Estimation of Maintenance Efficiency
in Imperfect Repair Models

(Estimation of maintenance efficiency)

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ABSTRACT. The aim of this paper is to study the estimation of repair efficiency in two imperfect repair models, called Arithmetic Reduction of Intensity and Arithmetic Reduction of Age models. These models have been proposed by Doyen and Gauzin (2004) and include many usual virtual age models. First, it is proved that the failure intensity of these models is asymptotically almost surely equivalent to a deterministic increasing function with a cumulative error proportional to a logarithm. Secondly, the almost sure convergence and asymptotic normality of several estimators (including maximum likelihood) of repair efficiency are derived, when the wear-out process without repair is known. Finally, the coverage rate of the asymptotic confidence intervals issued from those estimators is empirically studied.

Key words: estimation, likelihood, point process, reliability, repairable system, virtual age

1 Introduction

All important industrial systems are subjected to corrective maintenance actions or repairs. The basic assumptions on repair efficiency are known as minimal repair or As Bad As Old (ABAO) and perfect repair or As Good As New (AGAN). Reality is between these two extreme cases: standard repair effect is imperfect. Many models have already been proposed for imperfect repair effects (see for example a review in Pham & Wang, 1996). Among them, the most used are the virtual age models proposed by Kijima (1989), in which repair is assumed to rejuvenate the system.

Only a few imperfect repair models have been statistically studied, especially regarding the estimation of repair efficiency. For virtual age models, some empirical studies on maximum likelihood estimators have been published (Shin et al., 1996, Yun & Choung, 1999, Kaminskiy & Krivtsov, 2000, Yanez et al., 2002, Gasmi et al., 2003, Doyen & Gauzin, 2004). All these papers are based on simulation results, but no theoretical statistical result on repair efficiency estimation in imperfect repair models have ever been published.

The aim of this paper is to propose such theoretical results in two particular imperfect repair models, proposed by Doyen & Gauzin (2004), and called the Arithmetic Reduction of Intensity model (ARI) and Arithmetic Reduction of Age model (ARA), both in the case of finite memory. ARA is a large class of virtual age models including the model proposed
by Kijima et al. (1988) and the model proposed by Brown et al. (1983). ARI models are not virtual age models.

The reduction of intensity and reduction of age classes of models are defined in section 2. Section 3 analyses the asymptotic behavior of the failure and cumulative failure intensities for models with finite memory. From these results, section 4 derives the asymptotic properties of some estimators of repair efficiency (including the maximum likelihood). The classical convergence theorems used in the paper are given in appendix.

2 Point processes and imperfect repair models

Let $0 = T_0 < T_1 < \ldots$ be the successive, nonexplosive $(T_n \overset{a.s.}{\to} \infty)$, failure times of a repairable system. The counting process associated to the observation of these failure times up to $t$ will be denoted by $N_t = \sum_{i=1}^{+\infty} \mathbb{I}_{\{T_i \leq t\}}$ and the inter-failure times by $X_i = T_i - T_{i-1}$ for $i \geq 1$.

A repair is supposed to be performed after each failure and the corresponding repair times are not taken into account. The filtration $\mathcal{F}_t$ in consideration will be the natural filtration associated to the failure times: $\mathcal{F}_t = \sigma(\{N_s\}_{s \leq t})$.

$N = \{N_t\}_{t \geq 0}$ has a failure intensity $\lambda_t$ if there exists a predictable process $\lambda_t$ verifying: $M_t = N_t - \int_0^t \lambda_s \, ds$, where $M_t$ is a martingale, representing the noise, that is to say, a right hand continuous with left hand limits, adapted stochastic process which is integrable and satisfies: $\forall s, t \geq 0, \ E[M_{t+s}|\mathcal{F}_s] = M_t$. If $\lambda_t$ is assumed to be left-continuous with right-hand limits, then one can show (Andersen et al., 1993) that $\lambda_t$ is unique (up to undistinguishability) and completely characterizes the failure process. Finally, the integral of the failure intensity $\Lambda_t = \int_0^t \lambda_s \, ds$ is called the cumulative failure intensity or compensator of $N$. The predictable variation process $\langle M \rangle$ is the compensator associated to $M^2$ and $\langle M \rangle_{+\infty}$ denotes its limit: $\langle M \rangle_{+\infty} = \lim_{t \to +\infty} \langle M \rangle_t$.

Let $\mathcal{P}$ be the set of the deterministic functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, and $\mathcal{P}^\dagger$ the subset of those that are also not always null and nondecreasing. Before the first failure, the failure intensity is supposed to be a function of time $\lambda(t) \in \mathcal{P}^\dagger$ called the initial intensity. Let $\Lambda(t) = \int_0^t \lambda(s) \, ds$.

In a Virtual Age (VA) model, repair is supposed to rejuvenate the system, in the sense that the failure intensity at time $t$ is equal to the initial intensity at time $A_t$, called the virtual age of the system, and verifying generally $A_t \leq t$. Then the effect of repair is to reduce the virtual age. The Arithmetic Reduction of Age (ARA) model (Doyen & Gaudoin,
is a particular VA model which assumes this reduction to be arithmetic. It leads to a failure intensity verifying \( \lambda_t = \lambda_A(t) \) with \( A_t = t - Z_{N_l} \), where \( Z \) is a predictable stochastic process. An Arithmetic Reduction of Age model with memory \( m \geq 1 \) (ARA\(_m\)) (Doyen & Gaudoin, 2004) have a failure intensity verifying

\[
\lambda_t = \lambda(t - \rho \sum_{j=0}^{m \wedge N_l} (1 - \rho)^j T_{N_l} - j)
\]

where \( N_l \) is the left hand limit of \( N_t \). There exists interesting particular cases of that class of models. The ARA\(_\infty\) model supposes that repair reduces the VA of an amount proportional to the current value of the VA. This model appears to be the same as the Brown et al. (1983) model also known as the Kijima (1989) type II model in the case of deterministic repair effect. The ARA\(_1\) model considers that repair actions cannot reduce the global VA of the system, but only the VA accumulated since the last repair. It leads to \( \lambda_t = \lambda(t - \rho T_{N_l}) \). This model appears to be the same as the Kijima et al. (1988) model and Shin et al. (1996) model or also the Kijima (1989) type I model in the case of deterministic repair effect. It has been first introduced by Malik (1979).

In the Arithmetic Reduction of Intensity (ARI) model (Doyen & Gaudoin, 2004), repair is supposed to reduce arithmetically not the age, but the failure intensity. This leads to a failure intensity verifying \( \lambda_t = \lambda(t) - Z_{N_t} \) where \( Z \) is a predictable stochastic process. Then ARI models can be defined by analogy with ARA models. An ARI model with memory \( m \geq 1 \) (ARI\(_m\)) has a failure intensity verifying

\[
\lambda_t = \lambda(t) - \rho \sum_{j=0}^{m \wedge N_l} (1 - \rho)^j \lambda(T_{N_l} - j)
\]

The ARI\(_\infty\) model appears to be the same as the Chan & Shaw (1993) model. The ARI\(_1\) model has a failure intensity \( \lambda_t = \lambda(t) - \rho \lambda(T_{N_l}) \). Then it is certainly one of the most simple imperfect repair models.

The parameter \( \rho \) in ARI\(_m\) and ARA\(_m\) models characterizes repair efficiency. When \( \rho \) is between 0 and 1, repair is efficient. When it is smaller than 0, repair is harmful. When \( \rho \) equals 0, repair is inefficient, that is to say ABAO. And finally, when \( \rho \) equals 1, repair is optimal but not necessary AGAN (there is no value of \( \rho \) for which repair is AGAN in ARI\(_m\) models). Then, assessing repair efficiency in ARI\(_m\) and ARA\(_m\) models is equivalent to estimate the parameter \( \rho \).

