STABILIZATION OF WALLS FOR NANO-WIRES OF FINITE LENGTH

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Abstract. We address the problem of control of the magnetic moment in a ferromagnetic nanowire of finite length by means of a magnetic field. Based on theoretical results for the 1D Landau-Lifschitz equation, we establish a stabilization result for the static solutions.

1. Introduction

This paper is concerned with stabilization of wall configurations in a mono-dimensional model of finite length nanowire. This kind of object can be found in nano electronic devices. The three dimensional model is the following (see [2], [10] and [17].) We denote by $m : \mathbb{R}_+^T \times \Omega \rightarrow \mathbb{R}^3$ the magnetic moment, defined on the ferromagnetic domain $\Omega$. This unitary vector field links the magnetic induction $B$ with the magnetic field $H$ by the relation $B = H + \overline{m}$, where $\overline{m}$ is the extension of $m$ by zero outside $\Omega$. The behaviour of $m$ is governed by the Landau-Lifschitz equation:

$$
\begin{align*}
\frac{\partial m}{\partial t} &= -m \wedge H_e - m \wedge (m \wedge H_e),
H_e &= \varepsilon^2 \Delta m + h_d(m) + H_a,
\frac{\partial m}{\partial n} &= 0 \text{ on } \partial \Omega,
\end{align*}
$$

(1)

where the demagnetizing field $h_d(m)$ is characterized by

$$
curl h_d(m) = 0 \text{ and } \text{div} (h_d(m) + \overline{m}) = 0 \text{ in } \mathbb{R}^3,
$$

(2)

where $H_a$ is an applied magnetic field and where $n$ is the outward unit normal on $\partial \Omega$.

Existence of weak solutions for (1) is established in [4], [12] and [16]. Existence of strong solutions is proved in [5] and [6]. Numerical simulations are performed.
in [13]. For thin domains, equivalent 2-d models are justified in [3], [1], [14]. For nanowires, 1-d models are discussed in [15], [7] and [8].

In this paper we deal with the following one-dimensional model of finite nanowires. After renormalization, the wire is assimilated to the segment $]-L/\varepsilon, L/\varepsilon[$, where $(e_1, e_2, e_3)$ is the canonical basis of $\mathbb{R}^3$, where $\varepsilon$ is the exchange length, and $L$ is the length of the wire. The magnetic moment $m$ is then defined on $\mathbb{R}^3 \times ]-L/\varepsilon, L/\varepsilon[$.

The equivalent demagnetizing field is given by

$$h_d(m) = -m_2 e_2 - m_3 e_3,$$

i.e. $-h_d$ is the orthogonal projection onto the plane orthogonal to the wire. In addition we assume that we apply a magnetic field in the direction of the wire axis. Therefore we consider the following system:

$$\begin{align*}
\frac{\partial m}{\partial t} &= -m \wedge H_e - m \wedge (m \wedge H_e), \\
H_e &= \varepsilon^2 \frac{\partial^2 m}{\partial x^2} - m_2 e_2 - m_3 e_3 + h e_1, \\
\frac{\partial m}{\partial x}(t, -L/\varepsilon) &= \frac{\partial m}{\partial x}(t, L/\varepsilon) = 0.
\end{align*}$$

(3)

This model will be justified by $\Gamma$-convergence arguments in Section 2.

Let us focus on the wall configurations, that is static solutions separating domains of almost constant magnetization. In [7] and [8], we studied wall configurations for an infinite nanowire. They were described by the “canonical” profile $(\text{th} x, 1/\text{ch} x, 0)$, and all its translations in $x$ and rotations around the wire axis. We proved the asymptotic stability and the controlability for these configurations.

Here, for a finite wire, the situation is quite different. We call a configuration obtained by rotation around the wire axis of a profile $(\sin \theta_0, \cos \theta_0, 0)$, where $\theta_0 : [-L/\varepsilon, L/\varepsilon] \to [-\pi/2, \pi/2]$ satisfies $\theta_0(-\pi/2) < 0 < \theta_0(\pi/2)$ (in the infinite wire case, the canonical profile is obtained taking $\theta_0(x) = \text{Arcsin th}(x/2)$.) The walls exist if and only if the wire is longer enough compared to the exchange length:

**Theorem 1.1.** If $L/\varepsilon > \pi$, there exists wall configurations. They are centered in the middle of the wire, that is $\theta_0(0) = 0$.

Concerning the stability, we can prove that the wall configurations given by Theorem 1.1 are unstable.

**Theorem 1.2.** Assume that $L/\varepsilon > \pi$. Let $\theta_0$ given by Theorem 1.1. The static solution $M_0 = (\sin \theta_0, \cos \theta_0, 0)$ in linearly unstable for the Landau-Lifschitz equation (3) with $h = 0$.

This phenomenon was expected. Let us consider a small translation of the centered wall. Without energetic cost, the Landau-Lifschitz equation induces then a displacement of the wall and pushes it outside the wire. Then the magnetic moment tends to $+\pi/2$ or $-\pi/2$ (i.e the minimizers of the ferromagnetism energy.) In the case of an infinite wire, obviously, this translation cannot make the wall disappear.

We prove now a stabilization result. We control the system by an applied field directed along the wire axis.
Theorem 1.3. Let $L$ and $\varepsilon$ as in Theorem 1.1, and let $M_0 = (\sin \theta_0, \cos \theta_0, 0)$ be the canonical profile given by this theorem. We consider the following control:

$$h(m(t,)) = \left[-\frac{\varepsilon}{2L} \int_{-L/\varepsilon}^{L/\varepsilon} m_1(t, s) ds\right].$$

Then $M_0$ is stable for the Landau Lifschitz equation controled with the applied field $H_a = h(m)e_1$.

Remark 1. The control given here is quite natural: when the wall is translated to the right hand side, the average of the profile first component in then negative and our applied field $he_1$ (with $h > 0$) pushes the wall to the left hand side.

