Semidefinite Approximations of Invariant Measure Supports and Reachable Sets for Discrete-time Polynomial Systems

Given a compact semialgebraic set $X_0$ included in $\mathbb{R}^n$, a polynomial map $f$ from $\mathbb{R}^n$ to $\mathbb{R}^n$, we consider the problem of approximating the union $X^*$ of $X_{inv}$ and $X_{invf}$, where $X_{inv}$ is the maximal support of Lebesgue dominated invariant measures w.r.t. $f$ and $X_{invf}$ is the reachable set for the discrete-time polynomial system $x_{t+1} = f(x_t)$, $t \in \mathbb{N}$, $x_0 \in X_0$. Assuming that $X^*$ is included in $X$, with $X$ being a “simple” set (box or ellipsoid), we provide a method to compute certified outer approximations of $X^*$.

Here, the proposed method consists of building a hierarchy of relaxations for the infinite dimensional moment problem whose optimal value is the volume of $X^*$ and whose optimum is the restriction of the Lebesgue measure on $X^*$. Then, one can outer approximate $X^*$ as closely as desired with a hierarchy of super level sets of the form $X^r := \{x \in X : \nu r(x) \geq 0\}$, for some polynomials $\nu r$ of increasing degrees $2r$. For each fixed $r$, finding the coefficients of the polynomial $\nu r$ boils down to computing the optimal solution of a semidefinite program. We provide guarantees of strong convergence to $X^*$ in $L1(X) - norm$, when the degree of the polynomial approximation tends to infinity and prove in particular that $\lim [\text{vol}(X^r \cap X^*)] = 0$ when $r$ tends to infinity. We also present some application examples together with numerical results.