ARA\(_m\) and ARI\(_m\) models have also an interesting property: there exists what Doyen & Gaudoin (2004) called a minimal wear intensity, denoted by \( \lambda_{min}(t) \). The minimal wear
intensity is a maximal lower bound for failure intensity. It can be proved that, for an ARI\(_m\) model, \(\lambda_{\text{min}}(t) = [1 \wedge (1 - \rho)^m] \lambda(t)\) and for an ARA\(_m\) model, \(\lambda_{\text{min}}(t) = \lambda(1 \wedge (1 - \rho)^m) t\).

By analogy with the minimal wear intensity, a \textit{maximal wear intensity} can be defined, which is a minimal upper bound for failure intensity. It is denoted by \(\lambda_{\text{max}}(t)\). It can be proved that, for an ARI\(_m\) model, \(\lambda_{\text{max}}(t) = [1 \vee (1 - \rho)^m] \lambda(t)\) and for an ARA\(_m\) model, \(\lambda_{\text{max}}(t) = \lambda(1 \vee (1 - \rho)^m) t\).

All the results of the following are based on the existence of a deterministic minimal wear intensity that belongs to \(\mathcal{P}^\dagger\). In fact, in that case, the following theorem 1 proves that times between failures are negligible with regard to failure times. Then, models with infinite memories are no longer considered \((m < \infty)\) since their minimal wear intensities are null for efficient repair. For the same reason, the repair efficiency is supposed to be not perfect: \(\rho < 1\).

**Theorem 1**

If there exists \(\lambda_{\text{min}} \in \mathcal{P}^\dagger\), such that for all \(t \geq 0\), \(\lambda_t \geq \lambda_{\text{min}}(t)\), then for all \(j \geq 0\):

\[
t - T_{N_{t-j}} = o(t)
\]

**Proof.** Since \(\lambda_{\text{min}} \in \mathcal{P}^\dagger\), there exists \(C > 0\) and \(t_0 > 0\) such that, for all \(t \geq t_0\), \(\lambda_{\text{min}}(t) \geq C\). Then \(\Lambda_t\) diverges, so \(T_{N_t}\) is almost surely divergent (Bremaud, 1981). So, there almost surely exists \(t_1 > 0\) such that, for all \(t \geq t_1\), the failure process verifies: \(T_{N_{t-j}} \geq t_0\). By using the differential martingale writing of \(N\), we almost surely have, for all \(t \geq t_1\):

\[
t - T_{N_{t-j}} = \int_{T_{N_{t-j}}}^{t} \frac{1}{\lambda_s} d\Lambda_s = \int_{T_{N_{t-j}}}^{t} \frac{1}{\lambda_s} dN_s - \int_{T_{N_{t-j}}}^{t} \frac{1}{\lambda_s} dM_s
\]

and

\[
t - T_{N_{t-j}} \leq j + \frac{1}{C} + 2 \max \left( \left| \int_{t_0}^{t} \frac{1}{\lambda_s} dM_s \right| , \left| \int_{t_0}^{T_{N_{t-j}}} \frac{1}{\lambda_s} dM_s \right| \right)
\]

For all \(t \geq t_0\), \(\int_{t_0}^{t} (1/\lambda_s)^2 d\Lambda_s \leq (t - t_0)/C\), then thanks to corollary 3 (see appendix),

\[
\int_{t_0}^{t} \frac{1}{\lambda_s} dM_s \stackrel{a.s.}{=} o\left( \int_{t_0}^{t} \frac{1}{\lambda_s} ds \right) + O(1) = o\left( \frac{t - t_0}{C} \right) + O(1) = o(t)
\]

That result remains true by replacing \(t\) by \(T_{N_{t-j}}\). Then, since \(o(T_{N_{t-j}}) = o(t)\), the lemma is proved.

Figure 1 shows, for an initial intensity \(\lambda(t) = 3t^2\), the failure intensity of an ARI\(_3\) model in the case of efficient repair \((\rho = 0.3)\), in solid thin line, and harmful repair \((\rho = -0.2)\), in solid bold line. The dashed line represents the initial intensity that corresponds to the
maximal wear intensity for the case of efficient repair and the minimal wear intensity for the
case of harmful repair. The dotted line is the maximal wear intensity for the case of harmful
repair, and the dashed-dotted line is the minimal wear intensity for the case of efficient repair.

![Graph](image)

Figure 1: $\text{ARI}_3$, thin line: $0 < \rho < 1$, bold line: $\rho < 0$

It seems, in the case of efficient repair, that the failure intensity and the minimal wear
intensity remain approximately equivalent. For harmful repair, the same property seems to
hold but with the maximal wear intensity. So we define the asymptotic intensity, denoted $\lambda_\infty$, as the minimal wear intensity for efficient repair and the maximal wear intensity for harmful repair. Then, in the $\text{ARI}_m$ model the asymptotic intensity is $\lambda_\infty(t) = (1 - \rho)^m \lambda(t)$ and in the $\text{ARA}_m$ model it is $\lambda_\infty(t) = \lambda((1 - \rho)^m t)$. In addition, let $\Lambda_\infty(t) = \int_0^t \lambda_\infty(s) ds$. We will prove that the failure intensity and the asymptotic intensity are asymptotically almost surely equivalent.

3 Failure process behavior

For $\text{ARI}_m$ and $\text{ARA}_m$ models, it is already assumed that: $\lambda \in \mathcal{P}^\uparrow$, $m < \infty$ and $\rho < 1$. In addition, the initial intensity is supposed to be a regular variation function.

3.1 Regular variation functions

Regular variation functions (Embrechts et al., 1997) verify:

$$\exists \beta \in \mathbb{R}_+, \forall x > 0, \lim_{t \to +\infty} \frac{\lambda(x t)}{\lambda(t)} = x^\beta$$

(1)

Regular variation functions with $\beta = 0$ are slow variation functions. Power of logarithm functions ($\lambda(t) = \alpha \beta (\ln(1 + t))^{\beta - 1}, \alpha > 0, \beta > 1$) and constant functions ($\lambda(t) = \alpha, \alpha > 0$)
are slow variation functions. Increasing power functions \((\lambda(t) = \alpha \beta t^{\beta-1}, \alpha > 0, \beta > 1)\) are not slow variation functions, however, they are regular variation functions. Since \(\lambda \in \mathcal{P}^\dagger\), nonregular variation functions for the initial intensity correspond to quick divergent initial intensity, for example: \(\lambda(t) = \exp(t^\beta) \) with \(\beta > 0\). That kind of initial intensity seems not to be realistic for reliable industrial systems.

Since \(\lambda(t)\) is a regular variation function, it is obvious to prove that for all the constant \(c \in \mathbb{R}^*\), the functions \(c \lambda(t)\) and \(\lambda(ct)\) are also regular variation functions. Therefore, for ARI\(_m\) and ARA\(_m\) models, it is equivalent to assume that \(\lambda\) or \(\lambda_\infty\) are regular variation functions. The initial intensity is supposed to be a regular variation function because the following lemma is then verified.

