Chapitre 1
An Introduction to Reduced Basis Output Bounds Methods

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Conduction Heat Transfer

\[ \text{Bi: heat transfer coefficient} \]

\[ \psi^4 \]
\[ \psi^3 \]
\[ \psi^2 \]
\[ \psi^1 \]
\[ \Omega \subset \mathbb{R}^2 \quad \text{2-D} \]

\[ \Gamma_{\text{root}} \]

Root: Heat In

\[ \Omega \subset \mathbb{R}^3 \quad \text{3-D} \]

\[ \Gamma_{\text{tip}} \]

\[ \psi^4 \]
\[ \psi^3 \]
\[ \psi^2 \]
\[ \psi^1 \]

\[ \Gamma_{\text{root}} \]

Root: Heat In

\[ k^0 = 1 \]

\[ L \]

Mathematical Model
## Inputs and Outputs

<table>
<thead>
<tr>
<th>Inputs</th>
<th>[ \mu \equiv ( (k^i)_{i=1}^{4} , ; Bi , ; (t, \ell) ) ]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>thermal conductivities \hspace{5em} convective coefficient \hspace{5em} geometry</td>
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<tr>
<td>[ \mu \in \mathcal{D} \subset \mathbb{R}^7 ]</td>
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| Outputs | \[ s^1(\mu) \equiv \int_{\Gamma_{\text{root}}} T(x; \mu) = \ell^1(T(x; \mu); \mu) \] |
| | \( \text{T_ROOT}^\dagger \) |
| | \[ s^2(\mu) \equiv \int_{\Omega} = \ell^2(\mu) \] |
| | \( \text{VOLUME} \) |
| | \[ s^3(\mu) = \text{Bi} \] |
| | \( \text{"FAN\_POWER"} \) |

\[^\dagger\text{The temperature is defined relative to the ambient level, } T_{\text{amb}}.\]
Engineering Goals

We wish to maintain

\[ T_{\text{ROOT}} \left[ -T_{\text{amb}} \right] \leq T_{\text{per}} - T_{\text{amb}} \]

while minimizing

\[ \text{VOLUME, FAN\_POWER, COSTS} \]

Optimization should be performed both at \textit{design} and (adaptively) in \textit{operation}. 
Optimization Problem: Design

For DESIGN ITERATION 1, 2, …

Choose  \( k_1, k_2, k_3, k_4 \) materials;
        \( c_1, c_2 \) preferences;
        \( (T_{\text{per}} - T_{\text{amb}})_{\text{min}} \) scenarios.

Minimize  \( c_1 \text{ VOLUME}(\mu) + c_2 \text{ FAN\_POWER}(\mu) \).

FEASIBLE \( \{ Bi, (t, L) \in D_{Bi,(t,L)} \mid T_{\text{ROOT}}(\mu) \leq (T_{\text{per}} - T_{\text{amb}})_{\text{min}} \} \)

“Inspect” proposed optimal design.
### Optimization Problem: Operation

#### Configuration

\[(k_i)_{i=1,...,4}, \ t, \ L\] \text{NOW FIXED}

#### Real-Time

**For OPERATION (MISSION) 1, 2, ...**

- **"Measure"** \(T_{\text{amb}}\).
- **Minimize** \(\text{FAN\_POWER} (\mu)\). \((= \ Bi)\)

\[
\text{FEASIBLE} \left\{ \begin{array}{l}
T_{\text{ROOT}} (\mu) \leq T_{\text{per}} - T_{\text{amb}} \\
Bi \in D^{Bi}
\end{array} \right.
\]

**Perform** mission.
### 3D Thermal Fin

- nondimensional 3D Thermal Fin
- dimensional 3D Thermal Fin
- dimensional 3D Thermal Fin Optimization
MicroTruss: 3D Heat Exchanger

Conduction/Convection Heat Transfer, Fluid Flow & (Non)linear Elasticity

Γ_{stress} \quad \Gamma_{flux} \quad \text{heat, } q'' \quad \Gamma_{L} \quad \text{load} \\
coolant \quad P_{high}, T_{0} \quad F \\
P_{low}
MicroTruss : 3D Heat Exchanger