The paper is organized as follows. In Section 2 we justify the one dimensional model by Gamma convergence arguments. In Section 3 we prove the existence of wall profiles. We address the unstability of these profiles in Section 4 by lin-earization of the Landau-Lifschitz equation. The last section is devoted to the stabilization of the walls by a convenient applied magnetic field.

2. Modelization

In this article, we develop a model of nano-wires of finite length. The behavior of the magnetization in such wires is modeled by the following pde: for $u_0$ in $C^\infty([-\varepsilon, \varepsilon])$, find $u$ in $C^\infty([0,T] \times [-\varepsilon, \varepsilon])$

$$\begin{cases}
\frac{\partial u}{\partial t} = -u \wedge H_e - u \wedge (u \wedge H_e), & x \in [-\varepsilon, \varepsilon], \\
H_e = \frac{\partial^2 u}{\partial x^2} - u_2 e_2 - u_3 e_3 + h e_1, \\
\frac{\partial u}{\partial x}(-\varepsilon) = \frac{\partial u}{\partial x}(\varepsilon) = 0, \\
u(x,0) = u_0(x), & \forall x \in [-\varepsilon, \varepsilon],
\end{cases}$$

(4)

this model is obtain as limit of a three dimensional model in the case of a wire of circular section.

The dynamical equation is build from the static equation in order to define the various contributions of the effective field $H_e$. In the finite wire, the static problem is the following:

$$\{ \begin{array}{l}
\text{find } u \text{ in } H^1(\Omega_{\varepsilon,\eta}, S^2) \text{ such that } \\
E_{\varepsilon,\eta}(u) = \min_{v \in H^1(\Omega_{\varepsilon,\eta}, S^2)} E_{\varepsilon,\eta}(v),
\end{array} \}
$$

(5)

where, for $B_d(x, r)$ the sphere of radius $r$ and center $x$ in $\mathbb{R}^d$

$$\Omega_{\varepsilon,\eta} = [-\frac{L}{\varepsilon}, \frac{L}{\varepsilon}] \times B_d(0, \eta),$$

and for all $v$ in $H^1(\Omega_{\varepsilon,\eta}, \mathbb{R}^3)$ one has

$$E_{\varepsilon,\eta}(v) = \frac{1}{2} \int_{\Omega_{\varepsilon,\eta}} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\Omega_{\varepsilon,\eta}} |h_d(v)|^2 \, dx - h \int_{\Omega_{\varepsilon,\eta}} v \cdot e_1 \, \partial x.$$
To apply a gamma convergence result, we introduce the following rescaling: for all \( v \) in \( H^1(\Omega_{\varepsilon,\eta}) \), we set \( v \circ l(\eta) = v \) where \( l(\eta) \) is the stretching of ratio \( \eta \) in the direction of \( (e_2,e_3) \):

\[
\psi(x,y,z) = v(x,\eta y,\eta z).
\]

Clearly, \( v \) is an element of \( H^1(\Omega_{\varepsilon,1},\mathbb{R}^3) \). Then, one can introduce the following energy:

\[
E_{\varepsilon,\eta}(v) = \frac{1}{\eta^2}E_{\varepsilon,\eta}(v \circ l(\eta))
\]

**Proposition 1.** For all \( (\psi)_{\eta \in \mathbb{R}^+} \) in \( H^1(\Omega_{\varepsilon,1},S^2) \), sequence of minimizers of (5), \( (E_{\varepsilon,\eta}(\psi))_{\eta \in \mathbb{R}^+} \) is a bounded sequence of \( \mathbb{R} \).

**Proof.** In order to exhibit an upper bound for \( E_{\varepsilon,\eta}(\psi) \), we write:

\[
\forall \eta \in \mathbb{R}^+, E_{\varepsilon,\eta}(\psi \circ l(\eta)) \leq E_{\varepsilon,\eta}(e_1),
\]

then

\[
E_{\varepsilon,\eta}(\psi \circ l(\eta)) \leq -h\eta^2\pi \frac{L}{\varepsilon} - \frac{1}{2} \int_{\Omega_{\varepsilon,\eta}} h\psi \cdot e_1 \, dx \leq \eta^2\frac{L\pi}{\varepsilon}(2-h).
\]

Then, the lower bound is obtain by canceling the positive contributions of the energy and maximizing the external contribution:

\[
-h\frac{\pi L}{\varepsilon} \leq E_{\varepsilon,\eta}(\psi) \leq \frac{L\pi}{\varepsilon}(2-h).
\]

So, we can conclude

\[
-h\frac{\pi L}{\varepsilon} \leq E_{\varepsilon,\eta}(\psi) \leq \frac{L\pi}{\varepsilon}(2-h).
\]

Thanks to this proposition, one can write the following limit problem:

\[
\begin{aligned}
\text{find } u \text{ in } H^1(\Omega_{\varepsilon,1},S^2) \text{ such that } \\
E_{\varepsilon}(u) = \min_{v \in H^1(\Omega_{\varepsilon,1},S^2)} E_{\varepsilon}(v),
\end{aligned}
\]

where

\[
H^1(\Omega_{\varepsilon,1},\mathbb{R}^3) = \{ u \in H^1(\Omega_{\varepsilon,1},\mathbb{R}^3) \text{ such that } \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0 \},
\]
and for all \( v \) in \( H^1(\Omega_{e,1}, \mathbb{R}^3) \):
\[
\mathcal{E}_\epsilon(v) = \frac{1}{2\pi} \int_{\Omega_{e,1}} |\frac{\partial v}{\partial x}|^2 \, dx + \frac{1}{2\pi} \int_{\Omega_{e}} |v \cdot e_1|^2 \, dx - \frac{h}{\pi} \int_{\Omega_{e}} v \cdot e_1 \, dx,
\]
and the problem (5) is rescaled
\[
\mathcal{E}_{\epsilon,\eta}(u) = \min_{v \in H^1(\Omega_{e,1}, \mathbb{R}^3)} \mathcal{E}_{\epsilon,\eta}(v),
\]
Then, we state the following theorem