**Lemma 1**

Let \(\lambda\) be a not always null, monotonic, regular variation function. Then, it verifies:

\[
\lambda(t) - \lambda(t + o(1)) = o(\lambda(t))
\]

**Proof.** There exists \(t_0 > 0\) such that \(\lambda(t)\) is never null for \(t \geq t_0\). In addition, \(\lambda\) is a monotonic function, then:

\[
\forall \epsilon > 0, \exists t_1 > 0, \forall t \geq \max(t_0, t_1), \quad |1 - \frac{\lambda(t(1 + o(1)))}{\lambda(t)}| \leq \max_{u \in \{-1,1\}} |1 - \frac{\lambda(t(1 + u\epsilon))}{\lambda(t)}|
\]

But \(\lambda\) is also a regular variation function and with \(x = 1 + u\epsilon\) in (1) it verifies:

\[
\forall \epsilon > 0, \exists t_2 > 0, \forall t \geq t_2, \quad \left| \frac{\lambda(t(1 + u\epsilon))}{\lambda(t)} - (1 + u\epsilon)^\beta \right| \leq \epsilon^2
\]

Then,

\[
|1 - \frac{\lambda(t(1 + o(1)))}{\lambda(t)}| \leq \max_{u \in \{-1,1\}} \left[ \frac{\lambda(t(1 + u\epsilon))}{\lambda(t)} - (1 + u\epsilon)^\beta \right] + |1 - (1 + u\epsilon)^\beta|
\]

\[
\leq \epsilon^2 + \max_{u \in \{-1,1\}} |1 - (1 + u\epsilon)^\beta|
\]

And the lemma is proved.

### 3.2 Failure intensity first order asymptotic expansion

The result of this section is based on a rewriting of the failure intensity of ARI\(_m\) and ARA\(_m\) models. Thanks to the fact that \(\rho \sum_{j=0}^{m-1} (1 - \rho)^j = 1 - (1 - \rho)^m\), the failure intensity of an ARI\(_m\) model verifies for \(t \geq T_m\):

\[
\lambda_t = \lambda_\infty(t) - \frac{\rho}{(1 - \rho)^m} \sum_{j=0}^{m-1} (1 - \rho)^j [\lambda_\infty(t) - \lambda_\infty(t + (t - T_{N_1 - j}))]
\]

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And then, using the results of theorem 1 and lemma 1:

$$\lambda_t \stackrel{a.s.}{=} \lambda_\infty(t) - \frac{\rho}{(1-\rho)^m} \sum_{j=0}^{m-1} (1-\rho)^j \left[ \lambda_\infty(t) - \lambda_\infty(t + o(1)) \right] = \lambda_\infty(t) + o(\lambda_\infty(t))$$

Similarly, for the ARA_m model:

$$\lambda_t = \lambda_\infty(t) - \left[ \lambda_\infty(t) - \lambda_\infty(t + o(1)) \right] = \lambda_\infty(t) + o(\lambda_\infty(t))$$

**Proposition 1**

For ARI_m and ARA_m models, the failure intensity verifies:

$$\lambda_t \stackrel{a.s.}{=} \lambda_\infty(t) + o(\lambda_\infty(t))$$

### 3.3 Cumulative failure intensity second order asymptotic expansion

The difference between failure and asymptotic intensities can not be expressed, but the integral of that difference can be expressed. It is calculated using theorem 2 below.

**Theorem 2**

For any divergent and regular variation function $u \in \mathcal{P}^+$, the failure process of ARI_m and ARA_m models verifies:

$$\int_0^t \frac{u(s) - u(T_{N_t-j})}{u(s)} \lambda_\infty(s) ds \stackrel{a.s.}{=} (j + 1) \ln(u(t)) + o(\ln(u(t)))$$

**Proof.** There almost surely exists $t_0 \in \mathbb{R}_+^*$ such that $\lambda_{\min}$, $\lambda_\infty$ and $u$ are non null for $t \geq t_0$ and $N_{t_0} > j$. Let $U_j^t = (u(t) - u(T_{N_t-j}))/u(t)$, since $u$ is a regular variation function, theorem 1 and lemma 1 imply that:

$$U_j^t = \frac{u(t) - u(t - (t - T_{N_t-j}))}{u(t)} a.s. u(t) - u(t + o(1)) = o(1)$$

Using the martingale writing of the counting process $N$, the failure intensity verifies for all $s \geq t_0$: $ds = dN_s/\lambda_s - dM_s/\lambda_s$. Then,

$$\int_0^t U_j^s \lambda_\infty(s) ds = \int_0^{t_0} U_j^s \lambda_\infty(s) ds + \int_t^{t_0} U_j^s \lambda_\infty(s) \frac{dN_s}{\lambda_s} - \int_0^t U_j^s \lambda_\infty(s) dM_s$$

(2)

Since $|U_j^s| \leq 1$, the first term verifies:

$$\int_0^{t_0} U_j^s \lambda_\infty(s) ds = O(1)$$

(3)
Since \( \int_{t_0}^{t} (U_j^j \lambda_\infty(s))^2/\lambda_s ds \leq t \lambda_\infty(t)^2/\lambda_{\min}(t_0) < \infty \), corollary 3 (in appendix) implies that:

\[
\int_{t_0}^{t} U_j^j \frac{\lambda_\infty(s)}{\lambda_s} dM_s \eqa \frac{a_s}{\lambda_s} \int_{t_0}^{t} \frac{\lambda_\infty(s)^2}{\lambda_s} ds + O(1) \eqa \frac{a_s}{\lambda_s} \int_{t_0}^{t} U_j^j \lambda_\infty(s) ds + O(1)
\]  \hspace{1cm} (4)

Finally, \( \ln(u(T_{N_s-j}))/u(s)) \eqa U_j^j(-1 + o(1)) \). And, since \( T_n \) is almost surely finite for all \( n \geq 0 \):

\[
\int_{t_0}^{t} \ln\left(\frac{u(s)}{u(T_{N_s-j})}\right) dN_s \eqa \sum_{i=0}^{j} \ln\left(\frac{u(T_{N_s-j})}{u(T_{N_{s-j}^i})}\right) = (j+1) \ln(u(t)) + \sum_{i=0}^{j} \ln(1 - U_i^j) + O(1) \eqa (j+1) \ln(u(t)) + o(\ln(u(t)))
\]  \hspace{1cm} (5)

Then \( \int_{t_0}^{t} U_j^j \lambda_\infty(s)/\lambda_s dN_s \eqa (j+1) \ln(u(t)) + o(\ln(u(t))) \). Using (3), (4) and (5) in (2), the theorem is proved.

For \( t \geq T_m \), the cumulative failure intensity of AR1_m models can be written as:

\[
\Lambda_t = \Lambda_\infty(t) + \frac{\rho}{(1-\rho)^m} \sum_{j=0}^{m-1} (1-\rho)^j \int_{0}^{t} \lambda_\infty(s) - \lambda_\infty(T_{N_s-j}) ds
\]  \hspace{1cm} (6)

Let us suppose that the asymptotic intensity or equivalently the initial intensity is divergent:

- **A1**: \( \lambda(t) \to \infty \)

Then, theorem 2 can be applied with \( u(t) = \lambda_\infty(t) \). And the following result of proposition 2 is just a consequence of the fact that \( \lambda_\infty(t) = (1-\rho)^m \lambda(t) \) and:

\[
\frac{\rho}{(1-\rho)^m} \left( \sum_{j=0}^{m-1} (j+1)(1-\rho)^j \right) = 1 - \frac{(1+m\rho)(1-\rho)^m}{\rho(1-\rho)^m}
\]  \hspace{1cm} (7)

**Proposition 2**

For AR1_m model, under assumption A1, the cumulative failure intensity verifies:

\[
\Lambda_t \eqa \Lambda_\infty(t) + \frac{1 - (1+m\rho)(1-\rho)^m}{\rho(1-\rho)^m} \ln(\lambda(t)) + o(\ln(\lambda(t)))
\]

In the case of AR1_m models the assumption on the initial intensity must be more restrictive. The initial intensity is supposed to be an increasing power function:

