CLT for Crossings of random trigonometric Polynomials.

Jean-Marc Azaïs    José R. León

Abstract

Let define the random trigonometric polynomial

\[ X_N(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (a_n \sin nt + b_n \cos nt). \]

The coefficients \( a_n \) and \( b_n \) are independent standard Gaussian random variables. Thus

\[ E[X_N^2(t)] = 1, \quad X_N'(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} n(a_n \cos nt - b_n \sin nt), \]

and

\[ E(X_N'(t))^2 = \frac{1}{N} \sum_{n=1}^{N} n^2 = \frac{(N + 1)(2N + 1)}{6}. \]

Denoting by \( N^{X_N}_{[0,2\pi]}(u) \) the numbers of crossings at level \( u \) of \( X_N \) in the interval \([0,2\pi]\) by using the Rice’s formula, we get

\[ E[N^{X_N}_{[0,2\pi]}(u)] = \frac{2}{\sqrt{3}} \left( \frac{(N + 1)(2N + 1)}{2} \right) e^{-\frac{u^2}{2}}, \]

hence

\[ \lim_{N \to \infty} \frac{E[N^{X_N}_{[0,2\pi]}(u)]}{N} = \frac{2}{\sqrt{3}} e^{-\frac{u^2}{2}}. \]

By using an argument of symmetry it is enough to consider only the crossings in the interval \([0, \pi]\). The crossings in \([0, \pi]\) for the process \( X_N \) can be put in relation with these of the process \( Y_N(\tau) = X_N(\frac{\tau}{N}) \). Recalling that \( Y_N(\tau) = \frac{1}{N} X_N(\frac{\tau}{N}) \).

In this conference we expose a Central Limit Theorem for the sequence

\[ \frac{1}{\sqrt{N}} (N^{X_N}_{[0,\pi]}(u) - EN^{X_N}_{[0,\pi]}(u)). \]

Moreover by using the scaling we have the following equality

\[ \frac{1}{\sqrt{N}} (N^{X_N}_{[0,\pi]}(u) - EN^{X_N}_{[0,\pi]}(u)) = \frac{1}{\sqrt{N}} (N^{Y_N}_{[0,\pi]}(u) - EN^{Y_N}_{[0,\pi]}(u)). \]

Hence it is this last sequence that will be the object of our study. We use the recent studies of the CLT for functionals of the Wiener Chaos that simplify enormously the proofs.