4 Computation of the limit cycle

Computing the limit cycle is equivalent to finding the fixed point of the Poincaré map which is $x_0^*$. In other words, $x_0^*$ is the root of the equation

$$P'(x) = x - P(x) = 0\ . \qquad (11)$$

Therefore, if $dP'(x_0^*)$ is regular, Newton's method applied to the function $P'$ will converge to $x_0^*$. We consequently make the following assumption.

*Assumption 3.* The limit cycle is not singular; $dP(x_0^*)$ has no eigenvalue equal to 1.

Let $x_0^0$ be an approximation of $x_0^*$ on the plane $S_{q_p,q_1}$, under assumption 3, Newton's method applied to $P'$ becomes

$$\begin{aligned} x_0^{k+1} &= x_0^k - dP'^{-1}(x_0^k)P'(x_0^k) \\ &= x_0^k - [I - dP(x_0^k)]^{-1}(x_0^k - P(x_0^k))\ . \end{aligned} \qquad (12)$$

Consequently, the iterative scheme is

$$x_0^{k+1} = [I - dP(x_0^k)]^{-1}(P(x_0^k) - dP(x_0^k)x_0^k) \;. \quad (13)$$

At each iteration of the method, we have to compute the values of $P$ and $dP$. Computing $P(x_0^k)$ is done by simulation. We compute the solution of the hybrid system 1 with initial value $(x_0^k, q_1)$ until it comes back on the plane $S_{q_p,q_1}$. The final point is $P(x_0^k)$. During this simulation, and thanks to an accurate event detection process (Girard, 2002), we have computed the switching points and the time passed in each discrete state. Therefore, we are able to compute, $dP(x_0^k)$. Then, applying iteration 13, we can compute the value of $x_0^{k+1}$. This numerical scheme is convergent, indeed the classical result of convergence of Newton's method gives :

*Theorem 3.* Let $x_0^0$ be on $S_{q_p,q_1}$ and sufficiently near of $x_0^*$, then

$$\lim_{k \to \infty} x_0^k = x_0^* \quad (14)$$

moreover

$$||x_0^{k+1} - x_0^*|| = O(||x_0^k - x_0^*||^2) \;. \quad (15)$$

We can see, that the convergence of our method is only local. Thus, we first have to make a localization of the limit cycle. This can be done using simulation or qualitative analysis such as the hybrid Poincaré Bendixon theorem presented in (Simic *et al.*, 2002).
In the next section, we apply our method to an example.

## 4. EXAMPLE

The two-tank system (see figure 3) has been presented in (Rubensson *et al.*, 1998) as an illustration of limit cycles arising in hybrid systems. The stability analysis of this system has been done in (Rubensson and Lennartson, 2000) and (Hiskens, 2001*b*).
The system consists of two tanks and two valves. The first valve allows to add water in the first tank, while the second one allows to drain off the second tank. There are also a constant inflow in tank 1 and a constant outflow in tank 2. The system is obtained by linearization about an operating point. The objective is to keep the water levels within some limits using a feedback on/off switching strategy for the valves.

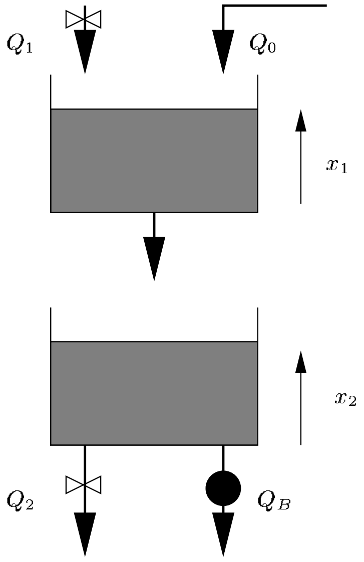

Fig. 3. Two tank system

The two valve settings result in four discrete states for our piecewise linear hybrid system (see table 1).

Table 1. Discrete states of the two tank system

| state of the hybrid system | valve 1 | valve 2 |
|---|---|---|
| $q(t) = 1$ | off | off |
| $q(t) = 2$ | on | off |
| $q(t) = 3$ | off | on |
| $q(t) = 4$ | on | on |

For an initial continuous state $(x_1(0), x_2(0))$ there is an associated discrete state $q(0)$ defined by

$$q(0) = \begin{cases} 1 \text{ if } x_1(0) \geq 0,\ x_2(0) < 0 \\ 2 \text{ if } x_1(0) < 0,\ x_2(0) < 0 \\ 3 \text{ if } x_1(0) \geq 0,\ x_2(0) \geq 0 \\ 4 \text{ if } x_1(0) < 0,\ x_2(0) \geq 0 \end{cases} \quad (16)$$

