A generalized Brown-Proschan model for preventive and corrective maintenance

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I. Introduction

The dependability of complex repairable systems depends strongly on the efficiency of maintenance actions.

**Corrective Maintenance (CM):** After a failure, put the system into a state in which it can perform its function again.

**Preventive Maintenance (PM):** When the system is operating, intends to slow down the wear process.

Basic maintenance effect assumptions:

- **As Bad As Old (ABAO):** restores the system in the same state it was just before maintenance.

- **As Good As New (AGAN):** restores the system as if it were new.

Reality is between these two extreme cases: Imperfect maintenance.
Brown-Proshchan model [83], CM is:

- **AGAN** with probability $p$,
- **ABAO** with probability $1 - p$.
- repair effects (AGAN or ABAO) are mutually independent and independent of already observed failure times.

CM effects can be characterized with random variables:

$$B_i = \begin{cases} 
1 & \text{if the } i^{th} \text{ CM is AGAN} \\
0 & \text{if the } i^{th} \text{ CM is ABAO} 
\end{cases}$$

**Statistical studies when the** $\{B_i\}_{i \geq 1}$ **are known:**
Whitaker-Samaniego[89], Hollander-Presnell-Sethuraman [92], Kvam-Singh-Whitaker [02], Bathe-Franz [96], Agustin-Pena[99],...
Practical purposes: \( \{B_i\}_{i \geq 0} \) are hidden variables.

- \( p = 0 \): ABAO
- \( p = 1 \): AGAN
- \( 0 < p < 1 \): imperfect

\( p \) characterizes maintenance efficiency:

Joint assessment of CM efficiency and intrinsic wear-out:
Lim [98]; Lim-Lie[00]; Lim-Lu-Park[98]; Langseth-Lindqvist [04].

Presentation aim:
- Generalize the BP model to PM effects
- Assess PM efficiency and intrinsic wear-out when PM effects are unknown
- Compute reliability indicators.
II/ A maintenance data set

PM and CM times of a subsystem of a fossil-fired thermal plant of EDF:

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III. Notations

\[ C_1' \quad C_2' \quad C_3' \quad C_4' \]
\[ u_1' = 0 \quad u_2' = 0 \quad u_3' = 1 \quad u_4' = 0 \]
IV. Model assumptions

- $\lambda(t)$ the failure rate of the new unmaintained system. \( \Lambda(t) = \int_0^t \lambda(s) \, ds \),
- Maintenance durations are not taken into account,
- PM are done at deterministic times, all the PM times are known
- CM are done at random times, CM are observed over \([c, T]\) \((c < T)\)
- CM effects are ABAO,
- PM effects follow a Brown-Proschchan model, i.e. PM effects are:
  - mutually independent and independent of previous failure times,
  - AGAN with probability $p$,
  - ABAO with probability $1 - p$.

\[
B_i = \begin{cases} 
1 & \text{if the } i^{\text{th}} \text{ PM is AGAN} \\
0 & \text{if the } i^{\text{th}} \text{ PM is ABAO}
\end{cases}
\]

\[
P(B_i = 1 \mid T_i, B_{i-1}) = P(B_i = 1) = p
\]
V. Joint assessment of intrinsic wear-out and PM efficiency

a/ Maximum likelihood method

**Likelihood** associated to a single observation of the failure process over $[c, t]$:

$$L_t(\theta) = f_{T_{n_t}} | N_t=n_t(t_{n_t}) P(N_t = n_t)$$

**Maximum Likelihood Estimator (MLE):**

$$\hat{\theta} = \arg \max_{\theta} L_T(\theta) = \arg \max_{\theta} \log(L_T(\theta))$$

**Notations:** $L_c(\theta) = 1$

$$L_{\tau,t}(\theta) = f_{T_{n_t-n_\tau},N_{t-\tau}=n_t-n_\tau}(t_{n_\tau+1 - \tau}, \ldots, t_{n_t} - \tau) \Rightarrow \text{the likelihood associated to the system new in } \tau \text{ and observed over } [c \lor \tau, t].$$
Let us denote: 

\[ D_t^m = (1 - p)^m \left( \prod_{c \vee \tau_m \leq t_j \leq t} \lambda(t_j - \tau_m) \right) \]