Inputs :

\[ \mu \equiv (H, W, \alpha, t_{\text{top}}, t_{\text{bot}}, t_{\text{truss}}, k_{\text{solid}}, P_{\text{high}} - P_{\text{low}}) \]
### MicroTruss: 3D Heat Exchanger

**Outputs**

\begin{align*}
  s^1(\mu) & \equiv \int_{\Omega_{\text{solid}}} \text{(VOLUME)} \\
  s^2(\mu) & \equiv \sum_{\text{channels}} \Delta PQ \quad \text{(PUMP\_POWER)} \\
  s^3(\mu) & \equiv \frac{1}{|\Gamma_{\text{flux}}|} \int_{\Gamma_{\text{flux}}} T(x; \mu) \quad \text{(T\_TOP)} \\
  s^4(\mu) & \equiv \frac{1}{|\Gamma_L|} \int_{\Gamma_L} u_2(x; \mu) \quad \text{(DEFLECTION)} \\
  s^5(\mu) & \equiv \frac{1}{|\Gamma_{\text{stress}}|} \int_{\Gamma_{\text{stress}}} \sigma_n(x; \mu) \quad \text{(STRESS)} \\
  s^{6,7}(\mu) & \equiv F_{\text{buckle}}(\mu)^{+/-} \quad \text{(BUCKLE\_LOAD)^{+/-}}
\end{align*}
Heat Transfer

MicroTruss: 3D Heat Exchanger

Temperature
Buckling Modes

MicroTruss: 3D Heat Exchanger

**Buckling Modes**

\[ t_{\text{top}} = 1.5, \quad t_{\text{bot}} = 0.5 \]

\[ t_{\text{top}} = 1.5, \quad t_{\text{bot}} = 1.0 \]
Objectives

Develop methodology to

Evaluate $s(\mu)^\dagger$ rapidly and reliably
in the limit of very many $\mu$-queries;

with application to optimal design and operation.

Enablers:

- Restriction to parametric dependence;
- Acceptance of large initial (off-line) cost.

\dagger Outputs of $\mu$-parametrized partial differential equations.
Field variables and associated PDEs

\(\mu\)-Parametrized Elliptic PDEs

A few examples

- temperature and steady conduction – Poisson (Linear)
- displacement and equilibrium – Helmholtz-elasticity (Linear)
- wavefunction and stationary Schrödinger via Hartree-Fock approximation (Nonlinear)
Applications: deployed systems

- assessment, evolution and accommodation of a crack in a critical component
- real-time characterisation and optimisation of the heat treatment protocol of a turbine disk

Require
- robust parameter estimation (inverse problem)
- adaptive design (optimisation)
Ingredients

- “Truth” FEM approximation: \( u_N(\mu) \)
- Reduced order approximation: \( s_N(\mu) \)
- Reliable and sharp error estimation: \( \Delta_N(\mu) \)
- Offline (expensive) / Online \( (O(1\text{ms})) \) decomposition
- \( S_N \) and \( W_N \) Generation strategies
“Truth” FEM Approximation

Let $\mu \in D^\mu$, evaluate

$$s^N(\mu) = \ell(u^N(\mu)),$$

where $u^N(\mu) \in Y^N$ satisfies

$$g(u^N, v; \mu) = 0 \quad \forall v \in Y^N$$

In the linear case, $g(w, v; \mu) \equiv a(u^N(\mu), v; \mu) - f(v)$ and $u^N(\mu)$ is solution of

$$a(u^N(\mu), v; \mu) = f(v), \quad \forall v \in Y^N.$$

Here $Y^N \subset Y$ is a Truth finite element approximation of dimension $N \gg 1$ equipped with an inner product $(\cdot, \cdot)_Y$ and induced norm $\|\cdot\|_Y$. Denote also $Y'$ and associated norm

$$\ell \in Y', \quad ||\ell||_Y' \equiv \sup_{v \in Y} \frac{\ell(v)}{\|v\|_Y}.$$
Purpose

- **Equate** \( u(\mu) \) and \( u_N(\mu) \) in the sense that

\[
\|u(\mu) - u_N(\mu)\|_Y \leq \text{tol} \quad \forall \mu \in D^\mu
\]

