A non local Schrödinger model for the propagation of waves in a photorefractive medium

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Overview

1. Derivation of the model
   - Physical modelling
   - From mesoscopic to macroscopic
   - Mathematical setting

2. Cauchy problem
   - Saturated NLS equation
   - Zozulya–Anderson model

3. Solitary waves
   - One-dimensional results
   - Solitary waves for (SNLS)
   - Solitary waves for (ZA)

4. Conclusion and perspectives
The photorefractive effect

- The propagation of an optical wave in insulating or semi-insulating electrooptical crystals induces a charge transfer.
- The new distribution of charges $\rho$ induces an electric field $E_{sc}$ such that $\nabla \cdot (\hat{\varepsilon}E_{sc}) = \rho$.
- This field derives from a potential $\varphi$ and produces a variation $\delta n$ of the refraction index in the principal direction of the photovoltaic effect in the crystal (chosen as $x$): $\delta n \propto \partial_x \varphi$. 
## Characteristics of the photorefractive effect

### Main characteristics

1. **Sensibility to energy** and not to the electric field (cf. Kerr effect, NLS).
2. **Nonlocal effect.** Charge distributions and the electric field are not located at the same position.
3. **Inertia.** Charges need a certain time to move, neglected here.
4. **Memory and reversibility** with applications to holography. Neglected here considering only electrons as moving charges.

### Equations

charge equation, evolution of ionized donor sites, charge transport, charge conservation, Poisson equation

$$\Rightarrow \text{mesoscopic description: Kukhtarev model.}$$
Derivation of the model

From mesoscopic to macroscopic

Description of the laser

- Paraxial approximation.
- Envelope approximation in the $z$ direction.

Other modelling tasks

- Heuristic asymptotic analysis.
- Dimensionless equations.

\[
\left[ \partial_z - i \nabla^2 \right] A(x) = \pm i \partial_x \varphi A(x).
\]

Special form of the $\hat{\varepsilon}$ tensor leads to

\[
\nabla^2_{\perp} \varphi + \nabla_{\perp} \ln(1 + |A|^2) \cdot \nabla_{\perp} \varphi = \partial_x \ln(1 + |A|^2).
\]

This is the Zozulya–Anderson model.
Mathematical setting

- Evolution variable $z \rightarrow t$
- Avoid logarithms in equations

\[ (ZA) \]
\[
i \partial_t A + \Delta A = -aA \partial_x \varphi, \quad A(x, 0) = A_0(x),
\]

\[
\text{div} \left( (1 + |A|^2) \nabla \varphi \right) = \partial_x(|A|^2),
\]

In reference to the cubic NLS equation:
- focusing case $a = 1$, defocusing case $a = -1$.

Close to the elliptic–elliptic Davey–Stewartson equation

\[ (DS) \]
\[
i \partial_t A + \Delta A = -aA \partial_x \varphi,
\]

\[
\Delta \varphi = \partial_x(|A|^2),
\]

for which there is a $L^2$ and $H^1$ local theory, and a global theory for small initial data.
But: not applicable here.
A one-dimensional model

Suppress the $y$ direction, and integrate the second equation

$$(1 + |A|^2) \partial_x \varphi = |A|^2 - C(t)$$

Two experimental contexts:

1. No external field applied: $C(t) \equiv 0$. Context of bright solitary waves
2. Context of dark solitary waves $C(t) = \lim_{x \to \pm \infty} |A|^2$, does not depend on $t$.

In both cases: Saturated NLS equation (derived for $\Delta = \partial_x^2$)

$$i \partial_t A + \Delta A = -a \frac{|A|^2 - |A_\infty|^2}{1 + |A|^2} A, \quad A(x, 0) = A_0(x).$$
Results

Cauchy problem \( (C \equiv 0) \)
- (SNLS): \( L^2 \) and \( H^1 \) global theory.
- (ZA): \( L^2 \) and \( H^1 \) global theory.