The continuous dynamics are given by

$$\begin{aligned} A_1 &= \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} & A_3 &= \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \\ b_1 &= \begin{pmatrix} -2 & 0 \end{pmatrix}^t & b_3 &= \begin{pmatrix} -2 & -5 \end{pmatrix}^t \\ A_2 &= \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} & A_4 &= \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \\ b_2 &= \begin{pmatrix} 3 & 0 \end{pmatrix}^t & b_4 &= \begin{pmatrix} 3 & -5 \end{pmatrix}^t \end{aligned} \quad (17)$$

The switching planes are

$$\begin{aligned} S_{1,2} = S_{3,4} &: x_1 + 1 = 0 \\ S_{1,3} = S_{2,3} &: x_2 - 1 = 0 \\ S_{3,1} = S_{4,2} &: x_2 \phantom{{}-1} = 0 \\ S_{4,3} \phantom{{}= S_{4,2}} &: x_1 - 1 = 0 \end{aligned} \quad (18)$$

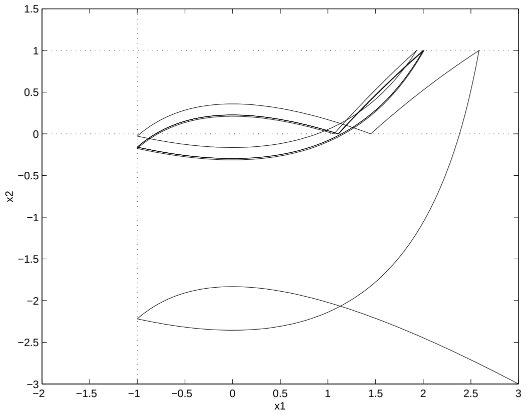

Fig. 4. Example of trajectory of the two tank system

On figure 4, an example of trajectory of the two tank system is shown. The axes correspond to the water levels in the tanks. The dashed lines correspond to the switching planes. It seems that the trajectory converges to a limit cycle. The discrete state sequence corresponding to this cycle is $\{1, 2, 3\}$. To initialize our algorithm we must choose a point near the limit cycle on the plane $S_{3,1}$, $x_0^0 = (1, 0)^t$ seems to be a reasonable initial point.

The results of our algorithm applied to the computation of the limit cycle are shown in table 2.

We can see that the convergence is quadratic. Only three iterations of our algorithm are needed to obtain an approximation of the limit cycle with a precision near of the one allowed by matlab's floating point arithmetics.

Table 2. Succesive approximations of the fixed point of the Poincaré map and associated errors

| $k$ | $(x_0^k)^t$ | $\|\|P(x_0^k) - x_0^k\|\|$ |
|---|---|---|
| 0 | [1.0000000000000000 0] | 0.1195483220052789 |
| 1 | [1.1086673728553014 0] | 0.0005631232587389 |
| 2 | [1.1081603057366756 0] | 0.0000000125312090 |
| 3 | [1.1081602944523914 0] | 0.0000000000000390 |

At the last iteration, the derivative of hybrid Poincaré map is

$$dP(x_0^k) = \begin{pmatrix} -0.1105 & -0.3099 \\ 0 & 0 \end{pmatrix} \qquad (19)$$

whose eigenvalues are 0 and -0.1105. Consequently, the limit cycle is stable. Moreover, its period is around 2.7732 and the time passed in each state are 1.1340 for state 1, 1.3868 for state 2 and 0.2524 for state 3.

On figure 5, the limit cycle of the two tank system is shown.

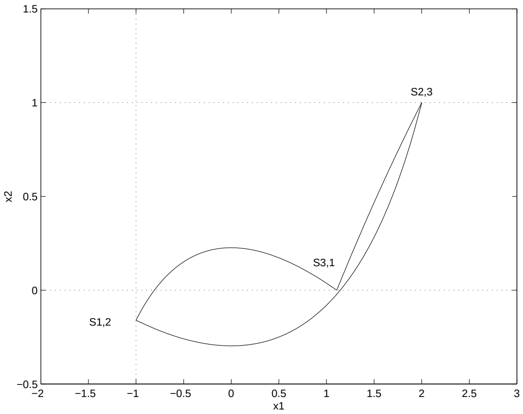

Fig. 5. Limit cycle of the two tank system

## 5. CONCLUSION

The stability analysis result given here is very similar to those of (Simic *et al.*, 2002), (Hiskens, 2001*a*), (Hiskens, 2001*b*). The main contribution of this paper lies in the computational aspect of our method. Indeed, we proposed here a numerical iterative scheme which allows an efficient accurate computation of the limit cycles in piecewise linear hybrid systems. Our method is easily implementable and should be generalizable to non-linear hybrid systems borrowing some techniques from (Guckenheimer and Meloon, 2000) or (Viswanath, 2001).