- **A2**: \( \lambda(t) = \alpha \beta t^{\beta-1}, \alpha > 0, \beta > 1 \)

**Proposition 3**

For AR1_m model under assumption A2, the cumulative failure intensity verifies:

\[
\Lambda_t \eqa \Lambda_\infty(t) + (\beta - 1) \frac{1 - (1+m\rho)(1-\rho)^m}{\rho(1-\rho)^m} \ln(t) + o(\ln(t))
\]
Proof. For $t \geq T_m$, the failure intensity of an $ARA_m$ model can be written:

$$\lambda_t = \lambda_\infty(t + R_t) \quad \text{with} \quad R_t = \frac{\rho}{(1 - \rho)^m} \sum_{j=0}^{m-1} (1 - \rho)^j (t - T_{N_s - j})$$

Let $M_1 = |1 - 1/(1 - \rho)^m|$. Then $R_t$ verifies:

$$0 \leq |R_t| \leq M_1 (t - T_{N_s - m+1}) \leq M_1 t$$

In addition, if $R_t < 0$, then necessarily $\rho < 0$ and $M_1 < 1$. So there exists $M_2 > 0$ such that:

$$M_2 t \leq t + R_t \leq (1 + M_1) t.$$ A second order Taylor-Lagrange expansion of $\lambda_t$ proves that:

$$\left| \lambda_t - \lambda_\infty(t) - R_t \lambda_\infty'(t) \right| \leq R_t^2 \max_{M_2 t \leq s \leq (1 + M_1) t} \left| \lambda_\infty''(s) \right|$$

But, $\lambda_\infty(t) = \alpha \beta((1 - \rho)^m t)^{\beta-1}$, then

$$R_t^2 \lambda_\infty''(t) \max_{M_2 t \leq s \leq (1 + M_1) t} |s^{\beta-3}| \leq O\left(\frac{R_t}{t}\right)^2 \lambda_\infty(t)$$

And by integrating this equation it is proved that:

$$\left| \Lambda_t - \Lambda_\infty(t) - (\beta - 1) \frac{\rho}{(1 - \rho)^m} \sum_{j=0}^{m-1} (1 - \rho)^j \int_0^t \frac{s - T_{N_s - j}}{s} \lambda_\infty(s) ds \right|$$

$$= O\left(\int_0^t \left(\frac{s - T_{N_s - m+1}}{s}\right)^2 \lambda_\infty(s) ds\right)$$

(9)

But, theorem 1 implies that: $(s - T_{N_s - m+1})/s \frac{a_s}{s} = o(1)$. Then

$$o\left(\int_0^t \frac{s - T_{N_s - m+1}}{s} \lambda_\infty(s) ds\right) + O(1)$$

The proposition is proved just by applying theorem 2 with $u(t) = t$ and using (7).

Proposition 3 is also verified by the $ARI_m$ model.

4 Repair efficiency estimation

The aim of this section is to study some estimators of repair efficiency in the case where the initial intensity is known. Then the failure intensity is supposed to depend on a single parameter $\rho \in \mathcal{F}$: $\lambda_t = \lambda_t(\rho)$. The true value of this parameter will be denoted $\rho_0$. First, the properties of the maximum likelihood estimator (MLE) are studied. Then, explicit estimators are introduced.
4.1 Maximum Likelihood Estimators

Since the cumulative failure intensities of $ARI_m$ and $ARA_m$ models are quite similar, a general theorem, including $ARI_m$ and $ARA_m$ cases, and possible others cases, is proposed.

4.1.1 Convergence Criteria for MLE

- **A'1**: $\mathcal{J}_0 = [\rho_1, \rho_2]$ is a known compact of $\mathbb{R}$, included in the interior of $\mathcal{J}$ and such that $\rho_1 < \rho_0 < \rho_2$.

- **A'2**: The derivative in $\rho$ of the failure intensity exists, it is denoted $\lambda'_t(\rho)$. Then $\lambda_{\min}, \lambda_{\max} \in \mathcal{P}^\uparrow$ are non null except in zero and such that there exists a positive constant $d$, verifying for all $\rho \in \mathcal{J}$ and $t \geq 0$: $\lambda_{\min}(t) \leq \lambda_t(\rho) \leq \lambda_{\max}(t)$ and $| \lambda'_t(\rho) | \leq d \lambda_{\min}(t)$.

- **A'3**: $\lambda \in \mathcal{P}^\uparrow$ and $g$ a strictly monotonic, continuous, differentiable function from $\mathcal{J}_0$ to $\mathbb{R}_+$ with continuous derivative $g'$, are such that:

  \[
  \lambda_t(\rho_0) \overset{a.s.}{=} g(\rho_0) \lambda(t) + o(\lambda(t)) \quad (10)
  \]

  \[
  \int_0^t \sup_{\rho \in \mathcal{J}_0} | \lambda_s(\rho) - g(\rho) \lambda(s) | \, ds \overset{a.s.}{=} o(\sqrt{\lambda(t)}) \quad (11)
  \]

  \[
  \int_0^t \sup_{\rho \in \mathcal{J}_0} | \lambda'_s(\rho) - g'(\rho) \lambda(s) | \, ds \overset{a.s.}{=} o(\sqrt{\lambda(t)}) \quad (12)
  \]

- **A'4**: $\Lambda(t) \to \infty$

$A'1$ is a classical assumption of maximum likelihood estimation. It supposes that the maximization of the likelihood is made in a bounded and closed interval that contains the true value of the repair efficiency parameter. In fact, the asymptotic behavior of the log-likelihood out of this interval can be very irregular. Assumption $A'2$ is necessary in particular to prove the existence of the partial derivative in $\rho$ of the log-likelihood. Then, since the log-likelihood verifies:

\[
\mathcal{L}_t(\rho) = \int_0^t \ln(\lambda_s(\rho)) \, dN_s - \int_0^t \lambda_s(\rho) \, ds \quad (13)
\]

its derivative in $\rho$ is equal to:

\[
\mathcal{L}'_t(\rho) = \int_0^t \frac{\lambda'_s(\rho)}{\lambda_s(\rho)} \, dN_s - \int_0^t \lambda'_s(\rho) \, ds \quad (14)
\]

The maximum likelihood estimator $\hat{\rho}^{ML}_t$ is the value of $\rho$ in $\mathcal{J}_0$ that maximizes the log-likelihood. Classically, it also nullifies the derivative of the log-likelihood. Finally, it will be
proved, in the following, that assumption A’3 is simply a consequence of the application of propositions 1, 2 and 3.

In order to prove convergence results for repair efficiency MLE, two lemmas are needed. The first one gives the asymptotic equivalence between the derivative of the log-likelihood and a linear combination of the counting process and the cumulative asymptotic intensity. This expression is easier to use than equation (13). The second lemma proves that, for $t$ big enough, the maximum of the likelihood on $\mathcal{J}_0$ corresponds to a zero of its derivative.