Solitary waves
- 1D (SNLS): First integrals (bright and dark cases).
- 2D (SNLS) and (ZA): Non existence (bright and dark cases).
- (SNLS): Existence result (bright case).
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\[(\text{SNLS}) \quad i \partial_t A + \Delta A = -a \frac{|A|^2 A}{1 + |A|^2}, \quad A(x, 0) = A_0(x).\]

**Theorem**

(i) \(A_0 \in L^2(\mathbb{R}^d)\). *Unique solution* \(A \in C(\mathbb{R}; L^2(\mathbb{R}^d))\) of *(SNLS)* such that \(\forall t \in \mathbb{R}\)

\[E(t) \equiv \int_{\mathbb{R}^d} |A(x, t)|^2 \, dx = E(0)\]

(ii) \(A_0 \in H^1(\mathbb{R}^d)\). *Above solution* \(A \in C(\mathbb{R}; H^1(\mathbb{R}^d))\) *and* \(\forall t \in \mathbb{R}\)

\[H(t) \equiv \int_{\mathbb{R}^d} \left[ |\nabla A(x, t)|^2 \, dx + a \ln(1 + |A(x, t)|^2) \right] \, dx = H(0).\]
(SNLS): sketch of proof

- Same as cubic NLS but easier.
- Conservation laws: formal, justified by usual truncation process.
- $S(t)$: group operator for the Schrödinger equation $i\partial_t A + \Delta A = 0$.
- Mild formulation

$$A(x, t) = S(t)A_0(x) - a \int_0^t S(t-s) \frac{|A(x, s)|^2}{1 + |A(x, s)|^2} A(x, s) \, ds.$$  

- Use of properties like $x \mapsto x/(1 + x)$ is Lipschitz.
- Contraction on a suitable ball of $C([0, T]; L^2(\mathbb{R}^d))$ for some $T > 0$ \implies local well-posedness.
- Global well-posedness from the conservation law $E$.
- The $H^1$ theory follows the same argument
- Since $\ln(1 + |A|^2) \leq |A|^2$, uniform bound $\forall t \in \mathbb{R}$

$$\int_{\mathbb{R}^d} |\nabla A(x, t)|^2 \, dx \leq \int_{\mathbb{R}^d} |A_0(x)|^2 \, dx + \int_{\mathbb{R}^d} |\nabla A_0(x)|^2 \, dx.$$  

No blow-up is occurs, whatever the sign of $a$.  

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(ZA), $d = 2$: potential estimate

- Try to mimic the proof (SNLS): express $\nabla \varphi$ in terms of $A \in L^2(\mathbb{R}^2)$.
- To ensure Lipschitz regularity we will have to assume $A \in H^2(\mathbb{R}^2)$.

Hilbert weighted homogeneous Sobolev space

$$H = \{ \varphi \in S'(\mathbb{R}^d), (1 + |A|^2)^{1/2} \nabla \varphi \in L^2(\mathbb{R}^d) \} / \mathbb{R}.$$

**Lemma**

(i) Let $A \in L^2(\mathbb{R}^2)$. There exists a unique $\varphi \in H$ solution of

$$\text{div}((1 + |A|^2) \nabla \varphi) = \partial_x(|A|^2) \quad \text{in } D'(\mathbb{R}^2)$$

such that

$$\int_{\mathbb{R}^2} (1 + \frac{1}{2}|A|^2)|\nabla \varphi|^2 \, dx \leq \frac{1}{2} \int_{\mathbb{R}^2} |A|^2 \, dx.$$

(ii) $A \in H^2(\mathbb{R}^2)$, then $\nabla \varphi \in H^2(\mathbb{R}^2)$ and there exists a polynomial $P$ such that

$$\| \nabla \varphi \|_{H^2(\mathbb{R}^2)} \leq P(\|A\|_{H^2(\mathbb{R}^2)}) \quad \text{and} \quad P(0) = 0.$$
(ZA), $d = 2$: result

\[
\begin{align*}
(\text{ZA})
\begin{cases}
    i\partial_tA + \Delta A &= -aA\partial_x\varphi, \\
    \text{div} \left( (1 + |A|^2)\nabla \varphi \right) &= \partial_x(|A|^2),
\end{cases}
\end{align*}
\]