Two equivalent equations for likelihood recursive computation:

\begin{itemize}
  \item \( L_t(\theta) = \left[ \sum_{m=0}^{m_t-} p^{1\{m \neq 0\}} D_t^m e^{-\left(\Lambda(t-\tau_m)-\Lambda((c\vee \tau_m)-\tau_m)\right)} L_{c\vee \tau_m}(\theta) \right] \Rightarrow \text{forward computation algorithm.} \)
  \item \( L_t(\theta) = \left[ \sum_{\tau \in \{\tau_1, \ldots, \tau_{m_t-}, t\}} p^{1\{\tau \neq t\}} D_{\tau_m}^0 e^{-\left(\Lambda(\tau_m \vee c)-\Lambda(c)\right)} L_{\tau_m, t}(\theta) \right] \Rightarrow \text{backward computation algorithm.} \)
\end{itemize}
Proof of the equation used for the forward computation:

\[
\{B_m = 1, \{B_j = 0\}_{m<j\leq m_{t-}}\}_{0\leq m\leq m_{t-}} \text{ is a partition of the probability space.}
\]

\(\tau_m\) is the last AGAN PM since \(t\)

Given \(B_m = 1\), the CM process can be divided into 2 independent processes.

\[
L_t(\theta) = \left[ \sum_{m=0}^{m_{t-}} p \left\{m \neq 0\right\} (1 - p)^{m_{t-} - m} \right]
\left[ \prod_{c \lor \tau_m \leq t_j \leq t} \lambda(t_j - \tau_m) e^{-(\Lambda(t - \tau_m) - \Lambda((c \lor \tau_m) - \tau_m))} \right]
\]

Given \(B_m = 1, \{B_j = 0\}_{m<j\leq m_{t-}}\), CM times after \(\tau_m\) follow a NHPP initialized in \(\tau_m\)

\[
L_{c \lor \tau_m}(\theta)
\]

Given \(B_m = 1\), maintenances before \(\tau_m\) follow a BP PM-ABAO CM model
**b/ MLE combined with moment estimation**: \( \lambda(t) = \alpha \beta t^{\beta-1} \)

**Power Law Process**: \( E[N_T] = \alpha (T^\beta - c^\beta) \Rightarrow \tilde{\alpha}_\beta = N_T / (T^\beta - c^\beta) \).

**BP model**, \( E[N_t] = \alpha S_t \), where
\[
S_t = \sum_{\tau \in \{\tau_{m+1}, \ldots, \tau_{m-} \}} \left( \sum_{k=0}^{m_\tau-} p^{\mathbb{1}_{k \neq 0}} (1 - p)^{m_\tau-k} ((\tau - \tau_k)^\beta - ((\tau_{m_\tau} \lor c) - \tau_k)^\beta) \right)
\]

**MLE combined with moment estimation of \( \alpha \)**: \( \tilde{\alpha}_{\beta,p} = \frac{N_T}{S_T} \)

\[
E[\tilde{\alpha}_{\beta,p}] = \alpha \quad (\tilde{\beta}, \tilde{p}) = \arg \max_{(\beta, p)} \log(L_T(\tilde{\alpha}_{\beta,p}, \beta, p)) \quad \text{and} \quad \tilde{\alpha} = \tilde{\alpha}_{\tilde{\beta}, \tilde{p}}
\]

\( \Rightarrow \) Dimension reduction of the likelihood maximization space.
c/ Individual PM efficiency assessment

$p$ characterized the **average global PM efficiency**.

The $m^{th}$ **PM effect** can be characterized by:

$$\pi^\theta_m = P(B_m = 1 \mid N_T = n_T, T_{n_T} = t_{n_T})$$

which verifies:

$$\pi^\theta_m = p \frac{L_{C\lor T_m}(\theta) L_{T_m,T}(\theta)}{L_T(\theta)}$$

It can naturally be estimated by: $\pi^\hat{\theta}_m$

- $L_{T_m}(\hat{\theta})$: intermediate computing values of the forward algorithm
- $L_{T_m,T}(\hat{\theta})$: intermediate computing values of the backward algorithm
d/ Expectation-Maximization algorithm

Complete likelihood:

\[
L_t^c(\theta) = f_{T_{n_t} \mid N_t=n_t, B_{n_t}=b_{n_t}}(t_{n_t}) P(N_t = n_t \mid B_{n_t} = b_{n_t}) P(B_{n_t} = b_{n_t})
\]

EM algorithm:

- **Expectation (E) step:** Compute \( Q(\theta \mid \theta_k) = E^{\theta_k} [\log(L_t^c(\theta)) \mid N] \)
- **Maximization (M) step:** \( \theta_{k+1} = \arg \max_{\theta} Q(\theta \mid \theta_k) \)

\[\theta_k \xrightarrow[k \to \infty]{\quad} \text{local likelihood maxima}\]
Complete likelihood:

\[
L_t^c(\theta) = \left[ \prod_{n=1}^{N_t} \lambda(A_{t_n}) \right] e^{-\int_c^t \lambda(A_s) \, ds} \left[ \prod_{m=1}^{m_t} p^B_{m}(1 - p)^{1-B_m} \right]
\]