Lemma 2

Under assumptions A’1 to A’4, the derivative of the log-likelihood verifies:

$$\sup_{\rho \in \mathcal{J}_0} | \mathcal{L}'_t(\rho) - g'(\rho) \left( \frac{N_t}{g(\rho)} - \Lambda(t) \right) | \overset{a.s.}{=} o(\sqrt{\Lambda(t)})$$

Proof. First of all, thanks to the assumptions on $g$ and the fact that $\mathcal{J}_0$ is a compact, it is easy to prove that:

$$\exists (a_1, a_2, a_3) \in \mathbb{R}_+^3, \forall \rho \in \mathcal{J}_0, \ a_1 \leq g(\rho) \quad \text{and} \quad a_2 \leq g'(\rho) \leq a_3 \quad (15)$$

In addition, using (14) and (12), we prove that almost surely:

$$\sup_{\rho \in \mathcal{J}_0} | \mathcal{L}'_t(\rho) - g'(\rho) \left( \frac{N_t}{g(\rho)} - \Lambda(t) \right) | \leq \int_0^t Y_s dN_s + o(\sqrt{\Lambda(t)})$$

where $Y_s = \sup_{\rho \in \mathcal{J}_0} | \lambda'_s(\rho) - \lambda_s(\rho) - g'(\rho) / g(\rho) |$. The differential martingale writing of $N$ implies that:

$$\int_0^t Y_s dN_s = \int_0^t Y_s \lambda_s(\rho_0) \, ds + \int_0^t Y_s dM_s$$

But

$$Y_s \leq \sup_{\rho \in \mathcal{J}_0} \left| \lambda'_s(\rho) - g'(\rho) \lambda(s) \right| + \sup_{\rho \in \mathcal{J}_0} \left| \lambda'_s(\rho) \left( \frac{1}{\lambda_s(\rho)} - \frac{g(\rho)}{g(\rho)} \lambda(s) \right) \right|$$

$$\leq \frac{1}{a_1} \left[ \sup_{\rho \in \mathcal{J}_0} \left| \lambda'_s(\rho) - g'(\rho) \lambda(s) \right| + \sup_{\rho \in \mathcal{J}_0} \left| \lambda'_s(\rho) \lambda(s) \right| + \sup_{\rho \in \mathcal{J}_0} \left| \lambda_s(\rho) - g(\rho) \lambda(s) \right| \right]$$

Then, thanks to (10), there almost surely is:

$$Y_s \lambda_s(\rho_0) \leq \frac{a_2}{a_1} \left[ \sup_{\rho \in \mathcal{J}_0} \left| \lambda'_s(\rho) - g'(\rho) \lambda(s) \right| + d \sup_{\rho \in \mathcal{J}_0} \left| \lambda_s(\rho) - g(\rho) \lambda(s) \right| \right]$$

And using (11) and (12), it is proved that $\int_0^t Y_s \lambda_s(\rho_0) \, ds \overset{a.s.}{=} o(\sqrt{\Lambda(t)}) + O(1)$.

Since $Y_s \leq d + a_3/a_1$, corollary 3 implies that $\int_0^t Y_s dM_s = o(\int_0^t Y_s^2 \lambda_s(\rho_0) \, ds) + O(1)$

$$\int_0^t Y_s^2 \lambda_s(\rho_0) \, ds \leq (d + a_3/a_1) \int_0^t Y_s \lambda_s(\rho_0) \, ds \overset{a.s.}{=} o(\sqrt{\Lambda(t)}) + O(1)$$

The lemma is proved, since $\Lambda(t) \to \infty$ and then $o(\sqrt{\Lambda(t)}) + O(1) = o(\sqrt{\Lambda(t)})$. 


Lemma 3

Under assumption A’1 and A’3, if \( L'_i(\rho) \) verifies:

\[
\forall \epsilon > 0, \quad \sup_{\rho \in \mathcal{J}_0} | L'_i(\rho) - \frac{g'(\rho)}{g(\rho)} (g(\rho_0) - g(\rho)) \Lambda(t) | \xrightarrow{a.s.} o(\Lambda(t)^{0.5+\epsilon})
\]

then, there almost surely exists \( T > 0 \), such that for all \( t \geq T \), \( L'_i(\rho_1) > 0 \) and \( L'_i(\rho_2) < 0 \).

Proof. Equation (16) implies that:

\[
| L'_i(\rho_1) - \frac{g'(\rho_1)}{g(\rho_1)} (g(\rho_0) - g(\rho_1)) \Lambda(t) | \xrightarrow{a.s.} o(\Lambda(t)^{0.5+\epsilon})
\]

Whatever \( g \) is a strictly increasing or decreasing function, it verifies:

\[
\frac{g'(\rho_1)}{g(\rho_1)} (g(\rho_0) - g(\rho_1)) > 0
\]

So, if \( L'_i(\rho_1) \) is supposed to be not positive it also verifies:

\[
L'_i(\rho_1) \xrightarrow{a.s.} \frac{g'(\rho_1)}{g(\rho_1)} (g(\rho_0) - g(\rho_1)) \Lambda(t) + o(\Lambda(t)^{0.5+\epsilon})
\]

Then, asymptotically \( L'_i(\rho_1) \) is necessarily almost surely strictly positive. The lemma is proved for \( \rho_1 \), a similar proof can be done for \( \rho_2 \).

Theorem 3

Let \( \mathcal{J} \) be an interval of \( \mathbb{R} \) with a non empty interior. Let us consider a counting process defined by a failure intensity \( \lambda_\mathcal{J}(\rho_0) \) for \( \rho_0 \in \mathcal{J} \) such that for all \( t \geq 0 \), \( \lambda_\mathcal{J}(\rho_0) \) is continuous, differentiable with continuous derivative in \( \rho_0 \) for \( \rho_0 \in \mathcal{J} \). Then, under assumptions A’1 to A’4 and for a single observation of the failure process over \([0,t]\), the maximum likelihood estimator \( \hat{\rho}_ML^t \) of \( \rho_0 \) verifies:

\[
\forall \epsilon > 0, \quad (g(\rho_0) - g(\hat{\rho}_ML^t)) \Lambda(t)^{0.5-\epsilon} \xrightarrow{a.s.} 0
\]

\[
\frac{g(\rho_0) - g(\hat{\rho}_ML^t)}{\sqrt{g(\rho_0) \Lambda(t)}} \xrightarrow{L} \mathcal{N}(0,1)
\]

Proof. Equation (11) implies that:

\[
| \Lambda_t(\rho_0) - g(\rho_0) \Lambda(t) | \leq \int_0^t \sup_{\rho \in \mathcal{J}_0} | \lambda_\mathcal{J}(\rho) - g(\rho) \lambda(s) | \; ds \xrightarrow{a.s.} o(\sqrt{\Lambda(t)})
\]

Then, \( \Lambda_t(\rho_0) \xrightarrow{a.s.} g(\rho_0) \Lambda(t) + o(\sqrt{\Lambda(t)}) \). In addition, since \( \Lambda(t) \) diverges and \( g(\rho_0) > 0 \), corollary 4 (in appendix) can be applied and \( \forall \epsilon > 0 \), \( M_t \xrightarrow{a.s.} o(\Lambda(t)^{0.5+\epsilon}) \). Then using A’4, \( N \) verifies:

\[
N_t - g(\rho_0) \Lambda(t) = \Lambda_t(\rho_0) - M_t - g(\rho_0) \Lambda(t) \xrightarrow{a.s.} o(\Lambda(t)^{0.5+\epsilon})
\]
But
\[
\sup_{\rho \in J_0} | \mathcal{L}'_i(\rho) - \frac{g'(\rho)}{g(\rho)}(g(\rho_0) - g(\rho))\Lambda(t) | \leq \sup_{\rho \in J_0} | \mathcal{L}'_i(\rho) - \frac{g'(\rho)}{g(\rho)}(\frac{N_t}{g(\rho)} - \Lambda(t)) | \\
+ [N_t - g(\rho_0)\Lambda(t)] \sup_{\rho \in J_0} | \frac{g'(\rho)}{g(\rho)} | 
\]

With the notations of (15): \( \sup_{\rho \in J_0} | \frac{g'(\rho)}{g(\rho)} | \leq \frac{\alpha_3}{\alpha_1}. \) Then, using the result of lemma 2 equation (16) is proved and lemma 3 can be applied. But, thanks to assumption A2 and the property of continuity of the Lebesgue integral, it can be proved that for all \( t \in \mathbb{R}_+ \) and \( \rho \in J_0 \), the log-likelihood and its derivative in \( \rho \) are continuous in \( \rho \) for \( \rho \) in \( J_0 \). So, for \( t \geq T \), the maximum likelihood estimator verifies \( \mathcal{L}'(\hat{\rho}_t^{ML}) \overset{a.s.}{=} 0. \) By using this result in equation (16), it is proved that the maximum likelihood estimator verifies for all \( \epsilon > 0: \)
\[
| g(\rho_0) - g(\hat{\rho}_t^{ML}) | \Lambda(t) \overset{a.s.}{=} o(\Lambda(t)^{0.5+\epsilon}). \]

And we obtain the first result of the property.