**Theorem**

Let $A_0 \in H^2(\mathbb{R}^2)$. Then there exists $T_0 > 0$ and a unique solution $(A, \nabla \varphi)$ of (ZA) such that $A \in C([0, T_0]; H^2(\mathbb{R}^2))$ and $\nabla \varphi \in C([0, T_0]; H^2(\mathbb{R}^2))$. Moreover

\[
\|A(\cdot, t)\|_{L^2(\mathbb{R}^2)} = \|A_0\|_{L^2(\mathbb{R}^2)}, \quad 0 \leq t \leq T_0
\]

and

\[
\int_{\mathbb{R}^2} \left( 1 + \frac{1}{2}|A|^2 \right)|\nabla \varphi|^2 d\mathbf{x} \leq \frac{1}{2} \int_{\mathbb{R}^2} |A_0|^2 d\mathbf{x}, \quad 0 \leq t \leq T_0.
\]

We do not know whether this local solution is global or not.
Cauchy problem

Zozulya–Anderson model

(ZA), \( d = 2 \): proof

- Formal \( H^2 \) a priori estimate \( T_0 < T \) sufficiently small

\[
\|A(\cdot, t)\|_{H^2(\mathbb{R}^2)} \leq C \left( \|A_0\|_{H^2(\mathbb{R}^2)} \right) \text{ for } 0 < t < T_0,
\]

- \( \varepsilon > 0 \). Approximation of (ZA).

\[
i \partial_t A^\varepsilon + \Delta A^\varepsilon = -a A^\varepsilon \partial_x \varphi^\varepsilon, \\
\text{div} \left( (1 + \varepsilon \Delta^2 + |A^\varepsilon|^2) \nabla \varphi^\varepsilon \right) = \partial_x (|A^\varepsilon|^2),
\]

globally well-posed in \( H^2(\mathbb{R}^2) \).

- Limit \( \varepsilon \to 0 \). Aubin-Lions compactness lemma.

\[
\nabla \varphi^\varepsilon \to \nabla \varphi \text{ in } L^\infty(0, T; H^2(\mathbb{R}^2)) \text{ weak-⋆ and } L^2([0, T] \times \mathbb{R}^2) \text{ weakly.} \\
A^\varepsilon \to A \text{ in } L^\infty(0, T; H^2(\mathbb{R}^2)) \text{ weak-⋆ and } L^2(0, T; H^1_{\text{loc}}(\mathbb{R}^2)). \\
\text{Bona–Smith approximation } \Rightarrow (A, \nabla \varphi) \in (C(0, T; H^2(\mathbb{R}^2)))^2.
\]
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First integrals: bright solitary waves

- \( A(x, t) = e^{i\omega t} u(x) \).
- \( u \) maximum at \( x = 0 \) (\( u(0) = u_m > 0 \) and \( u'(0) = 0 \)).
- \( x \to \infty, u(x) \to 0 \) and \( u'(x) \to 0 \).
- Then: unique possible frequency for the solitary wave, namely
  \[
  \omega = a \left( 1 - \frac{\ln(1 + u_m^2)}{u_m^2} \right)
  \]
  and
  \[
  [u'(x)]^2 = a \left( -\frac{u^2(x)}{u_m^2} \ln(1 + u_m^2) + \ln(1 + u^2) \right).
  \]
- Only possible for \( a = 1 \) (focusing case) and
  \[
  u'(x) = -\text{sign}(x) \sqrt{\ln(1 + u^2) - \frac{u^2}{u_m^2} \ln(1 + u_m^2)}.
  \]
First integrals: dark solitary waves

- $A(x, t) = u(x)$.
- $\lim_{x \to \pm \infty} u'(x) = 0$ and consistently with $A_\infty \neq 0$,
  $$\lim_{x \to +\infty} u(x) = - \lim_{x \to -\infty} u(x) = u_\infty.$$

Then

$$[u'(x)]^2 = a \left( -(u^2 - u_\infty^2) + (1 + u_\infty^2) \ln \left( \frac{1 + u^2}{1 + u_\infty^2} \right) \right).$$

At the origin $u(0) = 0$ and more generally $|u(x)| \leq |u_\infty|$.