- **Build** the reduced basis approximation using the FEM approximation

- **Measure** the error associated with the reduced basis approximation relative to the FEM approximation
Dual/Adjoint Problem: Linear case

Given $\mu \in D^\mu$, find $\Psi(\mu) \in Y^N$ such that

$$a(v, \Psi(\mu); \mu) = -\ell(v; \mu), \quad \forall v \in Y^N$$

Ensures

- good approximation
- good error characterisation of the output

Compliance

if $a$ symmetric and $\ell = f$, $\Psi(\mu) = -u(\mu)$
Approximation opportunities: Low-Dimension Manifold

\[ Y \equiv H^1(\Omega \subset \mathbb{R}^d) \]

To approximate \( u(\mu) \), and thus \( s(\mu) \), we need not represent all functions in \( Y \).
Approximation opportunities: Low-Dimension Manifold

\[ W = \{ u(\mu) \in Y ; \mu \in D^\mu \} \]

To approximate \( u(\mu) \), and thus \( s(\mu) \), we need only approximate functions in low-dimensional manifold

\[ W = \{ u(\mu) \in Y ; \mu \in D^\mu \} \]
Approximation opportunities: Low-Dimension Manifold

\[ W_N = \{ u(\mu^i) \in Y ; \mu^i \in \mathcal{D}^\mu \} \]

To approximate \( u(\mu) \), and thus \( s(\mu) \), we construct the approximation space

\[ W_N = \{ u(\mu^i) \in Y ; (\mu^i)_{i=1}^N \in \mathcal{D}^\mu \} \]
Computational opportunities: observations

Two observations:

- parameter dependence can be (often) expressed as (e.g. in the linear case)

\[
a(w, v; \mu) = \sum_{q=1}^{Q} \Theta^q(\mu) a^q(w, v)
\]

with

- \( \Theta^q(\mu) : \mathcal{D}^\mu \to \mathbb{R} \),
- \( a^q(w, v) : Y^N \times Y^N \to \mathbb{R}, 1 \leq q \leq Q \)

- “deployed” systems requires low marginal cost for each \( \mu \to s(\mu) \)
Computational opportunities: strategy

**Offline-Online decomposition**

A two stages strategy

- pre-deployment: computationally intensive initial preprocessing (offline)
- deployment: greatly reduced marginal cost (online)

applied to two main ingredients

- rapidly convergent over $\mathcal{D}^\mu$ reduced basis approximation
- associated rigorous sharp *a posteriori* error bounds
Formulation (Linear Case)

Sample : \( S_N = \{ \mu_1 \in D^\mu, \ldots, \mu_N \in D^\mu \} \).

Sample : \( S_{N_{du}}^{du} = \{ \mu^{du} \in D^\mu, \ldots, \mu^{du}_{N_{du}} \in D^\mu \} \).

Space : \( W_N = \text{span} \{ \zeta_n \equiv u(\mu^n), n = 1, \ldots, N \} \).

Space : \( W_{N_{du}}^{du} = \text{span} \{ \zeta_{du}^{n_{du}} \equiv \Psi(\mu^{du}_n), n = 1, \ldots, N^{du} \} \).

Sampling strategies?

- Equidistributed points in \( D^\mu \) (curse of dimensionality)
- Log-random distributed points in \( D^\mu \)
- See later for more efficient, adaptive strategies
Formulation (Linear Case): a Galerkin method

Galerkin Projection

Given \( \mu \in \mathcal{D}^\mu \) evaluate

\[
s_N(\mu) = \ell(u_N(\mu)) + g(u_N(\mu), \Psi_{N_{du}}(\mu); \mu); \quad (1)
\]

where \( u_N(\mu) \in \mathcal{W}_N \) and \( \Psi_{N_{du}}(\mu) \in \mathcal{W}_{N_{du}} \) satisfy

\[
a(u_N(\mu), v; \mu) = f(v), \quad \forall \ v \in \mathcal{W}_N.
\]

and

\[
a(v, \Psi_{N_{du}}(\mu); \mu) = -\ell(v), \quad \forall \ v \in \mathcal{W}_{N_{du}}.
\]
Formulation (Linear Case) : back to the compliant case

Recall that in compliance

- $a$ is symmetric
- $\ell = f$

such that $\Psi(\mu) = -u(\mu)$.