**Theorem 2.1.** \( \mathcal{E}_{\epsilon,\eta} \) gamma-converges to \( \mathcal{E}_\epsilon \) in sense of \( H^1(\Omega_{e,1}, \mathbb{R}^3) \) in \( H^1(\Omega_{e,1}, S^2) \), it is to say:

(i) (lower semi continuity) for all sequence \((v_\eta)_{\eta \in \mathbb{R}^+} \) of \( H^1(\Omega_{e,1}, S^2) \) such that
\[
\lim_{\eta \to 0} v_\eta = v_0 \text{ and } (\mathcal{E}_{\epsilon,\eta}(v_\eta))_{\eta \in \mathbb{R}^+} \text{ bounded,}
\]
the limit \( v_0 \) is an element of \( H^1(\Omega_{e,1}, B^3) \) such that:
\[
\liminf_{\eta \to 0} \mathcal{E}_{\epsilon,\eta}(v_\eta) \geq \mathcal{E}_\epsilon(v_0),
\]
(ii) (construction) for all \( u_0 \), solution of (6), there exists \((v_\eta)_{\eta \in \mathbb{R}^+}\), sequence of \( H^1(\Omega_{e,1}, S^2) \), such that:
\[
\lim_{\eta \to 0} v_\eta = u_0,
\]
and
\[
\limsup_{\eta \to 0} \mathcal{E}_{\epsilon,\eta}(v_\eta) \leq \mathcal{E}_\epsilon(u_0).
\]

**Proof.** (i) **Lower semi continuity:** Let \((u_\eta)_{\eta \in \mathbb{R}^+}\), a sequence of \( H^1(\Omega_{e,1}, \mathbb{R}^3) \) of limit \( u_0 \) and such that the \((\mathcal{E}_{\epsilon,\eta}(v_\eta))_{\eta \in \mathbb{R}^+}\) is bounded. Then, one has:
\[
|\frac{1}{\eta^2} \mathcal{E}_{\epsilon,\eta}(u_\eta \circ l(\eta))| = \int_{\Omega_{e,1}} |\frac{\partial u_\eta}{\partial x}|^2 \, dx + \int_{\mathbb{R}^3} |h_d(u_\eta \circ l(\eta)) \circ l(\frac{1}{\eta})|^2 \, dx
\]
\[
- \lambda \int_{\Omega_{e,1}} e_1 \cdot u_\eta \, dx + \frac{1}{\eta^2} \int_{\Omega_{e,1}} |\nabla_X u_\eta|^2 \, dx,
\]
where \( \nabla_X \) is the gradient operator through the directions \( e_2 \) and \( e_3 \). Then, we state that
\[
\lim_{\eta \to 0} \|\nabla_X u_\eta\|_{0, \Omega_{e,1}} = 0,
\]
so, the limit \( u_0 \) only variate in the \( e_1 \) direction. Then, using the fact that \( h_d \) is a \( L^p \) multiplier of order 0, then,
\[
\forall \eta \in \mathbb{R}^+, \quad \frac{1}{\eta^2} \int_{\mathbb{R}^3} |h_d(u_\eta \circ l(\eta))|^2 \, dx < \infty
\]
and, from the system (2) we get, in the sense of distributions
\[
\eta(\partial_x h_{d,1} - \partial_x u_{n,1}) = - \nabla_X \cdot \hat{h}_d(u_\eta) + \nabla_X \cdot \hat{u}_n,
\]
where, for all \( Z \) in \( \mathbb{R}^3 \), we set \( \hat{Z} = (Z \cdot e_2)e_2 + (Z \cdot e_3)e_3 \). Then we have
\[
\lim_{\eta \to 0} \|\nabla_X \cdot \hat{h}_d(u_\eta)\|_{0, \Omega_{e,1}} = 0.
\]
by the fact proved above, that
\[ \lim_{\eta \to 0} \| \nabla X \cdot \tilde{u}_\eta \|_{0,\mathbb{R}^3} = 0. \]

Then, we use that
\[ \text{curl}_X \tilde{h}_d(u_\eta) = 0, \]
and conclude that
\[ \lim_{\eta \to 0} \| \tilde{h}_d(u_\eta) \|_{0,\mathbb{R}^3} = 0. \]

Then, \( u_0 \) is an element of \( H^1(\Omega, \mathbb{R}^3) \) and we can conclude, using the positivity of the difference between \( E_{\varepsilon,\eta} \) and \( E_\varepsilon \) that
\[ \liminf_{\eta \to 0} E_{\varepsilon,\eta}(u_\eta) \geq E_\varepsilon(u_0). \]

(ii) **Reconstruction** In fact, this part is straightforward choosing, for a given solution \( u_0 \) of (6), the sequence \( (u_\eta)_{\eta \in \mathbb{R}^+} \) constant equal to \( u_0 \). In this case, trivially, one has
\[ \limsup_{\eta \to 0} E_{\varepsilon,\eta}(v_\eta) \leq E_\varepsilon(u_0). \]

This theorem gives the behavior of minimizers for the nanowire. Then one can state that the limit energy is the following, for all \( u \) in \( H^1(\Omega, \mathbb{R}^3) \):
\[ E_\varepsilon(u) = 2\pi \int_{\Omega_\varepsilon} \left| \frac{\partial v}{\partial x} \right|^2 dx + \int_{\Omega_\varepsilon} |v \cdot e_1|^2 dx - h \int_{\Omega_\varepsilon} v \cdot e_1 dx, \]
then, in order to find out the effective field, we write (see [10]):
\[ H_\varepsilon = \frac{dE_\varepsilon}{du}, \]
it is to say
\[ H_\varepsilon = \frac{\partial^2 u}{\partial x^2} - u_2 e_2 - u_3 e_3 + h e_1. \]

Then, the limit dynamic system is obtain using the Landau Lifchitz combined with the new effective field \( H_\varepsilon \). This system is then given by (4) as we expected. The boundary conditions comes naturally from the fact that the solutions are in \( H^1(\mathbb{R}^3) \).