Only possible for $a = -1$ (defocusing case).

$u(x)$ is a monotonous function and

$$u'(x) = \text{sign}(u_\infty) \sqrt{u^2 - u_\infty^2 - (1 + u_\infty^2) \ln \left( \frac{1 + u^2}{1 + u_\infty^2} \right)}.$$
Non-existence of bright solitary waves

Assume $A(x, t) = e^{i\omega t}U(x)$, where $U \in H^1(\mathbb{R}^d)$ is solution to

\[
(OS) - \Delta U + \omega U = a \frac{|U|^2 U}{1 + |U|^2}, \quad U \in H^1(\mathbb{R}^d).
\]

**Theorem**

No non-trivial solitary wave (solution of $(OS)$) exists when

(i) $a = -1$ (defocusing case), for $\omega \geq 0$.
(ii) $a = 1$ (focusing case) and $\omega \geq 1$.
(iii) $a = \pm 1$ if $\omega < 0$ provided $|U|^2 / (1 + |U|^2) = O(1/|x|^{1+\varepsilon})$, $\varepsilon > 0$ as $|x| \to +\infty$.

**Tools**

- (i) and (ii) Energy estimates and Pohozaev identity.
- (iii) Classical result of Kato on the absence of embedded eigenvalues.
Regularity

Corollary

*Solitary waves may exist only when* $a = 1$ *and* $0 < \omega < 1$.

Theorem

*Let* $a = 1$ *and* $0 < \omega < 1$. Then any* $U \in H^1(\mathbb{R}^d)$ *solution of (OS) satisfies*

\[
U \in H^\infty(\mathbb{R}^d),
\]

\[
e^{\delta|x|} U \in L^\infty(\mathbb{R}^d) \text{ for any } \delta < \omega/2.
\]
Existence results

Assume $0 < \omega < 1$. Look for radial $H^2$ solutions $U(x) = u(|x|) \equiv u(r)$ of

$$-\Delta U + \omega U = \frac{|U|^2 U}{1 + |U|^2}$$

solution to

$$\begin{cases} 
-u'' - \frac{d - 1}{r} u' + \omega u = \frac{u^3}{1 + u^2}, \\
u \in H^2(]0, \infty[), \quad u'(0) = 0.
\end{cases}$$

Theorem

If $a = 1$ and $0 < \omega < 1$, there exists a non-trivial positive solution of (Rad).

Stems from a classical result of Berestycki, Lions and Peletier.

$u$ satisfies the decay rate (regularity result).
Solitary waves

(ZA): non existence results

Solutions of the form \((e^{i\omega t} U(x), \phi(x))\) with \(U \in H^1(\mathbb{R}^d)\) and \(\phi \in H\). 

\((U, \phi)\) should satisfy the system

\[
\begin{align*}
-\Delta U + \omega U &= aU \partial_x \phi, \\
\text{div}\left((1 + |U|^2) \nabla \phi\right) &= \partial_x (|U|^2).
\end{align*}
\]

The existence of non-trivial solutions of (OS) is an open problem. It does not seem to be the Euler–Lagrange equation associated to a variational problem.

**Theorem**

(i) Let \(a = -1\) (defocusing case). Then no non-trivial solution of (OS2) exists for \(\omega \geq 0\).

(ii) Let \(a = 1\) (focusing case). No non-trivial solution of (OS2) exists for \(\omega \geq 1\).

(iii) Let \(a = \pm 1\). No non-trivial solution of (OS2) exists if \(\omega < 0\) provided \(\partial_x \phi = O\left(\frac{1}{|x|^{1+\varepsilon}}\right), \varepsilon > 0\) as \(|x| \to +\infty\).
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Conclusion and perspectives

Done

- Full but heuristic derivation of Zozulya–Anderson model.
- Cauchy problem.
- Existence and non existence of solitary waves.

To do

- "Lighter" Cauchy problem in 2D. Global? in $H^1$?
- Larger range of phenomena ... memory effects.
- Transverse stability of 1D solitons in the 2D model.
- Zhidkov like boundary conditions.
- Numerics.