We may take $N_{du}^u = N$, $S_{N}^{du} = S_{N}$ and $W_{N}^{du} = W_{N}$ and get

$$\Psi_{N}(\mu) = -u_{N}(\mu)$$

Compliant case

- The dual problem is never formed/solved
- We simply identify $\Psi_{N}(\mu) = -u_{N}(\mu)$
- We get a 50% cost reduction
Formulation (Linear Case) : offline-online decomposition

Expand our RB approximations:

\[ u_N(\mu) = \sum_{j=1}^{N} u_{Nj}(\mu) \zeta_j \]  
\[ (2) \]
\[ \Psi_{N^{du}}(\mu) = \sum_{j=1}^{N_{du}} \Psi_{Nj}(\mu) \zeta_{j^{du}} \]  
\[ (3) \]

Express \( s_N(\mu) \), thanks to (1), (2) and (3)

\[ s_N(\mu) = \sum_{j=1}^{N} u_{Nj}(\mu) \ell(\zeta_j) + \sum_{j=1}^{N_{du}} \Psi_{Nj}(\mu) f(\zeta_{j^{du}}) \]
\[ + \sum_{j=1}^{N} \sum_{j'=1}^{N_{du}} \sum_{q=1}^{Q} u_{Nj}(\mu) \Psi_{Nj'}(\mu) \Theta^q(\mu) a^q(\zeta_j, \zeta_{j'}) \]  
\[ (4) \]
Formulation (Linear Case) : offline-online decomposition

In (4), \( u_{Ni}(\mu), 1 \leq i \leq N \) and \( \Psi_{Ni}(\mu), 1 \leq i \leq N^{du} \) satisfy

\[
\sum_{j=1}^{N} \left\{ \sum_{q=1}^{Q} \Theta^q(\mu) a^q(\zeta_i, \zeta_j) \right\} u_{Nj}(\mu) = f(\zeta_i), \quad 1 \leq i \leq N \tag{5}
\]

\[
\sum_{j=1}^{N^{du}} \left\{ \sum_{q=1}^{Q} \Theta^q(\mu) a^q(\zeta_i^{du}, \zeta_j^{du}) \right\} \Psi_{N^{du}j}(\mu) = -\ell(\zeta_i^{du}), \quad 1 \leq i \leq N^{du} \tag{6}
\]
Formulation (Linear Case) : matrix form

Solve

$$A_N(\mu) \, u_N(\mu) = F_N$$  \hspace{1cm} (7)

and

$$A_{Ndu}^{du}(\mu) \, \psi_{Ndu}(\mu) = -L_N$$  \hspace{1cm} (8)

where

$$(A_N)_{ij}(\mu) = \sum_{q=1}^{Q} \Theta^q(\mu) \, a^q(\zeta_i, \zeta_j), \quad F_{Ni} = f(\zeta_i) .$$  \hspace{1cm} (9)

$$1 \leq i, j \leq N \quad 1 \leq i \leq N$$

and

$$(A_{Ndu}^{du})_{ij}(\mu) = \sum_{q=1}^{Q} \Theta^q(\mu) \, a^q(\zeta_i^{du}, \zeta_j^{du}), \quad L_{Ni} = \ell(\zeta_i^{du}) .$$  \hspace{1cm} (10)

$$1 \leq i, j \leq N^{du} \quad 1 \leq i \leq N^{du}$$
Formulation (Linear Case): complexity analysis

**Offline: independent of $\mu$**

Solve: $N + N^{du}$ FEM system depending on $N$
Form and store: $f(\zeta_i), \ell(\zeta_i), f(\zeta_i^{du}), \ell(\zeta_i^{du})$
Form and store: $a^q(\zeta_i, \zeta_j), a^q(\zeta_i^{du}, \zeta_j^{du})$

**Online: independent of $N$**

Given a new $\mu \in D^\mu$
Form and solve $A_N(\mu): O(QN^2)$ and $O(N^3)$
Form and solve $A_{N^{du}}(\mu): O(QN^{du2})$ and $O(N^{du3})$
Compute $s_N(\mu)$

