A barrier on convex cones with parameter equal to the dimension

Roland Hildebrand

Université Grenoble 1 / CNRS

August 23 / ISMP 2012, Berlin
Outline

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   - Logarithmically homogeneous barriers
   - Universal barrier

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Definition (Nesterov, Nemirovski 1994)

Let $K \subset \mathbb{R}^n$ be a regular convex cone. A (self-concordant logarithmically homogeneous) barrier on $K$ is a smooth function $F : K^o \to \mathbb{R}$ on the interior of $K$ such that

- $F(\alpha x) = -\nu \log \alpha + F(x)$ (logarithmic homogeneity)
- $F''(x) \succ 0$ (convexity)
- $\lim_{x \to \partial K} F(x) = +\infty$ (boundary behaviour)
- $|F'''(x)[h, h, h]| \leq 2(F''(x)[h, h])^{3/2}$ (self-concordance)

for all tangent vectors $h$ at $x$.

The homogeneity parameter $\nu$ is called the barrier parameter.
Theorem (Nesterov, Nemirovski 1994)

Let $K \subset \mathbb{R}^n$ be a regular convex cone and $F : K^0 \to \mathbb{R}$ a barrier on $K$ with parameter $\nu$. Then the Legendre transform $F^*$ is a barrier on $-K^*$ with parameter $\nu$.

The graph $G_F = \{(x, F'(x)) | x \in K^0\}$ is a smooth $n$-dimensional submanifold of $K \times (-K^*) \subset \mathbb{R}^n \times \mathbb{R}_n$.

$F$ is determined by this graph up to an additive constant.
**Universal barrier**

**Theorem (Nesterov, Nemirovski 1994)**

There exists an absolute constant $c > 0$ such that

$$F(x) = c \log \text{Vol}\{p \in K^* | \langle x, p \rangle < 1\}$$

is a $(c \cdot n)$-self-concordant barrier on $K \subset \mathbb{R}^n$.

**Lemma (Güler 1996)**

$$F(x) = c \log \varphi(x) \text{ up to an additive constant, where}$$

$$\varphi(x) = \int_{K^*} e^{-\langle x, p \rangle} dp$$

is the characteristic function of the cone.
the universal barrier is

- invariant under the action of $SL(\mathbb{R}^n)$
- fixed under unimodular automorphisms of $K$
- additive under the operation of taking products
- has barrier parameter $O(n)$

not invariant with respect to duality
Projective images of cones

let $\mathbb{R}P^{n-1}$, $\mathbb{R}P_{n-1}$ be the primal and dual real projective space — lines and hyperplanes through the origin of $\mathbb{R}^n$

let $K \subset \mathbb{R}^n$ be a regular convex cone

the canonical projection $\Pi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}P^{n-1}$ maps

- $K \setminus \{0\}$ to a compact convex subset $C \subset \mathbb{R}P^{n-1}$
- $\partial K \setminus \{0\}$ to the spherical boundary $\partial C \subset \mathbb{R}P^{n-1}$

the canonical projection $\Pi^* : \mathbb{R}_n \setminus \{0\} \to \mathbb{R}P_{n-1}$ maps

- $K^* \setminus \{0\}$ to a compact convex subset $C^* \subset \mathbb{R}P_{n-1}$
- $\partial K^* \setminus \{0\}$ to the spherical boundary $\partial C^* \subset \mathbb{R}P_{n-1}$
between elements of $\mathbb{RP}^{n-1}$, $\mathbb{RP}_{n-1}$ there is no scalar product, but an orthogonality relation

the set

$$\mathcal{M} = \left\{ (x, p) \in \mathbb{RP}^{n-1} \times \mathbb{RP}_{n-1} \mid x \not\perp p \right\}$$

is dense in $\mathbb{RP}^{n-1} \times \mathbb{RP}_{n-1}$

$$\partial \mathcal{M} = \left\{ (x, p) \in \mathbb{RP}^{n-1} \times \mathbb{RP}_{n-1} \mid x \perp p \right\}$$

is a submanifold of $\mathbb{RP}^{n-1} \times \mathbb{RP}_{n-1}$ of codimension 1
Product of projective spaces

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Define the set

\[ \delta_K = \{ (\Pi(x), \Pi^*(p)) \mid x \in \partial K \setminus \{0\}, \ p \in \partial K^* \setminus \{0\}, \ x \perp p \} \]

\( \subset \ \partial M \cap (\partial C \times \partial C^*) \)

- The projections of \( \delta_K \) on \( \partial C \) and \( \partial C^* \) are surjective.
- If \( K \) is smooth, then \( \delta_K \) is homeomorphic to \( S^{n-2} \).