In addition, lemma 2 also implies:
\[
\frac{g(\rho_0)}{g'(\rho_0)} \mathcal{L}'_i(\rho_0) - \frac{g(\hat{\rho}_t^{ML})}{g'(\hat{\rho}_t^{ML})} \mathcal{L}'_i(\hat{\rho}_t^{ML}) \overset{a.s.}{=} [g(\rho_0) - g(\hat{\rho}_t^{ML})] \Lambda(t) + o(\sqrt{\Lambda(t)})
\]

And since the maximum likelihood estimator nullifies \( \mathcal{L}'_i \), the same lemma also implies:
\[
\frac{g(\rho_0)}{g'(\rho_0)} \mathcal{L}'_i(\rho_0) - \frac{g(\hat{\rho}_t^{ML})}{g'(\hat{\rho}_t^{ML})} \mathcal{L}'_i(\hat{\rho}_t^{ML}) \overset{a.s.}{=} \sqrt{g(\rho_0)\Lambda(t)} \left( - \frac{M_t}{\sqrt{g(\rho_0)\Lambda(t)}} + o(1) \right)
\]

Finally, corollary 4 can be applied and:
\[
M_t/\sqrt{g(\rho_0)\Lambda(t)} \overset{L}{\rightarrow} \mathcal{N}(0,1). \]

Then the second result of the theorem is proved.

For ARI_m and ARA_m models, A2 to A4 are verified under the same assumptions as in propositions 2 and 3. Then to apply theorem 3, only A1 imposes stronger assumption:

- **A0:** The MLE is searched in a known interval \( J_0 = [\rho_1, \rho_2] \) such that \(-\infty < \rho_1 < \rho_0 < \rho_2 < 1. \)

As it will be seen in the next section, in practice, the log-likelihood seems to be regular enough in order that we can maximize it on the whole interval \(-\infty, 1. \) So, it seems that in practice, for ARI_m and ARA_m models, we have no matter of assumption A1.

### 4.1.2 The ARI_m Model

**Proposition 4**

For ARI_m model and under assumptions A0 and A1, the maximum likelihood estimator of repair efficiency parameter verifies, for a single observation of the failure process over \([0, t]:\)
\[
\forall \epsilon > 0, \quad | \rho_0 - \hat{\rho}_t^{ML} | \Lambda(t)^{0.5-\epsilon} \overset{a.s.}{=} 0
\]

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\[
\sqrt{\frac{\Lambda(t)}{(1 - \rho_0)^m}}[(1 - \rho_0)^m - (1 - \hat{\rho}_t^{ML})^m] \xrightarrow{c} \mathcal{N}(0, 1)
\]

**Proof.** In all the proof let us consider \( \rho \in [\rho_1, \rho_2] \). Using the minimal and maximal wear intensities proposed in section 2, it is proved that:

\[
[1 \land (1 - \rho_2)^m] \lambda(t) = \lambda_{min}(t) \leq \lambda_t(\rho) \leq \lambda_{max}(t) = [1 \lor (1 - \rho_1)^m] \lambda(t)
\]

Let \( g(\rho) = (1 - \rho)^m \), then (10) corresponds to the result of proposition 1. Using the expression (6) of the cumulative failure intensity of AR1 models we also have:

\[
\int_0^t \sup_{\rho \in \mathcal{J}_0} | \lambda_s(\rho) - g(\rho) \lambda(s) | \, ds \leq \sup_{\rho \in \mathcal{J}_0} \rho \sum_{j=0}^{m-1} (1 - \rho)^j | \int_0^t \lambda(s) - \lambda(T_{N_{\rho} - m+1}) \, ds
\]

First, \( \sup_{\rho \in \mathcal{J}_0} | \rho \sum_{j=0}^{m-1} (1 - \rho)^j | \leq 1 + (1 - \rho_2)^m \), secondly theorem 2 with \( u(t) = \lambda(t) \) implies that: \( g(\rho_0) \int_0^t \lambda(s) - \lambda(T_{N_{\rho} - m+1}) \, ds \overset{a.s.}{=} O(ln(\lambda(t))) \). Finally, Karamata’s theorem (Embretches et al., 1997) proves the following asymptotic equivalence for regular variation functions with \( \beta > -1 \):

\[
\Lambda(t) \sim (\beta + 1)^{-1} t \lambda(t)
\]

Then, \( ln(\lambda(t)) = o(\sqrt{\Lambda(t)}) \) and equation (11) is proved.

For \( t \geq T_m \), the partial derivative in \( \rho \) of the failure intensity is:

\[
\lambda'_t(\rho) = g'(\rho) \lambda(t) + \sum_{j=0}^{m-1} (1 - (j + 1)\rho)(1 - \rho)^j \lambda(t) - \lambda(T_{N_{\rho} - j})
\]

But \( |g'(\rho)| \leq m (1 - \rho_1)^{m-1} \) and since \( [\rho_1, \rho_2] \) is a compact of \( ] - \infty, 1[ \) there exists \( M > 0 \) such that \( \sum_{j=0}^{m-1}(1 - \rho)^j | \rho(j + 1) - 1 | \leq M \). Then, the upper bound for \( |\lambda'_t(\rho)| \) in \( \Lambda'2 \) is realized with \( d = (m (1 - \rho_1)^{m-1} + M)/[1 \land (1 - \rho_2)^m] \). Finally, just notice that:

\[
\int_0^t \sup_{\rho \in \mathcal{J}_0} | \lambda'_t(\rho) - g'(\rho) \lambda(s) | \, ds \leq M \int_0^t \lambda(s) - \lambda(T_{N_{\rho} - m+1}) \, ds
\]

And using theorem 2, equation (12) is proved.

Therefore, theorem 3 can be applied. It first implies the convergence in distribution of \( (1 - \rho_0)^m - (1 - \hat{\rho}_t^{ML})^m \) to a centered Gaussian variable with the wanted standard deviation. Secondly it also implies the almost sure convergence of \( \Lambda(t)^{0.5 - \epsilon}((1 - \rho_0)^m - (1 - \hat{\rho}_t^{ML})^m)/(1 - \rho_0)^{0.5 + \epsilon} \) to zero for all \( \epsilon > 0 \). Then, \( (1 - \hat{\rho}_t^{ML})^m \overset{a.s.}{=} (1 - \rho_0)^m + o(\Lambda(t)^{-0.5 + \epsilon}) \) and the first result of the property can be easily deduced by an asymptotic expansion of this expression.
4.1.3 The ARA<sub>m</sub> model

**Proposition 5**

For ARA<sub>m</sub> model and under assumptions A0 and A2, the maximum likelihood estimator of repair efficiency parameter verifies, for a single observation of the failure process over [0, t]:

\[ \forall \epsilon > 0, \quad |\hat{\rho}_0 - \hat{\rho}_t^{ML}| \leq \epsilon^{1/2} \Rightarrow \frac{\lambda_{\min}(t)}{\lambda_{\max}(t)} \leq \lambda([1 \land (1 - \rho_t)^m] t) \]