**Online: $N, N^{du} << N$**

Online we realize often orders of magnitude computational economies relative to FEM in the context of many $\mu$-queries
Questions

- Is there a solution \( u(\mu) \) near \( u_N(\mu) \)? (particularly relevant in the nonlinear context)
- Is \( |s(\mu) - s_N(\mu)| \leq \epsilon_{\text{tol}} \)?
- Is \( s(\mu) \leq C \) satisfied? (not only \( s_N(\mu) \))
- Is \( N \) too large? will \( |s(\mu) - s_N(\mu)| \leq \epsilon_{\text{tol}} \) with a \( O(N^3) \) complexity impact on computation efficiency?
Definitions: inner products and norms

- Denote $Y \equiv Y^N$ our Hilbert space and denote $(\cdot, \cdot)_Y$, $\| \cdot \|_Y$ the associated inner product and norm.
- Denote the dual norm of any bounded linear functional $h$

\[
\|h\|_{Y'} = \sup_{v \in Y} \frac{h(v)}{\|v\|_Y} \tag{11}
\]

- Recall that we are interested in a 2nd order elliptic PDEs, then $H^1_0(\Omega) \subset Y^e \subset H^1(\Omega)$, a typical choice for $(\cdot, \cdot)_Y$ is then

\[
(w, v)_Y = \int_{\Omega} \nabla w \cdot \nabla v + w v \tag{12}
\]
Definitions: Operator

- Introduce $T^\mu : Y \rightarrow Y$ such that for any $w \in Y$
  \[
  (T^\mu w, v)_Y = a(w, v; \mu), \quad \forall v \in Y
  \] (13)

- Define
  \[
  \sigma(w; \mu) = \frac{||T^\mu w||_Y}{||w||_Y}
  \] (14)

- Note
  \[
  \beta(\mu) = \inf_{w \in Y} \sup_{v \in Y} \frac{a(w, v; \mu)}{||w||_Y ||v||_Y} = \inf_{w \in Y} \sigma(w; \mu)
  \] (15)
  \[
  \gamma(\mu) = \sup_{w \in Y} \sup_{v \in Y} \frac{a(w, v; \mu)}{||w||_Y ||v||_Y} = \sup_{w \in Y} \sigma(w; \mu)
  \] (16)

Remarks

- $\beta(\mu)$ is the Babuska “inf-sup” stability constant
  \[
  \beta(\mu) ||w||_Y ||T^\mu w||_Y \leq a(w, T^\mu w; \mu), \quad \forall w \in Y
  \] (17)

- $\gamma(\mu)$ is the continuity constant
Hypothesis

- $\gamma(\mu)$ is bounded $\forall \mu \in D^\mu$
- $\beta(\mu) \geq \beta_0 \geq 0$, $\forall \mu \in D^\mu$

Remark: the symmetric coercive bilinear form case

we have $\beta(\mu) = \alpha_c(\mu)$ where

$$\alpha_c(\mu) = \inf_{w \in Y} \frac{a(w, w; \mu)}{\|w\|_Y^2} \quad (18)$$

is the coercivity constant
Given our RB approximation \( u_N(\mu) \), we have

\[
e(\mu) \equiv u(\mu) - u_N(\mu)
\]

that satisfies

\[
a(e(\mu), v; \mu) = -g(u_N(\mu), v; \mu), \forall v \in Y
\]

where \(-g(u_N(\mu), v; \mu) = f(v) - a(u_N(\mu), v; \mu)\) in the linear case is the residual. We have then from (11), (17) and (20)

\[
\|e(\mu)\|_Y \leq \frac{\|g(u_N(\mu), v; \mu)\|_{Y'}}{\beta(\mu)} = \frac{\varepsilon_N(\mu)}{\beta(\mu)}
\]
Given \( \tilde{\beta}(\mu) \) a nonnegative lower bound of \( \beta(\mu) \):

\[
\beta(\mu) \geq \tilde{\beta}(\mu) \geq \epsilon_\beta \beta(\mu), \quad \epsilon_\beta \in ]0, 1[, \quad \forall \mu \in D^\mu
\]  \hspace{2cm} (22)