Call \( \delta_K \) the boundary frame corresponding to the cone \( K \).
the boundary frame $\delta_K$ consists of pairs $z = (x, p) \in \partial \mathcal{M}$ where

- the line $x$ contains a ray in $\partial K$
- $p$ is a supporting hyperplane at $x$
Images of barriers

Let $F : K^o \to \mathbb{R}$ be a barrier

Let $M_F$ be the projective image of the graph $G_F$ of the Legendre transform

$M_F = \{(\Pi(x), \Pi^*(F'(x))) \mid x \in K^o\} \subset \mathbb{RP}^{n-1} \times \mathbb{RP}_{n-1}$

is a smooth $(n - 1)$-dimensional submanifold of $\mathcal{M}$
the manifold $M_F$ consists of pairs $(x, p)$ where

- $x$ is a line through a point $y \in K^o$
- $p$ is parallel to the hyperplane which is tangent to the level surface of $F$ at $y$

if $y \to \hat{y} \in \partial K$, then $p$ tends to a supporting hyperplane at $\hat{y}$

$$\Rightarrow \partial M_F = \delta_K$$
Two-point function on $\mathcal{M}$

let $z = (x, p), z' = (x', p') \in \mathcal{M} \subset \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}$

let $y, y' \in \mathbb{R}^n, s, s' \in \mathbb{R}^n$ be representatives of $x, x', p, p'$

the quantity

$$(z; z') = (z'; z) := 1 - \frac{\langle y, s'\rangle \langle y', s \rangle}{\langle y, s \rangle \langle y', s' \rangle} \in \mathbb{R}$$

does not depend on the choice of the representatives
There exists a complete pseudo-metric $g$ of neutral signature on $\mathcal{M}$ such that for all $z, z' \in \mathcal{M}$ linked by a geodesic $\sigma$,

- if $(z; z') > 0$, then the velocity vector of $\sigma$ has positive square and $d(z, z') = \arcsin \sqrt{(z; z')}$;
- if $(z; z') = 0$, then $\sigma$ is light-like;
- if $(z; z') < 0$, then the velocity vector of $\sigma$ has negative square and $d(z, z') = \arcsinh \sqrt{-(z; z')}$. 

**Theorem (H., 2012)**
$M_F$ is a **Riemannian submanifold** of $\mathcal{M}$ bounded by $\delta_K \subset \partial \mathcal{M}$
Minimal surface in Euclidean space
Minimal submanifolds

Definition

Let $\mathcal{M}$ be a pseudo-Riemannian manifold. Then $M \subset \mathcal{M}$ is a **minimal** submanifold if $M$ is a stationary point of the volume functional with respect to variations with compact support.

A submanifold is minimal if and only if its *mean curvature* vanishes.
Affine hyperspheres

Theorem (H., 2011)

Let $K \subset \mathbb{R}^n$ be a regular convex cone and $F : K^o \to \mathbb{R}^n$ a barrier on $K$. Then the submanifold $M_F \subset \mathcal{M}$ is minimal if and only if the level surfaces of $F$ are affine hyperspheres.
the affine normal is the tangent to the curve made of the gravity centers of the sections
an affine hypersphere is a hypersurface such that all affine normals meet in a point
Theorem (Fefferman 1976, Cheng-Yau 1986, Li 1990, and others)

Let $K \subset \mathbb{R}^n$ be a regular convex cone. Then there exists a unique foliation of $K^0$ by a homothetic family of affine hyperspheres which are asymptotic to $\partial K$.
Corollary

Let \( K \subset \mathbb{R}^n \) be a regular convex cone. Then there exists a unique (up to affine scaling) logarithmically homogeneous locally strongly convex function \( F \) on \( K \) such that \( M_F \) is a minimal submanifold of \( \mathcal{M} \) with \( \partial M_F = \delta_K \).

call \( F \) the **Einstein-Hessian** function

classified as convex solution of the PDE

\[
\log \det F'' = \frac{2n}{\nu} F, \quad F|_{\partial K} = +\infty
\]

already conjectured by O. Güler
Theorem (Calabi, 1972)

*The Ricci curvature of an affine hypersphere is negative semi-definite.*

by a splitting theorem of [Loftin, 2002] we get

Corollary

*The Ricci curvature of the Hessian metric $F''$ defined by the Einstein-Hessian function is negative semi-definite.*
for $\nu = n$ non-positivity of the Ricci curvature amounts to

$$4 F_{,ij} \geq \sum_{u,v,r,s} F_{,uv} F_{,rs} F_{,iur} F_{,jvs},$$

where $F_{,ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$, $F_{,ijk} = \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}$, $\sum_k F_{,ik} F_{,kj} = \delta^i_j$

**Theorem (H., 2012; independently by D. Fox)**

*For $\nu = n$ the Einstein-Hessian function is self-concordant.*

call it the **Einstein-Hessian barrier**
Properties

the Einstein-Hessian barrier is

- invariant with respect to the action of $SL(\mathbb{R}^n)$
- fixed under unimodular automorphisms of $K$
- additive under the operation of taking products
- invariant with respect to duality
- has barrier parameter not greater than $n$
let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and consider the power cone

$$K_p = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x \geq 0, \ y \geq 0, \ z^2 \leq x^{1/p} y^{1/q} \right\}$$

**Theorem (H., 2011)**

*The barrier parameter of the Einstein-Hessian barrier on the power cone $K_p$ is given by $\frac{3 \max(p, q)}{\max(p, q) + 1}$.***
dotted red: barrier parameter of the Einstein-Hessian barrier
solid blue: lower bound on the barrier parameter
References

- Centro-affine differential geometry, Lagrangian submanifolds of the reduced paracomplex projective space, and conic optimization. *Differential geometry 2012*, Będlewo, Poland.
Thank you