3. Existence of particular equilibrium states: the walls

In this section we are interested in characterizing equilibrium states of the magnetization in a finite nano-wire when \( h = 0 \), it is to say when there is no external magnetic field. In this case, we look for solutions which can be written as follows:
\[ M_0(x) = \begin{pmatrix} \sin \theta_0 \\ \cos \theta_0 \\ 0 \end{pmatrix}, \quad \forall x \in ] - \frac{L}{\varepsilon}, \frac{L}{\varepsilon} [, \]
where \( \theta_0 \) is a map from \( ] - \frac{L}{\varepsilon}, \frac{L}{\varepsilon} [ \) into \( \mathbb{R} \) such that \( M_0 \) is a stationary solution to (4). In fact, we want \( M_0 \) to verify:
\[ -M_0 \wedge H_\varepsilon - M_0 \wedge (M_0 \wedge H_\varepsilon) = 0, \quad \forall x \in ] - \frac{L}{\varepsilon}, \frac{L}{\varepsilon} [, \]

with
\[ H_x = \frac{\partial^2 M_0}{\partial x^2} - \cos \theta \, e_2, \]
then, one has the following relation
\[ -\theta''_0 - \sin \theta \cos \theta = 0, \quad \forall x \in \left[ -\frac{L}{\varepsilon}, \frac{L}{\varepsilon} \right], \]
with, on the boundaries
\[ \theta'_0\left( -\frac{L}{\varepsilon} \right) = \theta'_0\left( \frac{L}{\varepsilon} \right) = 0. \]

![Figure 2. Phase portrait.](image)

Setting \( -\gamma_0 = \theta_0(0) \) \((\gamma_0 > 0)\), we have, integrating the equation (8) and using (9):
\[ (\theta'_0)^2 + \sin^2 \theta_0 = \sin^2 \gamma_0. \]
The length of the nano-wire has to be such that the function \( \theta_0 \) goes from \( -\gamma_0 \) to \( \gamma_0 \). From formula (9), we deduce the length:
\[ \ell(\gamma_0) = \int_{-\gamma_0}^{\gamma_0} \frac{d\theta}{\sqrt{\sin^2 \theta_0 - \sin^2 \theta}}. \]

Then, using the length expression computed above, we deduce the following theorem

**Theorem 3.1** (Existence of equilibrium states). For every \((\varepsilon, L)\) in \(\mathbb{R}_+^*\) such that \(\frac{L}{\varepsilon} > \pi\), there exists an equilibrium state for (4).

**Proof.** In order to ensure that an equilibrium state as defined above, we must verify that the length given by the solution is at least greater to the effective length \(2\frac{L}{\varepsilon}\). First of all, one has
\[ \ell(\gamma_0) = \int_{-1}^{1} \frac{\gamma_0}{\sin \gamma_0} \frac{d\theta}{\sqrt{1 - \frac{\sin^2 \theta}{\sin^2 \gamma_0}}}, \]
we then see that
\[ \lim_{\gamma_0 \to \frac{\pi}{2}} \ell(\gamma_0) = +\infty, \]
The first remark is that \( \gamma_0 \) is in the interval \( [0, \frac{\pi}{2}] \). Now, computing the derivative of \( \ell(\gamma_0) \) in \( \gamma_0 \), we conclude that \( \ell \) is strictly increasing on \( [0, \frac{\pi}{2}] \). Then, the comparison
could be done at the limit $\gamma_0 = 0$:

$$
\lim_{\gamma_0 \to 0} \ell(\gamma_0) = \lim_{\gamma_0 \to 0} \int_{-1}^{1} \frac{\gamma_0}{\sin \gamma_0} \frac{du}{\sqrt{1 - \frac{\sin^2(u \gamma_0)}{\sin^2 \gamma_0}}}
$$

$$
= \int_{-1}^{1} \frac{du}{\sqrt{1 - u^2}}
$$

$$
= 2\pi,
$$

then, we require that $2\frac{L}{\varepsilon} > 2\pi$.

\[\square\]

4. Unstability of walls without applied field

In this section, we consider that $L$ and $\varepsilon$ given such that $\frac{L}{\varepsilon} > \pi$ and also a given wall $\theta_0$. We consider, as in [7] and [8], the mobile frame $(M_0(x), M_1(x), M_2)$ given by:

$$
\forall x \in [-\frac{L}{\varepsilon}, \frac{L}{\varepsilon}], M_0(x) = \begin{pmatrix}
\sin \theta_0(x) \\
\cos \theta_0(x)
\end{pmatrix},
M_1(x) = \begin{pmatrix}
-\cos \theta_0(x) \\
\sin \theta_0(x)
\end{pmatrix},
M_2(x) = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}.
$$

Then, we describe the small perturbations of the static wall $M_0$ as follows:

$$
\forall t \in \mathbb{R}^+, \forall x \in [-\frac{L}{\varepsilon}, \frac{L}{\varepsilon}], u(t, x) = r_1 M_1 + r_2 M_2 + \sqrt{1 - r_1^2 - r_2^2} M_0.
$$

We denote $r = (r_1, r_2)$. In these coordinates, one can write:

$$
H_c = M_0 \left( 2\partial_x r_1 \partial_x \theta_0 + r_1 (\partial_x \theta_0)^2 - r_1 \sin \theta_0 \cos \theta_0 + \sin^2 \theta_0 \right)
$$

$$
+ M_1 \left( \partial_x r_1 - r_1 ((\partial_x \theta_0)^2 - \cos^2 \theta_0) - \partial_x \theta_0 - \sin \theta_0 \cos \theta_0 \right)
$$

$$
+ M_2 (\partial_x r_2) + Q(r),
$$

where $Q(r)$ is the non linear part in $r$ of $H_c$. Then, we can state that

$$
\partial_t r = J \left( (\mathcal{L} - \cos^2 \gamma_0) r_1 \right) + F(r, \partial_x r, \partial_{xx} r),
$$

with $J = \begin{pmatrix}
-1 & -1 \\
1 & -1
\end{pmatrix}$, and, $\mathcal{L} r = -\partial_{xx} r_2 + g_0 r_2$, where $g_0(x) = \sin^2 \theta_0 - (\theta_0')^2$.