**Definition : Energy error bound**

\[
\Delta_N(\mu) \equiv \frac{\varepsilon_N(\mu)}{\tilde{\beta}(\mu)} \hspace{2cm} (23)
\]

**Definition : Effectivity**

\[
\eta_N(\mu) \equiv \frac{\Delta_N(\mu)}{\|e(\mu)\|_Y} \hspace{2cm} (24)
\]
\( u_N(\mu) \): Rigorous sharp error bounds

One can prove that

\[
1 \leq \eta_N(\mu) \leq \frac{\gamma(\mu)}{\tilde{\beta}(\mu)}, \quad 1 \leq N \leq N_{\text{max}}, \quad \forall \mu \in D^\mu
\] (25)

**Remarks**

- **Rigorous**: Left inequality ensures rigorous upper bound measured in \( \| \cdot \|_Y \), i.e. \( \| e(\mu) \|_Y \leq \Delta_N(\mu), \quad \forall \mu \in D^\mu \)

- **Sharp**: Right inequality states that \( \Delta_N(\mu) \) overestimates the “true” error by at most \( \gamma(\mu)/\tilde{\beta}(\mu) \)
\[ \Psi_N(\mu) : \text{error bounds} \]

We have a similar result for the dual problem

\[ \| \psi(\mu) - \psi_{N^{\text{du}}} \|_Y \leq \Delta_N^{\text{du}}(\mu), \quad 1 \leq N^{\text{du}} \leq N_{\text{max}}^{\text{du}}, \quad \forall \mu \in D^{\mu} \quad (26) \]

where

\[ \Delta_N^{\text{du}}(\mu) \equiv \frac{\varepsilon_N^{\text{du}}(\mu)}{\tilde{\beta}(\mu)} \equiv \frac{\| - \ell(\cdot) - a(\cdot, \Psi_{N^{\text{du}}}(\mu); \mu) \|_Y}{\tilde{\beta}(\mu)} \quad (27) \]

\( \varepsilon_N^{\text{du}}(\mu) \) is the dual norm of the residual.
$s_N(\mu)$: error bounds

It follows from (24) and (27)

$$|s(\mu) - s_N(\mu)| \leq \Delta^s_N(\mu), \quad \mu \in \mathcal{D}_\mu$$  

(28)

where

$$\Delta^s_N(\mu) \equiv \varepsilon_N(\mu)\Delta^\text{du}_N(\mu)(\mu) = \tilde{\beta}(\mu)\Delta_N(\mu)\Delta^\text{du}_N(\mu)$$  

(29)

Rapid convergence of the error in the output

Note that the error in the output vanishes as the product of the error in the primal and dual error

Back to compliance: a symmetric and $\ell = f$

We obtain

$$\Delta^s_N(\mu) \equiv \frac{\varepsilon^2_N(\mu)}{\tilde{\beta}(\mu)}, \quad \forall \mu \in \mathcal{D}_\mu$$  

(30)
Offline-Online decomposition (Primal problem)

- Dual problem: similar treatment
- Denote $\hat{e}(\mu) \in Y$

$$||\hat{e}(\mu)||_Y = \varepsilon_N(\mu) = ||g(u_N(\mu), \cdot; \mu)||_Y$$ (31)

such that

$$(\hat{e}(\mu), v)_Y = -g(u_N(\mu), v; \mu), \quad \forall v \in Y$$ (32)

- Recall that

$$-g(u_N(\mu), v; \mu) =$$

$$f(v) - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu) u_{Nn}(\mu) a^q(\zeta_n, v), \quad \forall v \in X$$ (33)
Offline-Online decomposition (Primal problem)

• It follows next that \( \hat{e}(\mu) \in Y \) satisfies

\[
(\hat{e}(\mu), v)_Y = f(v) - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu) u_{Nn}(\mu) a^q(\zeta_n, v), \quad \forall v \in X
\]  

(34)

• Observe then that the rhs is the sum of products of parameter dependent functions and parameter independent linear functionals, thus invoking linear superposition

\[
\hat{e}(\mu) = C - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu) u_{Nn}(\mu) L_n^q
\]  