where \( A_s \) is the virtual age, i.e. the time elapsed since the last perfect PM

\[
\forall s \in [\tau_m, \tau_{m+1}] \cap [c, T] \quad \lambda(A_s) = \prod_{i=0}^{m} [\lambda(s - \tau_{m-i})]^{\mathbb{1}_{\{B_{\tau_m}^{-i}\}}}
\]

where \( B_{\tau_m}^{-i} \) = “at time \( \tau_m \), the last AGAN PM time is \( \tau_{m-i} \)”
Complete likelihood:

\[
E^{\theta}[\log(L^c_t(\theta)) \mid N] = Q(p \mid \theta)
\]

\[
L^c_t(\theta) = \left[ \prod_{n=1}^{N_t} \lambda(A_{t_n}) \right] e^{-\int_c^t \lambda(A_s) \, ds}
\]

\[
E^{\theta}[\log(\cdot) \mid N] = Q_2(\lambda \mid \theta)
\]

\[
E^{\theta}[\log(\cdot) \mid N] = Q_1(p \mid \theta)
\]

where \( A_s \) is the virtual age, i.e. the time elapsed since the last perfect PM

\[
\forall s \in [\tau_m, \tau_{m+1}] \cap [c, T] \quad \lambda(A_s) = \prod_{i=0}^{m} \left[ \lambda(s - \tau_{m-i}) \right]\{B_i^{-1}\}
\]

where \( B_i^{-1} \) = “at time \( \tau_m \), the last AGAN PM time is \( \tau_{m-i} \)"
Complete likelihood:

\[
E^\theta[\log(L^c_t(\theta)) \mid N] = Q(p \mid \theta)
\]

\[
L^c_t(\theta) = \left[ \prod_{n=1}^{N_t} \lambda(A_{tn}) \right] e^{-\int_c^t \lambda(A_s) \, ds}
\]

where \( A_s \) is the virtual age, ie the time elapsed since the last perfect PM

\[
\forall s \in [\tau_m, \tau_{m+1}] \cap [c, T] \quad \lambda(A_s) = \prod_{i=0}^{m} [\lambda(s - \tau_{m-i})] \cdot \mathbb{1}_{\{B_{\tau_m}^{-i}\}}
\]

where \( B_{\tau_m}^{-i} = "at \ time \ \tau_m, \ the \ last \ AGAN \ PM \ time \ is \ \tau_{m-i}" \)

- \( Q_2 \) function of \( \pi_{\theta m, i} = E^\theta \left[ \mathbb{1}_{\{B_{\tau_m}^{-i}\}} \mid N_T = n_T, T_{nT} = t_{nT} \right] \)

- \( Q_1 \) function of \( \pi_{\theta m, 0} = \pi_{m} = E^\theta \left[ B_{m} \mid N_T = n_T, T_{nT} = t_{nT} \right] \)
**E step:** Compute for $0 \leq m \leq m_T$ and $0 \leq i \leq m$,

$$\pi_{m,i}^\theta = E^\theta \left[ \mathbb{1}_{\{B_{\tau_m}^-\}} \mid N_T = n_T, T_{n_T} = t_{n_T} \right]$$

- $\pi_{0,0}(\theta) = 1$
- for $m \in \{1, \ldots, m_T\}$, $\pi_{m,0}^\theta = \pi_m^\theta$
- for $0 \leq m < m_T$ and $0 \leq i \leq m$,

$$\pi_{m+1,i+1}^\theta = \pi_{m,i}^\theta - p^1 + \mathbb{1}_{\{m>i\}} (1-p)^i \prod_{(c \vee \tau_{m-i}) < t_j \leq \tau_{m+1}} \lambda(t_j - \tau_{m-i})$$

$$e^{-\Lambda((c \vee \tau_{m+1}) - \tau_{m-i}) - \Lambda((c \vee \tau_{m-i}) - \tau_{m-i})} \frac{L_{c \vee \tau_{m-i}}(\theta)L_{\tau_{m+1},T}(\theta)}{L_T(\theta)}$$
**M step:** \( \lambda(t) = \alpha \beta t^{\beta-1} \)

- \( p_{k+1} = \left[ \sum_{m=1}^{m_T} \pi_{m,0}^k \right] / m_T \)

- \( \beta_{k+1} = \arg \max_{\beta} \left[ n_T (\log(n_T \beta) - \log(S_k(\beta))) + (\beta - 1) \right] \)

\[
\left[ \sum_{\tau \in \{ \tau_{m_c+1}, \ldots, \tau_{m_T-1}, T \}} \sum_{i=0}^{m_{\tau-}} \pi_{m_{\tau-},m_{\tau-}-i}^k \sum_{i} \log(t_j - \tau_i) \right]
\]

with \( S_k(\beta) = \sum_{\tau \in \{ \tau_{m_c+1}, \ldots, \tau_{m_T-1}, T \}} \sum_{i=0}^{m_{\tau-}} \pi_{m_{\tau-},m_{\tau-}-i}^k ((\tau - \tau_i)\beta - ((\tau \lor m_{\tau}) - \tau_i)\beta) \)