The linear unstability of the wall structure computed in the previous section is given by the study of the operator $\mathcal{L}$.

**Proposition 2.** $\mathcal{L}$ is a linear, positive operator. Its first eigenvalue, 0, is associated to the eigenfunction $\cos \theta_0$ and its second eigenvalue, 1, is associated to the eigenfunction $\sin \theta_0$.

**Proof.** We set: $f = \theta'_0 \tan \theta_0$, then

$$
\mathcal{L} = \ell' \ell, \text{ where } \ell = \partial_x + f,
$$

then, we can conclude that $\mathcal{L}$ is a positive operator and that $\cos \theta_0$ is in the kernel of $\mathcal{L}$. Thus 0 is the first eigenvalue of $\mathcal{L}_2$.

Furthermore, we have:

$$
\mathcal{L}(\sin \theta_0) = \sin \theta_0,
$$
it is to say that 1 is eigenvalue of $\mathcal{L}$ associated to the eigenfunction $\sin \theta_0$. In addition we remark that $\sin \theta_0$ vanishes once in the domain, so by Sturm-Louville theorem, 1 is the second eigenvalue of $\mathcal{L}_2$. \hfill \square

We can now prove Theorem 1.2.

Proof. From the previous proposition, since $-\cos^2 \gamma_0 < 0$, we conclude that $\mathcal{L} - \cos^2 \gamma_0$ has got one eigenvalue strictly negative, then, zero is unstable for the linearized of (3) around $M_0$. \hfill \square

5. Stabilization of walls

Now, we discuss the stabilization of $M_0$ by the command $H_a$ (the applied field). We recall that we introduced the following command:

$$h(m) = -\frac{\varepsilon}{2L} \int_{-L/\varepsilon}^{L/\varepsilon} m_1(t,s) \, ds.$$ We want to prove that the profile $M_0$ is a stable stationary solution for the following system:

$$\begin{aligned}
\frac{\partial m}{\partial t} &= -m \wedge H_e - m \wedge (m \wedge H_e), \\
\frac{\partial m}{\partial x} (-L/\varepsilon) &= \frac{\partial m}{\partial x} (L/\varepsilon) = 0, \\
H_e &= \frac{\partial^2 m}{\partial x^2} - m_2 e_2 - m_3 e_3 + h(m) e_1.
\end{aligned}$$

(11)

Proof. To start with, let us introduce $M_0 = (\sin \theta_0, \cos \theta_0, 0)$ given in Theorem 1.1. We recall that

$$\theta_0'' + \sin \theta_0 \cos \theta_0 = 0,$$

$$\theta'(L/\varepsilon) - \theta'(-L/\varepsilon) = 0,$$

Furthermore, on $[-L/\varepsilon, L/\varepsilon]$,

$$\cos^2 \theta_0 - (\theta_0')^2 = \cos^2 \gamma_0,$$

where $\gamma_0 = \theta_0(L/\varepsilon)$.

Since $h(M_0) = 0$, we remark that $M_0$ is a stationary solution of (11).

First step: moving frame.

As in the previous section, in the spirit of [7], we will describe the problem in the moving frame

$$(M_0(x), M_1(x), M_2(x)).$$

We write the solutions to (11) as:

$$m(t,x) = M_0(x) + r_1(t,x)M_1(x) + r_2(t,x)M_2(x) + \nu(r(t,x))M_0(x)$$

where $\nu(r) = \sqrt{1 - r_1^2 - r_2^2} - 1$. In this moving frame, we get

$$H_e = (g_0 + a_0)M_0 + (a_1 + \tilde{a}_1)M_1 + a_2 M_2.$$
with

\[
g_0(x) = \sin^2 \theta_0 - (\theta'_0)^2
\]

\[
a_0 = 2\theta'_0 \partial_x r_1 + r_1 \theta'_0 - \cos \theta_0 \partial_x r_1 + \partial_{xx} \nu - \nu(\theta'_0)^2 + \nu \sin^2 \theta_0 + h(r) \sin \theta_0,
\]

\[
a_1 = \partial_{xx} r_1 + \cos^2 \gamma_0 r_1 - S(r_1) \cos \theta_0,
\]

\[
\tilde{a}_1 = -2\partial_x \nu \theta'_0 - \varphi(r) \cos \theta_0,
\]

\[
a_2 = \partial_{xx} r_2.
\]

where

\[
S(r_1) = \frac{\varepsilon}{2L} \int_{-\frac{L}{2}}^{\frac{L}{2}} r_1 \cos \theta_0 \, ds, \quad \varphi(r) = -\frac{\varepsilon}{2L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \nu(r) \sin \theta_0 \, ds, \quad \text{and} \quad h(r) = S(r_1) + \varphi(r).
\]

Using these coordinates in the Landau-Lifchitz equation (9) and projecting on \(M_1\) and \(M_2\) yield:

(12) \[ \partial_t r = \Lambda r + F(x, r, \partial_x r, \partial_{xx} r), \]

where

\[ \Lambda r = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \tilde{L}_1(r_1) \\ \tilde{L}(r_2) \end{pmatrix} \]

with

\[ \tilde{L}_1 = \mathcal{L} = -\partial_{xx} + g_0, \]

\[ \tilde{L}_1 = \mathcal{L} - \cos^2 \gamma_0 + \cos \theta_0 S, \]

and where the non linear part \(F\) is given by

\[
F(x, r, \partial_x r, \partial_{xx} r) = F_1(r) \partial_{xx} r + F_2(r)(\partial_x r, \partial_x r) + F_3(x, r) \partial_x r + F_4(r) + F_5(r),
\]

with

- \(F_1 \in C^\infty(\mathbb{R}^2; M_2(\mathbb{R}))\):

\[
F_1(r) = \begin{pmatrix} (r_2)^2 + (\nu(r))^2 + 2\nu(r) & \nu(r) - r_1 r_2 \\ r_1 r_2 - \nu(r) & (r_1)^2 + (\nu(r))^2 + 2\nu(r) \end{pmatrix}
\]

\[+ \begin{pmatrix} -r_1 - r_2 - \nu(r)r_1 \\ r_1 - r_2 - \nu(r)r_2 \end{pmatrix} \nu'(r), \]

