Optimal domains
for spectral functionals with perimeter and volume penalizations

Bozhidar Velichkov

The results in this talk are jointly obtained with Guido De Philippis (Zurich), Michel Pierre (Rennes) and Jimmy Lamboley (Dauphine).

We consider shape optimization problems of the form
\[
\min \left\{ \mathcal{F}(\Omega) + \alpha P(\Omega) + \beta |\Omega| : \Omega \subset D \right\},
\]
where \(\alpha > 0, \beta \geq 0, \mathcal{F}\) is a functional depending on the the Dirichlet Laplacian on \(\Omega\), \(P(\Omega)\) is the generalized perimeter in sense of De Giorgi and \(|\Omega|\) is the Lebesgue measure of \(\Omega\). The geometric obstacle \(D \subset \mathbb{R}^d\) is a bounded convex set or the entire space \(\mathbb{R}^d\).

We consider functionals \(\mathcal{F}\) on the Borel sets in \(\mathbb{R}^d\), which are “decreasing and Hölder continuous with respect to the \(\gamma\) distance”, typical examples being:

- \(\mathcal{F}(\Omega) = \sum_{i=1}^{k} \lambda_i(\Omega)\) or \(\mathcal{F}(\Omega) = \lambda_k(\Omega)\), where \(\lambda_k(\Omega)\) is the \(k\)th eigenvalue of the Dirichlet Laplacian on \(\Omega\).
- \(\mathcal{F}(\Omega) = E_f(\Omega)\), for some \(f \in L^p(\mathbb{R}^d)\), where the Dirichlet Energy \(E_f\) is defined as
  \[
  E_f(\Omega) = \min \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} f u \, dx : u \in H_0^1(\Omega) \right\}.
  \]

Our goal is to prove the following results:

- the existence of a solution \(\Omega\) of (1) in the class of open sets;
- the \(C^{1,\alpha}\) regularity of the free boundary \(\partial^* \Omega\).

In order to obtain the regularity of the optimal domains, we will introduce the set of positive curvature in viscosity sense and we will use the theory of the quasi-minimizer of the perimeter, i.e. the sets \(\Omega \subset \mathbb{R}^d\) satisfying
\[
P(\tilde{\Omega}) \leq P(\Omega) + \Lambda r^d, \quad \text{for every } \tilde{\Omega} \subset \mathbb{R}^d \quad \text{such that } \tilde{\Omega} \Delta \Omega \subset B_r(x).
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References


\(^1\)Post-doc University of Pisa, Homepage: www.velichkov.it