**Proof.** As in the case of ARI<sub>m</sub> models, the failure intensity verifies:

\[ \lambda([1 \land (1 - \rho_t)^m] t) = \lambda_{\min}(t) \leq \lambda_t(\rho) \leq \lambda_{\max}(t) = \lambda([1 \lor (1 - \rho_t)^m] t) \quad (18) \]

Let \( g(\rho) = (1 - \rho)^m(\beta - 1) \). Then, thanks to proposition 1, equation (10) is verified. Similarly to the proof of proposition 3 and with the same notation \( R_t(\rho) = R_t \), the failure intensity verifies, for \( t \geq T_m \), \( \lambda_t(\rho) = g(\rho)\lambda(t + R_t(\rho)) \). Let \( M_1 = [1 \land (1 - \rho_2)^m] / (1 - \rho_1)^m > 0 \) and \( M_2 = [1 \lor (1 - \rho_1)^m] / (1 - \rho_2)^m > 0 \), then with \( \lambda(t) = t \) in (18) it is proved that:

\[ M_1 t \leq \frac{1 \land (1 - \rho_2)^m}{(1 - \rho_1)^m} t + R_t(\rho) \leq \frac{1 \lor (1 - \rho_1)^m}{(1 - \rho_2)^m} t \leq M_2 t \]

So a first order Taylor-Lagrange inequality implies that:

\[ |\lambda_t(\rho) - g(\rho)\lambda(t)| \leq |R_t(\rho)| \max_{M_1 t \leq s \leq M_2 t} |g(\rho)\lambda'(s)| \]

Since \( \lambda(t) = \alpha t^{\beta - 1} \)

\[ \max_{M_1 t \leq s \leq M_2 t} |g(\rho)\lambda'(s)| \leq (\beta - 1) g(\rho_1) t \lambda(t) \max_{M_1 t \leq s \leq M_2 t} |s^{\beta - 2}| \]

Let \( M_3 = [1 - 1/(1 - \rho_1)^m] \lor [1 - 1/(1 - \rho_2)^m] \). Then \( |R_t(\rho)| \leq M_3(t - T_{N_m - m + 1}) \). So

\[ \int_0^t \sup_{\rho \in \mathcal{J}_{\rho_0}} |\lambda_s(\rho) - g(\rho)\lambda(s)| ds \leq C \int_0^t \frac{s - T_{N_m - m + 1}}{s} \lambda(s) ds \]

where \( C = M_3 (\beta - 1) g(\rho_1) \max_{M_1 t \leq s \leq M_2 t} |s^{\beta - 2}| > 0 \). Finally, theorem 2 with \( u(t) = t \) implies that \( g(\rho_0) \int_0^t (s - T_{N_m - m + 1}) / s \lambda(s) ds \sim_{a.s.} O(ln(t)) \). Then, equation (11) is proved.

For \( t \geq T_m \), the partial derivative in \( \rho \) of the failure intensity is:

\[ \lambda'_t(\rho) = g'(\rho)\lambda(t + R_t(\rho)) + g(\rho)\lambda'(t + R_t(\rho)) \frac{\partial R_t(\rho)}{\partial \rho} \]

\[ = \lambda_t(\rho) \left[ g'(\rho) + \frac{(\beta - 1)}{t + R_t(\rho)} \sum_{j=0}^{m-1} (1 - (j - m + 1)\rho)(1 - \rho)^{j-m-1}(t - T_{N_m - j}) \right] \]
Since $[\rho_1, \rho_2]$ is a compact of $]-\infty, 1[$, there exists two constants $M_4 > 0$ and $M_5 > 0$ such that $|g'(\rho)| \leq M_4$ and $\sum_{j=0}^{m-1} |1 - (j - m + 1)\rho| (1 - \rho)^j \leq M_5$. Then
\[
|\lambda'(\rho)| \leq \left[ \frac{M_4}{g(\rho_1)} + (\beta - 1) \frac{M_5}{M_1} \right] \lambda(\rho) = M_6 \lambda(\rho)
\]
Then, the upper bound for $|\lambda'(\rho)|$ in $A^2$ is realized with $d = M_6/[1 \vee (1 - \rho_1)^m]^{\beta - 1}$. And
\[
\int_0^t \sup_{\rho \in \mathcal{F}_0} |\lambda'(\rho) - g'(\rho)\lambda(s)| \, ds \leq (\beta - 1) \frac{M_6}{M_1} g(\rho_1) \int_0^t \frac{s - T_n_{n-1}^{n+1}}{s} \lambda(s) \, ds
\]
So, equation (12) is proved. Therefore theorem 3 can be applied and the proposition is proved as it has been done for proposition 4.

4.2 Explicit Estimators

There also exists explicit estimators (EE) with the same asymptotic properties as MLE. These EE are based on the fact that the cumulative failure intensity can be asymptotically approximated by the asymptotic cumulative intensity with an error proportional to a logarithm. In this section we will use the more simple notation $\lambda_t = \lambda_t(\rho_0)$.

**Proposition 6**

For $ARI_m$ model, under assumption A1 and for a single observation of the failure process over $[0, t]$, the explicit estimator:
\[
\hat{\rho}_t^E = 1 - \left( \frac{N_t}{\Lambda(t)} \right)^{1/m}
\]
verifies the same convergence properties as the MLE of proposition 4.

**Proof.** Since A1 is assumed, proposition 2 is verified and so $\Lambda_t \overset{a.s.}{=} (1 - \rho_0)^m \Lambda(t) + O(ln(\lambda(t)))$. Then, by replacing this equation in the martingale writing of $N_t$ and thanks to equation (17), it can be proved for all $\epsilon \geq 0$ that:
\[
\Lambda(t)^{0.5-\epsilon}(1 - \hat{\rho}_t^E)^m - (1 - \rho_0)^m \overset{a.s.}{=} \frac{M_t}{\Lambda(t)^{0.5+\epsilon}} + o(1)
\]
Moreover, $\Lambda_t \overset{a.s.}{=} (1 - \rho_0)^m \Lambda(t) + o(\Lambda(t))$. So corollary 4 implies that $M_t/((1 - \rho_0)^m \Lambda(t))^{0.5+\epsilon}$ converges almost surely to zero for all $\epsilon > 0$ and converges in distribution to a standard Gaussian variable for $\epsilon = 0$. By using these two results in the previous equation, the property is proved.

Similarly, an explicit estimator can be found for the $AR\alpha_m$ model with strictly increasing power function for the initial intensity.
Proposition 7

For ARA\textsubscript{n} model, under assumption A2 and for a single observation of the failure process over \([0, t]\), the explicit estimator:

\[
\hat{\rho}_t^E = 1 - \left( \frac{N_t}{\alpha t^\beta} \right)^{1/(m(\beta-1))}
\]

verifies the same convergence properties as the MLE of proposition 5.

4.3 Asymptotic Confidence Intervals

Thanks to the asymptotic normality of all previous estimators (MLE and EE), we can obtain asymptotic confidence intervals (ACI) for repair efficiency. For a same model, MLE and EE verify the same normality property, then they define the same asymptotic confidence interval.

And the following corollary is easily proved.

Corollary 1

For an ARI\textsubscript{m} model, under the assumptions A0 and A1 for MLE and A1 for EE an asymptotic confidence interval for \((1 - \rho_0)^m\) at level \(\eta\) is:

\[
\left[ (1 - \hat{\rho})^m + \frac{u_\eta^2 - \sqrt{\Delta}}{2 \Lambda(t)}, (1 - \hat{\rho})^m + \frac{u_\eta^2 + \sqrt{\Delta}}{2 \Lambda(t)} \right]
\]

where \(u_\eta\) is the \(1 - \frac{\eta}{2}\) percentile of the standard Gaussian distribution, \(\hat{\rho}\) is the MLE or EE and \(\Delta = u_\eta^2[4\Lambda(t)(1 - \hat{\rho})^m + u_\eta^2]\).