- \(F_2 \in C^\infty(\mathbb{R}^2; \mathcal{L}_2(\mathbb{R}^2; \mathbb{R}^2))\):

\[
F_2(r)(\partial_x r, \partial_x r) = \begin{pmatrix} -r_1 - r_2 - \nu(r)r_1 \\ r_1 - r_2 - \nu(r)r_2 \end{pmatrix} \nu''(r)(\partial_x r, \partial_x r),
\]
\( F_3 \in \mathcal{C}^\infty([-L/\varepsilon, L/\varepsilon] \times \mathbb{R}^2; \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2)):\)

\[
F_3(x, r) \partial_x r = 2 \theta'_0(x) \begin{pmatrix} -r_1 - r_2 - \nu(r) r_1 \\ r_1 - r_2 - \nu(r) r_2 \end{pmatrix} \partial_x r_1 - 2 \theta'_0(x) \begin{pmatrix} (r_2)^2 + (\nu(r))^2 + 2\nu(r) + 1 \\ r_1 r_2 - \nu(r) - 1 \end{pmatrix} \nu'(r)(\partial_x r),
\]

\( F_4(r) \in \mathcal{C}^\infty([-L/\varepsilon, L/\varepsilon] \times \mathbb{R}^2; \mathbb{R}^2):\)

\[
F_4(r) = (\cos^2 \gamma_0 r_1) \begin{pmatrix} (r_2)^2 + (\nu(r))^2 + 2\nu(r) \\ r_1 r_2 - \nu(r) \end{pmatrix} - g_0 \nu(r) \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}
+ ((\theta'_0 - \cos \theta_0 \sin \theta_0)r_1 + g_0 \nu(r)) \begin{pmatrix} -r_1 - r_2 - \nu(r) r_1 \\ r_1 - r_2 - \nu(r) r_2 \end{pmatrix}.
\]

\( F_5 \) is given by

\[
F_5(r) = -h(r) \cos \theta_0 \begin{pmatrix} (r_2)^2 + (\nu(r))^2 + 2\nu(r) \\ r_1 r_2 - \nu(r) \end{pmatrix} - \varphi(r) \cos \theta_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}
+ h(r) \sin \theta_0 \begin{pmatrix} -r_1 - r_2 - \nu(r) r_1 \\ r_1 - r_2 - \nu(r) r_2 \end{pmatrix}.
\]

**Remark 2.** The command \( h \) makes the linear part of (12) positive. Indeed, on one hand, we know that \( \mathcal{L} \geq 0 \) with \( \operatorname{Ker} \mathcal{L} = \mathbb{R} \cos \theta_0 \). On the other hand, \( \mathcal{L}_1 = \mathcal{L} + \cos \theta_0 S - \cos^2 \gamma_0 \). On \( \mathbb{R} \cos \theta_0 \), \( \mathcal{L}_1 \cos \theta_0 = \alpha_0 \cos \theta_0 \) with

\[
\alpha_0 = \frac{\varepsilon}{2L} \int_{-L/\varepsilon}^{L/\varepsilon} \cos^2 \theta_0(x) dx - \cos^2 \gamma_0 > 0
\]

since for \( x \in ]-L/\varepsilon, L/\varepsilon[ \), \( \cos^2 \theta_0(x) > \cos^2 \gamma_0 \), and on \( (\cos \theta_0)^\perp \),

\( \mathcal{L}_1(\cos \theta_0)^\perp = \mathcal{L} - \cos^2 \theta_0 \geq 1 - \cos^2 \theta_0 \)

since \( \mathcal{L} \geq 1 \) on \( (\cos \theta_0)^\perp \) (see Proposition 2).

**Second step: new coordinates.**

The Landau Lifschitz equation (11) is invariant by rotation around the wire axis, so we can build a family of static solutions. For \( \tau \in \mathbb{R} \) let us introduce the rotation around the \( x \)-axis given by

\[
\rho_\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \tau & -\sin \tau \\ 0 & \sin \tau & \cos \tau \end{pmatrix}.
\]

We denote \( M_\tau(x) = \rho_\tau M_0(x) \), and \( R_\tau \) its projection on the moving frame:

\[
R_\tau(x) = \begin{pmatrix} M_\tau(x).M_1(x) \\ M_\tau(x).M_2(x) \end{pmatrix} = \begin{pmatrix} \cos \theta_0(x) \sin \theta_0(x)(\cos \tau - 1) \\ \cos \theta_0(x) \sin \tau \end{pmatrix}.
\]
Since $M_r$ is solution to (11), $R_r$ is a static solution of (12), that is
\begin{equation}
\Lambda R_\theta + F(x, R_\theta, \partial_x R_\theta, \partial_{xx} R_\theta) = 0.
\end{equation}

Now in order to avoid the problems due to the zero eigenvalue of $\Lambda$, we describe $r$ in the new coordinates $(\tau, \sigma, W)$ defined by
\begin{equation}
r(t, x) = R_{\tau(t)}(x) + \sigma(t) \begin{pmatrix} \cos \theta_0(x) \\ 0 \end{pmatrix} + W(t, x),
\end{equation}
where $(\tau, \sigma) \in C^1(\mathbb{R}^+; \mathbb{R}^2)$ and $W \in C^1(\mathbb{R}^+; H^2)$ such that both coordinates of $W$ are in $(\cos \theta_0)^\perp$.