- \( \alpha_{k+1} = n_T / S_k(\beta_{k+1}) \)
VI. Reliability indicators

• **Failure intensity:** $\lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P\left(N_t + \Delta t - N_t^- \mid T \sim N_t^-, N_t^-\right)$

$$\lambda_t = \frac{\sum_{m=0}^{m_{t^-}} D_m C_{K_{t^-}} e^{-\Lambda(t-\tau_m) - \Lambda((c \vee \tau_m) - \tau_m) L_{c \vee \tau_m}(\theta)}}{\sum_{m=0}^{m_{t^-}} D_m C_{K_{t^-}}} \lambda(t - \tau_m) e^{-\Lambda(t-\tau_m) - \Lambda((c \vee \tau_m) - \tau_m) L_{c \vee \tau_m}(\theta)}$$

• **Cumulative failure intensity:** $\Lambda_t = \int_0^t \lambda_s ds$

$$\Lambda_t = - \left[ \sum_{k=1}^{K_t} \log \left( \frac{L_{C_k^-}(\theta)}{L_{C_{k-1}^+}(\theta)} \right) \right] - \log \left( \frac{L_t^-(\theta)}{L_{C_{K_t}^+}(\theta)} \right)$$
- **Reliability**: \( P(T_{N_T+1} > s \mid T_{N_T}, N_T) = \exp(-(\Lambda_s - \Lambda_T)) \)

- **Expected cumulative number of failures**: \( E[N_s \mid T_{N_T}, N_T] = \)

\[
N_T + \left[ \sum_{i=m_T+1}^{m_s} (\Lambda(s - \tau_i) + E[N_{\tau_i} \mid T_{N_T}, N_T] - N_T)p(1 - p)^{m_s-i} \right] \\
+ \left[ \sum_{i=0}^{m_T} (\Lambda(s - \tau_i) - \Lambda(T - \tau_i))(1 - p)^{m_s-m_T}\pi^{\theta}_{m_T, m_T-i} \right]
\]
VII. Application

The BP PM-ABAO CM model is implemented in **MARS** (Maintenance Assessment of Repairable Systems): a free software developed by LJK (Grenoble university) and EDF (French electricity utility).
• MLE combined with moment estimation:

\[ \hat{\alpha} = 1.84 \times 10^{-6}, \quad \hat{\beta} = 1.95, \quad \hat{p} = 0.602, \quad \log(L_T(\tilde{\alpha}, \tilde{\beta}, \tilde{p})) = -150.900 \]

• MLE computed with EM algorithm or direct maximization:

\[ \hat{\alpha} = 1.87 \times 10^{-6}, \quad \hat{\beta} = 1.94, \quad \hat{p} = 0.614, \quad \log(L_T(\hat{\alpha}, \hat{\beta}, \hat{p})) = -150.902 \]

\[ \pi_{\hat{\theta}_m} : \quad 0.61 \quad 0.62 \quad 0.69 \quad 0.73 \quad 0.07 \quad 0.00 \quad 1.00 \quad 0.99 \quad 0.96 \quad 0.46 \]

ABAO    ABAO    AGAN    AGAN    AGAN

Wearing out state at time \( c \) \( \Rightarrow \) failures surge
Quality of estimation: $\theta = (1.9e - 6, 1.9, 0.61)$

NEB: Normalized Empirical Bias  
NESD: Norm. Emp. Standard Deviation  
$\theta^*$: MLE estimator when the $B_i$ are known

Initial intensity: $\lambda(t) = \alpha \beta t^{\beta-1}$ or $\lambda(t) = \beta/\eta(t/\eta)^{\beta-1}$

EM is more robust than direct likelihood maximization

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<td>$\theta^*$</td>
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Quality of individual PM efficiency estimation:

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<td>Mean</td>
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$\Rightarrow$ Individual PM efficiency estimation is relevant after $c$
Failure intensity for $\theta = \hat{\theta}$

Cumulative failure intensity and number of failures for $\theta = \hat{\theta}$
Forecast reliability for $\theta = \hat{\theta}$

![Reliability graph]

Forecast cumulative number of failures for $\theta = \hat{\theta}$

![Cumulative failures graph]
VIII. Prospects

- Maintenance times optimization

- Consider a more general distribution over \([-\infty, 1]\) for the \(B_i\).

- Develop confidence intervals and tests for the BP model

- ...
References