(35)

where

• \( C \in Y \) satisfies

\[
(C, v) = f(v), \quad \forall v \in Y
\]  

(36)

• \( L \in Y \) satisfies

\[
(L_n^q, v)_Y = -a^q(\zeta_n, v), \quad \forall v \in Y, \; 1 \leq n \leq N, \; 1 \leq q \leq Q
\]  

(37)

(37) are parameter independent Poisson problems
Offline-Online decomposition: Error bounds

From (35) we get

$$\|\hat{e}(\mu)\|_Y^2 = (C, C)_Y + \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu) u_{Nn}(\mu) \left\{ 2(C, L^q_n)_Y + \sum_{q'=1}^{Q'} \sum_{n'=1}^{N'} \Theta^{q'}(\mu) u_{Nn'}(\mu) (L^q_n, L^{q'}_{n'})_Y \right\}$$

(38)

Remark

In (38), $\|\hat{e}(\mu)\|_Y^2$ is the sum of products of

- parameter dependent (simple/known) functions and
- parameter independent inner-product,

the offline-online for the error bounds is now clear.
### Offline-Online decomposition: steps and complexity

**Offline:**

Solve for $C$ and $L^q_n$, $1 \leq n \leq N$, $1 \leq q \leq Q$

Form and save $(C, C)_Y$, $(C, L^q_n)_Y$ and $(L^q_n, L^{q'}_{n'})_Y$, $1 \leq n, n' \leq N$, $1 \leq q, q' \leq Q$

**Online**

Given a new $\mu \in D^\mu$

Evaluate the sum (38) in terms of $\Theta^q(\mu)$ and $u_{Nn}(\mu)$

Complexity in $O(Q^2N^2)$ independent of $N$
The linear symmetric coercive case

- We require a lower bound $\tilde{\beta}(\mu)$ for $\beta(\mu) = \alpha_c(\mu)$, $\forall \mu \in D^\mu$
- If
  - $\Theta^q(\mu) > 0$, $\forall \mu \in D^\mu$ and
  - $a^q(v, v) \geq 0$, $\forall v \in Y$, $1 \leq q \leq Q$

We choose

$$\tilde{\beta}(\mu) = \left( \min_{q=1 \ldots Q} \frac{\Theta^q(\mu)}{\Theta^q(\bar{\mu})} \right) \alpha_c(\bar{\mu}) \quad (39)$$

for some $\bar{\mu} \in D^\mu$

Remarks

- restrictive hypotheses
- more complicated recipes must often be pursued
Offline-Online Scenarios

Offline

Given a tolerance $\tau$, build $S_N$ and $W_N$ s.t.

$$\forall \mu \in \mathcal{P} \equiv \mathcal{D}^\mu, \Delta_N(\mu) < \tau$$

Online

Given $\mu$ and a tolerance $\tau$, find $N^*$ and thus $s_{N^*}(\mu)$ s.t.

$$N^* = \text{arg max}_N \Delta_N(\mu) < \tau$$

or given $\mu$ and a max execution time $T$, find $N^*$ and thus $s_{N^*}(\mu)$ s.t.

$$N^* = \text{arg min}_N \Delta_N(\mu) \text{ and execution time } < T$$
Offline Generation

Given a tolerance $\epsilon$, set $N = 0$ and $S_0 = \emptyset$

While $\Delta^\text{max}_N > \epsilon$

$N = N + 1$

If $N == 1$; then Pick ((log-)randomly) $\mu_1 \in D^\mu$

Build $S_N := \{\mu_N\} \cup S_{N-1}$

Build $W_N := \{\xi = u(\mu_N)\} \cup W_{N-1}$

Compute $\Delta^\text{max}_N := \max_{\mu \in D^\mu} \Delta_N(\mu)$

$\mu^{N+1} := \arg \max_{\mu \in D^\mu} \Delta_N(\mu)$

End While

Condition number

the $\zeta_n$ are orthonormalized, this ensures that the condition number
will stay bounded by $\gamma(\mu)/\beta(\mu)$
Online Algorithm I

\[ \mu \text{ adaptive online} \]

Given \( \mu \in \mathcal{D}^\mu \), compute \( (s_N^*(\mu), \Delta_N^*(\mu)) \) such that \( \Delta_N^*(\mu) < \tau \).