Indeed, as in [11], we can prove that for a given $r \in H^2(-L/\varepsilon, L/\varepsilon)$ in a neighbourhood of 0, there exists a unique $(\tau, \sigma, W) \in \mathbb{R} \times \mathbb{R} \times \mathcal{W}$ such that
\begin{equation}
\rho = R_\theta + \sigma \begin{pmatrix} \cos \theta_0 \\ 0 \end{pmatrix} + W,
\end{equation}
where
\[ \mathcal{W} = \left\{ W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \in H^2, \int_{-L/\varepsilon}^{L/\varepsilon} W_1(x) \cos \theta_0(x) dx = \int_{-L/\varepsilon}^{L/\varepsilon} W_2(x) \cos \theta_0(x) dx = 0 \right\}. \]

**Remark 3.** On $(\cos \theta_0)^\perp$, the operators $\mathcal{L}$ and $\mathcal{L} - \cos^2 \theta_0$ are non negative, so we introduce the following norms on $\mathcal{W}$, respectively equivalent to the $H^2$ and the $H^3$ norms:
\begin{align*}
\| W \|_{H^2} & = \left( \| \mathcal{L}^{\frac{1}{2}} (\mathcal{L} - \cos^2 \gamma_0) \frac{1}{2} W_1 \|_{L^2}^2 + \| \mathcal{L} W_2 \|_{L^2}^2 \right)^{\frac{1}{2}}, \\
\| W \|_{H^3} & = \left( \| \mathcal{L}^{\frac{3}{2}} (\mathcal{L} - \cos^2 \gamma_0) W_1 \|_{L^2}^2 + \| \mathcal{L}^{\frac{3}{2}} W_2 \|_{L^2}^2 \right)^{\frac{1}{2}}.
\end{align*}

Plugging the decomposition (15) in (12) and using (14) yield the following equivalent form for the Landau-Lifschitz equation in the coordinates $(\tau, \sigma, W)$, valid for little perturbations of $M_0$. Indeed we have:
\[ \partial_t r = \frac{d\tau}{dt} R'_\tau(x) + \frac{d\sigma}{dt} \begin{pmatrix} \cos \theta_0 \\ 0 \end{pmatrix} + \partial_t W, \]
where
\[ R'_\tau(x) = \begin{pmatrix} -\sin \tau \cos \theta_0 \sin \theta_0 \\ \cos \tau \cos \theta_0 \end{pmatrix}. \]

In addition,
\[ \Lambda r = \Lambda R_\tau + \begin{pmatrix} -\alpha_0 \cos \theta_0 \\ \alpha_0 \cos \theta_0 \end{pmatrix} \sigma + \Lambda(W) \]
(see Remark 2 for the definition of $\alpha_0$), and
\[ F(x, r, \partial_x r, \partial_{xx} r) = F(x, R_\tau, \partial_x R_\tau, \partial_{xx} R_\tau) + G(x, \tau, \sigma, W, \partial_x W, \partial_{xx} W). \]
The last term $G$ is obtained from $F$ with the Taylor formula around $R_\tau$:
\[
G(x, \tau, \sigma, W, \partial_x W, \partial_{xx} W) = F_1(r)(\partial_{xx} w) + \tilde{F}_1(r)(w)(\partial_{xx} R_\tau) + 2F_2(r)(\partial_x w, \partial_x R_\tau) + F_2(r)(w)(\partial_{xx} R_\tau, \partial_x R_\tau)
\]
\[
+ F_3(x, r)(\partial_x w) + \tilde{F}_3(x, r)(w)\partial_x R_\tau + \tilde{F}_4(r)w + F_5(r),
\]
where $w = W + \begin{pmatrix} \cos \theta_0 \\ 0 \end{pmatrix} \sigma$, and where for $i = 1, 4, \tilde{F}_i(r) \in L^2(\mathbb{R}^2; \mathbb{R}^2)$ is given by
\[
\tilde{F}_i(r) = \int_0^1 F'_i(R_\tau + sw)ds.
\]
From straightforward calculations, we see that:
\[
F_1(r) = O(|r|^2), \quad \tilde{F}_1(r) = O(|r|^2)
\]
\[
F_2(r) = O(|r|), \quad \tilde{F}_2(r) = O(1)
\]
\[
F_3(x, r) = O(|r|), \quad \tilde{F}_3(x, r) = O(1)
\]
\[
\tilde{F}_4(r)O(|r|).
\]
In addition, $\partial_x R_\tau = O(|r|)$. Concerning $F_5$, on one hand we remark that
\[
S(r_1) = S(W_1) + \sigma S(\cos \theta_0)
\]
(since $S(R_\tau) = 0$.)

On the other hand, $\varphi(R_\tau) = 0$ and
\[
\varphi(r) = -\frac{\varepsilon}{2L} \int_{-L/\varepsilon}^{L/\varepsilon} \tilde{\nu}(r)w \sin \theta_0,
\]
where
\[
\tilde{\nu}(r) = \int_0^1 \nu'(R_\tau + sw)ds = O(|r|).
\]
Therefore with all these estimates, if $||r||_{H^2}$ is sufficiently small, we have
\[
||G||_{L^2} \leq K||r||_{L^\infty} [||\sigma|| + ||W||_{H^2}],
\]
\[
||\partial_x G||_{L^2} \leq K||r||_{L^\infty} [||\sigma|| + ||W||_{H^3}]
\]
(see Remark 3.)