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## APPENDIX

*Proof of lemma 1*

Let $\Phi_{q_i}$ be the flow associated to the differential equation 7. We have $x_i = \Phi_{q_i}(x_{i-1}, s_i)$. Since $x_i$ is on the plane $S_{q_i,q_{i+1}}$, $x_{i-1}$ and $s_i$ are related according to the equation

$$\begin{aligned}\mathcal{H}_i(x_{i-1}, s_i) &= k^t_{q_i,q_{i+1}} \Phi_{q_i}(x_{i-1}, s_i) - d_{q_i,q_{i+1}} \\ &= 0\end{aligned} \tag{20}$$

$H_i(x_{i-1}, s_i)$ is continuous and differentiable and its derivatives are :

$$\begin{aligned}\frac{\partial \mathcal{H}_i}{\partial x_{i-1}}(x_{i-1}, s_i) &= k^t_{q_i,q_{i+1}} e^{s_i A_{q_i}} \\ \frac{\partial \mathcal{H}_i}{\partial s_i}(x_{i-1}, s_i) &= k^t_{q_i,q_{i+1}} (A_{q_i} x_i + b_{q_i})\end{aligned} \tag{21}$$

Moreover, $\mathcal{H}_i(x^*_{i-1}, s^*_i) = 0$ and according to assumption 2

$$\frac{\partial \mathcal{H}_i}{\partial s_i}(x^*_{i-1}, s^*_i) \neq 0 \ . \tag{22}$$

Thus, the theorem of implicit functions applies, therefore there exists a function $\mathcal{S}_i$ defined on a neighborhood of $x^*_{i-1}$ such that $s_i = \mathcal{S}_i(x_{i-1})$ is a solution of equation 20. Moreover, $\mathcal{S}_i$ is differentiable and its derivative is

$$\begin{aligned}\frac{\partial \mathcal{S}_i}{\partial x_{i-1}}(x_{i-1}) &= -\frac{\frac{\partial \mathcal{H}_i}{\partial x_{i-1}}(x_{i-1}, s_i)}{\frac{\partial \mathcal{H}_i}{\partial s_i}(x_{i-1}, s_i)} \\ &= \frac{-k^t_{q_i,q_{i+1}} e^{s_i A_{q_i}}}{k^t_{q_i,q_{i+1}} (A_{q_i} x_i + b_{q_i})} \ .\end{aligned} \tag{23}$$

It follows that $P_i(x_{i-1}) = \Phi_{q_i}(x_{i-1}, \mathcal{S}_i(x_{i-1}))$ is defined on a neighborhood of $x^*_{i-1}$. Moreover it is differentiable and its derivative is

$$\begin{aligned}dP_i(x_{i-1}) &= \frac{\partial \Phi_{q_i}}{\partial x_{i-1}}(x_{i-1}, \mathcal{S}_i(x_{i-1})) \\ &+ \frac{\partial \Phi_{q_i}}{\partial s_i}(x_{i-1}, \mathcal{S}_i(x_{i-1})) \frac{\partial \mathcal{S}_i}{\partial x_{i-1}}(x_{i-1})\end{aligned} \tag{24}$$

$$\begin{aligned}dP_i(x_i) &= e^{s_i A_{q_i}} \\ &+ (A_{q_i} x_i + b_{q_i}) \frac{-k^t_{q_i,q_{i+1}} e^{s_i A_{q_i}}}{k^t_{q_i,q_{i+1}} (A_{q_i} x_i + b_{q_i})} \ .\end{aligned} \tag{25}$$

*Proof of theorem 3*

First, note that equation 13 is equivalent to

$$x_0^{k+1} = P(x_0^k) + dP(x_0^k)(x_0^{k+1}) - x_0^k) \tag{26}$$

From theorem 1 and since $x^*_0$ is fixed point of the Poincaré map $P$ we have :

$$\begin{aligned}x^*_0 &= P(x^*_0) = P(x^k_0 + (x^*_0 - x^k_0)) \\ &= P(x^k_0) + dP(x^k_0)(x^*_0 - x^k_0) \\ &\quad + O(\|x^*_0 - x^k_0\|^2)\end{aligned} \tag{27}$$

Now, substracting equation (26) to equation (27), we have

$$\begin{aligned}x^*_0 - x^{k+1}_0 &= dP(x^k_0)(x^*_0 - x^{k+1}_0) \\ &\quad + O(\|x^*_0 - x^k_0\|^2) \ .\end{aligned} \tag{28}$$

Consequently

$$(I - dP(x^k_0))(x^*_0 - x^{k+1}_0) = O(\|x^*_0 - x^k_0\|^2) \tag{29}$$

and since $I - dP(x^k_0)$ is regular,

$$\|x^*_0 - x^{k+1}_0\| = O(\|x^*_0 - x^k_0\|^2) \ . \tag{30}$$