By analogy, we have also an asymptotic confidence interval for the ARA\textsubscript{n} model.

Corollary 2

For an ARA\textsubscript{m} model, under the assumptions A0 and A2 for MLE and A2 for EE an asymptotic confidence interval for \((1 - \rho_0)^m(\beta-1)\) at level \(\eta\) is:

\[
\left[ (1 - \hat{\rho})^m(\beta-1) + \frac{u_\eta^2 - \sqrt{\Delta}}{2 \alpha t^\beta}, (1 - \hat{\rho})^m(\beta-1) + \frac{u_\eta^2 + \sqrt{\Delta}}{2 \alpha t^\beta} \right]
\]

where \(\Delta = u_\eta^2[4\alpha t^\beta(1 - \hat{\rho})^m(\beta-1) + u_\eta^2]\).

4.4 Coverage Rate of Asymptotic Confidence Intervals

In order to test the quality (convergence speed) of the ACI, we have calculated over 10000 simulations, and for an initial intensity \(\lambda(t) = \alpha t^\beta\), the coverage rate (CR) of the previous confidence intervals at level 95\% for \(m \in \{1, 2, 3\}\), \(\alpha \in \{1, 10\}\), \(\beta \in \{1.5, 2, 3\}\), \(\rho \in \{-1, -0.4, 0, 0.2, 0.4, 0.6, 0.8\}\) and \(n \in \{5, 10, 20, 40, 60, 80, 100\}\). The CR is the proportion of simulations for which the true value of the parameter is in the ACI.
The following notations are used in figure 2 that represents the CR versus the real value of the repair efficiency parameter.

- **MLE**
- **ARI**
- **ARA**

Figure 2: CR vs. $\rho$ for $\beta = 3$ and 40 observed failures

The value of the repair efficiency parameter $\rho_0$ modifies considerably the CR of EE. EE provides very good ACI for repair efficiency close to the ABAO case ($\rho_0$ close to 0): even for a very few number of failures (almost 10 for model with memory 1), ACI are very good approximations. When repair efficiency is too different from the ABAO case, CR of EE are converging dramatically slowly. This is a consequence of the fact that EE are based on an approximation of the cumulative failure intensity by the cumulative asymptotic intensity. The error done with this approximation is asymptotically almost surely equivalent to:

$$\frac{|1 - (1 + m\rho)(1 - \rho)^m|}{|\rho|(1 - \rho)^m} \ln(\lambda(t))$$

(propositions 2 and 3). This expression is null for $\rho = 0$ (in fact in that case $\Lambda_t = \Lambda_\infty(t)$) and increases as the distance between $\rho$ and 0 increases.

CR of MLE are less sensible to the value of $\rho_0$, but there are also less efficient, for repair efficiency close to the ABAO case. This homogeneity of the CR of MLE is certainly...
a consequence of the fact that this estimator is not only based on a first order asymptotic expansion of the cumulative failure intensity, but on the global asymptotic behavior of the failure process. For model with memory one, 20 failures is almost enough in order that ACI are good approximations for practical value of repair efficiency. In addition, on simulation, we noticed that it is not necessary to maximize the log-likelihood on a compact of $]-\infty, 1]$ containing $\rho_0$. In fact, we just have to prevent $\rho$ to be greater than one.

Comparing CR for different models with the same parameters is not necessarily judicious. In fact, we compare confidence intervals of different width and, in addition, for point processes that are not comparable since they have different asymptotic wear-out speed. An idea could be to compare models with similar asymptotic intensity (Doyen & Gaudoin, 2004).

5 Conclusion

This paper proves that ARI$_m$ and ARA$_m$ models with finite memories are adapted to systems in which repair efficiency is not strong enough to stop the wear-out. In fact they are asymptotically equivalent to a Non Homogeneous Poisson Process with nondecreasing failure intensity. Thanks to this property, convergence results of MLE estimators and explicit estimator of repair efficiency can be derived. Then, asymptotic confidence intervals can be built for repair efficiency.

It seems (but it has not been already proved) that ARA$_\infty$ models are more adapted to systems in which repair efficiency manages to restrain the wear-out. Then, maybe, memory can also be viewed as a repair efficiency parameter, that can be estimated. In the case of ARI$_\infty$ models, a kind of asymptotic intensity seems to appear on the simulations of the failure intensity. But unlike for finite memory models, this asymptotic wear-out is not proportional to the initial intensity.

References


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Appendix: Convergence theorems

Theorem 4 is a weak version of a classical theorem (Andersen et al., 1993) on stochastic integration.

**Theorem 4**

Let us consider a predictable stochastic process $Z = \{Z_t\}_{t \geq 0}$ such that $\int_0^t Z_s^2 \, d\Lambda_s$ is finite for all $t \geq 0$ (it is automatically true if $Z_s$ and $\lambda_s$ are bounded over $[0, t]$). Then, $\int_0^t Z_s \, dM_s$ is a local square integrable martingale with predictable variation process: $\int_0^t Z_s^2 \, d\Lambda_s$.

Theorems 5 and 6 (Coccozza, 1997) are two convergence results, one issued from the law of large numbers, and the other one from the central limit theorem.

**Theorem 5**

If $\{\langle M \rangle_t \}_{t < +\infty}$, $M_t$ converges almost surely to a finite random variable as $t$ grows to infinity. If $\{\langle M \rangle_t \}_{t = +\infty}$, for all $\epsilon > 0$: $M_t/\langle M \rangle_t^{0.5+\epsilon} \xrightarrow{a.s.} 0$.

**Theorem 6**

Let $Z$ be a locally bounded predictable process.

Then, under the following assumptions:

- there exists $c_t > 0$ and $\sigma$ such that: $\int_0^t Z_s^2 \lambda_s \, ds/c_t^2 \xrightarrow{P} \sigma^2$,

- for all $\delta > 0$ $\int_0^t Z_s^2 \mathbb{1}_{\{|Z_s| > \delta c_t\}} \lambda_s \, ds/c_t^2 \xrightarrow{P} 0$ (it is automatically true if $Z_t$ is almost surely finite and $c_t$ grows to the infinity)

the random variable: $\int_0^t Z_s \, dM_s/c_t$ converges in law to a Gaussian centered variable with standard deviation $\sigma$.

In particular, theorems 4 and 5 imply the first following corollary and theorems 5 and 6 imply the second following corollary. Both deal with martingale asymptotic behavior.

**Corollary 3**

Let $Z$ be a bounded predictable process such that $\int_0^t Z_s^2 \, d\Lambda_s$ is finite for all $t \geq 0$. Then: $\int_0^t Z_s \, dM_s \xrightarrow{a.s.} o(\int_0^t Z_s^2 \, d\Lambda_s) + O(1)$.

**Corollary 4**

If there exists a positive, nondecreasing, divergent function $\Lambda_{\infty}(t)$ such that the cumulative failure intensity verifies almost surely $\Lambda_t = \Lambda_{\infty}(t) + o(\Lambda_{\infty}(t))$, then for all $\epsilon > 0$, $M_t \xrightarrow{a.s.} o(\Lambda_{\infty}(t)^{0.5+\epsilon})$ and $M_t/\sqrt{\Lambda_{\infty}(t)} \xrightarrow{L} \mathcal{N}(0, 1)$.