Hence, using (14), we have obtained
\[
\frac{d\tau}{dt} R'_\tau + \frac{d\sigma}{dt} \begin{pmatrix} \cos \theta_0 \\ 0 \end{pmatrix} + \partial_t W = \sigma \begin{pmatrix} -\alpha_0 \cos \theta_0 \\ \alpha_0 \cos \theta_0 \end{pmatrix} + \Delta W + G(x, \tau, \sigma, W, \partial_x W, \partial_{xx} W).
\]
In order to separate the unknowns $\tau$, $\sigma$ and $W$, we first take the inner product of (17) with $\begin{pmatrix} \cos \theta_0 \\ 0 \end{pmatrix}$ and with $\begin{pmatrix} 0 \\ \cos \theta_0 \end{pmatrix}$. We remark that both $\partial_t W$ and $\Lambda W$ are orthogonal to these vectors, so that we obtain:

$$\rho_0 \sigma' = -\alpha_0 \rho_0 \sigma + \mathcal{G}_1,$$

$$g_\tau \tau' = \alpha_0 \rho_0 \sigma + \mathcal{G}_2,$$

where

$$\rho_0 = \int^{L/\varepsilon}_{-L/\varepsilon} \cos^2 \theta_0, \quad \mathcal{G}_1 = \int^{L/\varepsilon}_{-L/\varepsilon} G(x, \tau, \sigma W, \partial_x W, \partial_{xx} W) \cdot \begin{pmatrix} \cos \theta_0 \\ 0 \end{pmatrix} \, dx,$$

$$g_\tau = \int^{L/\varepsilon}_{-L/\varepsilon} R'_\tau (x) \cdot \begin{pmatrix} 0 \\ \cos \theta_0 \end{pmatrix} \, dx,$$

$$\mathcal{G}_2 = \int^{L/\varepsilon}_{-L/\varepsilon} G(x, \tau, \sigma W, \partial_x W, \partial_{xx} W) \cdot \begin{pmatrix} 0 \\ \cos \theta_0 \end{pmatrix} \, dx.$$

By subtraction, we have:

$$\partial_t W = \Lambda W + \tilde{G}$$

with

$$\tilde{G} = G - \frac{\mathcal{G}_2}{g_\tau} R'_\tau - \frac{\mathcal{G}_1}{\rho_0} \begin{pmatrix} \cos \theta_0 \\ 0 \end{pmatrix} + \alpha_0 \sigma \left[ \begin{pmatrix} 0 \\ \cos \theta_0 \end{pmatrix} - \frac{\rho_0}{g_\tau} R'_\tau \right].$$

We are then led to study the following equation

(18) $$\tau' = \frac{\rho_0}{g_\tau} \sigma + \frac{1}{g_\tau} \mathcal{G}_2,$$

together with the system coupling:

(19) $$\sigma' = -\alpha_0 \sigma + \frac{1}{\rho_0} \mathcal{G}_1,$$

with

(20) $$\partial_t W = \begin{pmatrix} -(\mathcal{L} - \cos^2 \gamma_0) W_1 - \mathcal{L} W_2 \\ (\mathcal{L} - \cos^2 \gamma_0) W_1 - \mathcal{L} W_2 \end{pmatrix} + \tilde{G}.$$

From (16), with Remark 3 we have

(21) $$\left| \frac{1}{\rho_0} \mathcal{G}_1 \right| (t) \leq K \| r \|_{L^\infty} \left[ |\sigma(t)| + \| W(t) \|_{H^2} \right].$$

In addition, since $g_\tau = \rho_0 + \mathcal{O}(\tau^2)$, since $R'_\tau = \begin{pmatrix} 0 \\ \cos \theta_0 \end{pmatrix} + \mathcal{O}(\tau)$, we get:

(22) $$\| \tilde{G} \|_{H^1} \leq K \| r \|_{L^\infty} \left[ |\sigma(t)| + \| W(t) \|_{H^0} \right].$$
Taking the inner product of (20) with \( \left( \mathcal{L}(\mathcal{L} - \cos^2 \gamma_0)W_1 \right) \), we obtain using (22) that
\[
\frac{d}{dt} \left( \|W\|_{H^2}^2 + \|W\|_{H^2}^2 \right) \leq K\|r\|_{L^\infty} \left[ \|\sigma(t)\| + \|W(t)\|_{H^2}^2 \right].
\]
Multiplying (19) by \( \sigma \), (21) yields:
\[
\frac{d}{dt} \sigma^2 + \alpha_0 \sigma^2 \leq K\|r\|_{L^\infty} \left[ \|\sigma(t)\| + \|W(t)\|_{H^2}^2 \right].
\]
Summing up the previous estimates, we have:
\[
\frac{d}{dt} \left[ \|\sigma(t)\|^2 + \|W(t)\|_{H^2}^2 \right] + \alpha_0 \left[ \|\sigma(t)\|^2 + \|W(t)\|_{H^2}^2 \right] \leq \left( 1 - K\|r\|_{L^\infty} \right) \leq 0.
\]
So there exists \( \delta > 0 \) such that while \( \|r\|_{L^\infty} \leq \frac{\alpha_0}{2K} \),
\[
\frac{d}{dt} \left[ \|\sigma(t)\|^2 + \|W(t)\|_{H^2}^2 \right] + \delta \left[ \|\sigma(t)\|^2 + \|W(t)\|_{H^2}^2 \right] \leq 0,
\]
that is
\[
\|\sigma(t)\|^2 + \|W(t)\|_{H^2}^2 \leq \|\sigma_0\|^2 + \|W_0\|_{H^2}^2 e^{-\delta t}.
\]
Now, with equation (18), we have
\[
|\tau'| \leq K|\sigma| + K \left[ \|\sigma(t)\| + \|W(t)\|_{H^2}^2 \right],
\]
so while \( \|r\|_{L^\infty} \leq \frac{\alpha_0}{2K} \),
\[
|\tau| \leq |\tau_0| + K \left[ \|\sigma_0\| + \|W_0\|_{H^2}^2 \right] e^{-\delta t/2}.
\]
Therefore, if \( \tau_0 \), \( \sigma_0 \) and \( \|W_0\|_{H^2}^2 \) are small enough, we remain in the domain \( \{\|r\|_{L^\infty} \leq \frac{\alpha_0}{2K}\} \) and all the previous estimates remain valid for all times. This concludes the proof of Theorem 1.3.

\[\square\]

References


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