\( N = 2 \)

While \( \Delta_N(\mu) > \tau \)

Compute \( (s_N(\mu), \Delta_N(\mu)) \) using \( (S_N, W_N) \)

\( N = N \times 2 \)

use the (very) fast convergence properties of RB

End While
## Online Algorithm II

### Offline

While \( i \leq \text{Imax} \) \( >> 1 \)

Pick log-randomly \( \mu \in \mathcal{D}^\mu \)

Store in table \( \mathcal{T}, \Delta_N(\mu) \) if worst case for \( N = 1, \ldots, N^{\text{max}} \)

\( i = i + 1; \) End While

### Online Algorithm II – \( \mu \) adaptive online – worst case

Given \( \mu \in \mathcal{D}^\mu \), compute \((s_{N^*}(\mu), \Delta_{N^*}(\mu))\) such that \( \Delta_{N^*}(\mu) < \tau \).

\( N^* := \arg \max_{\mathcal{T}} \Delta_N(\mu) < \tau \)

Use \( \mathcal{W}_{N^*} \) to compute \((s_{N^*}(\mu), \Delta_{N^*}(\mu))\)
Online Algorithm II

Offline

While $i \leqslant \text{Imax} > 1$
Pick log-randomly $\mu \in \mathcal{D}^\mu$
Store in table $\mathcal{T}, \Delta_N(\mu)$ if worst case for $N = 1, \ldots, N_{\text{max}}$
$i = i + 1$; End While

Online Algorithm II – $\mu$ adaptive online – worst case

Given $\mu \in \mathcal{D}^\mu$, compute $(s_{N^*}(\mu), \Delta_{N^*}(\mu))$ such that $\Delta_{N^*}(\mu) < \tau$.
$N^* := \arg\max_{\mathcal{T}} \Delta_N(\mu) < \tau$
Use $\mathcal{W}_{N^*}$ to compute $(s_{N^*}(\mu), \Delta_{N^*}(\mu))$
Online Algorithm II

**Offline**

While $i \leq \text{Imax} >> 1$

Pick log-randomly $\mu \in \mathcal{D}^\mu$

Store in table $\mathcal{T}, \Delta_N(\mu)$ if worst case for $N = 1, \ldots, N_{\text{max}}$

$i = i + 1$; End While

**Online Algorithm II – $\mu$ adaptive online – worst case**

Given $\mu \in \mathcal{D}^\mu$, compute $(s_{N^*}(\mu), \Delta_{N^*}(\mu))$ such that $\Delta_{N^*}(\mu) < \tau$.

$N^* := \text{argmax}_T \Delta_N(\mu) < \tau$

Use $W_{N^*}$ to compute $(s_{N^*}(\mu), \Delta_{N^*}(\mu))$
Outlook I

- How large can $P$ be?
  that would depend on the problem: $N$ growth is modest as $P$, however the complexity for the inf-sup lower bound is problematic.

- Non-affine, non-linear?
  use of new empirical interpolation approach
  replace $\mathcal{H}(u; x; \mu)$ by a collateral RB, i.e.

\[
\mathcal{H}(u_N(x; \mu); x; \mu) = \sum_{m=1}^{M} d_m(\mu) \, \xi_m(x) \tag{40}
\]

where $\xi_m(x) = \mathcal{H}(u_N(x; \mu^H_m); x; \mu^H_m)$, $\mu^H_m \in \mathcal{S}^H_M$ and $d_m(\mu)$ is obtained using a stable inexpensive interpolation procedure.

An error estimator is also developed albeit not always provably rigorous.
Outlook II

• class of PDEs
  • Parabolic: most elliptic ingredients apply except that time appears a parameter
  • Hyperbolic: may perform well in certain problems, but the underlying smoothness in $\mu$ and stability won’t be obtained. As a result approximation properties and error estimators will suffer

• Offline stage is often complicated and computationally intensive. So these techniques will be viable only in the context of real-time certified response