Inference for the Wiener process with random initiation time

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2 Parameters estimation
3 Time-to-failure estimation
4 Application to a dataset
5 Bibliography
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Introduction

- **Objective:** study of stochastic models to a better understanding of component/system ageing
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- **Degradation models vs. Lifetime models?** highly reliable components, use of complex preventive maintenance policies, etc.

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References:
- Guo et al. (13)
- Nelson (10)
Introduction

- **Objective**: study of stochastic models to a better understanding of component/system ageing

- **Degradation models vs. Lifetime models?** highly reliable components, use of complex preventive maintenance policies, etc.

- **Current models**: component degradation initiated when put in service!
**Objective:** study of stochastic models to a better understanding of component/system ageing

**Degradation models vs. Lifetime models?** highly reliable components, use of complex preventive maintenance policies, etc.

**Current models:** component degradation initiated when put in service!

**Need of some new models:** models with an initiation period (deterministic or random)
See Guo *et al.* (13), Nelson (10)
Degradation model

Degradation model with random initiation period \((X(t))_{t \geq 0}\):

\[
X(t) = \left[ \mu(t - S) + \sigma B(t - S) \right] \mathbb{I}_{t \geq S}
\]

where

- \(t = 0\) is the instant where the component is put in service
- \((B(t))_{t \geq 0}\) is a standard Brownian motion
- \(S\) is an absolutely continuous and positive random variable, independent of \((B(t))_{t \geq 0}\)
For degradation model, time-to-failure $T_c = \text{first-time to reach a given and known critical level } c$:

$$T_c = \inf\{ t \geq 0; X(t) \geq c \}$$
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Special case: $S$ exponentially distributed, see Schwarz (01, 02) with an application in psychology
Simulation of three sample paths:
black circles = degradation initiations
red dash line = critical level.
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Statistical model

Observations?
Statistical model

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- $n$ independent components: $n$ copies $X_1, \ldots, X_n$ of $X$
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- discrete-time sampling at regular instants $0, \delta, 2\delta, \ldots, m\delta = \tau$
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Consequence: random number of non-null observations
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Consequence: random number of non-null observations

Model assumptions? Parametric model for the distribution of $S$, with unknown parameter $\theta \in \Theta \subseteq \mathbb{R}^p$
Random variable $R_i$ such $(R_i - 1)\delta < S_i \leq R_i\delta$. 
Notations (1/3)

Random variable $R_i$ such $(R_i - 1)\delta < S_i \leq R_i\delta$.

1. if $R_i > m$, $S_i > m\delta = \tau$ and $X_i(j\delta) = 0$ for any $j \in \{0, \ldots, m\}$
Notations (1/3)

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1. If $R_i > m$, $S_i > m\delta = \tau$ and $X_i(j\delta) = 0$ for any $j \in \{0, \ldots, m\}$
   Information only on $\theta$ (right-censoring)
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2. If $R_i = m$, $(m - 1)\delta < S \leq m\delta$ and $X_i(j\delta) = 0$ for any $j \in \{0, \ldots, m - 1\}$ but $X_i(m\delta) \neq 0$
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   Information only on $\theta$ (interval-censoring)

3. if $R_i < m$, at least two non-null degradation measures observed
   Information on $\theta$ (interval-censoring), $\mu$ and $\sigma^2$
Three random subsets of the individuals:

- $N_0$: set of individuals with zero non-null degradation measure: $N_0 = \{ i; R_i > m \} \subseteq \{1, \ldots, n\}$ and $N_0 = \divides N_0 \divides$

- $N_1$: set of individuals with exactly one non-null degradation measure: $N_1 = \{ i; R_i = m \} \subseteq \{1, \ldots, n\}$ and $N_1 = \divides N_1 \divides$

- $N_2^+$: set of individuals with exactly at least two degradation measures: $N_2^+ = \{ i; R_i < m \} \subseteq \{1, \ldots, n\}$ and $N_2^+ = \divides N_2^+ \divides$
Notations (2/3)

Three random subsets of the individuals:

- $\mathcal{N}_0$: set of individuals with zero non-null degradation measure:
  
  $$
  \mathcal{N}_0 = \{ i; R_i > m \} \subseteq \{ 1, \ldots, n \}
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- $\mathcal{N}_{2+}$: set of individuals with exactly at least two degradation measures:
  \[ \mathcal{N}_{2+} = \{ i; R_i < m \} \subseteq \{1, \ldots, n\} \quad \text{and} \quad N_{2+} = |\mathcal{N}_{2+}| \]
Random vector $\mathcal{K} = (\mathcal{K}_r)_{r \in \mathbb{N}^*}$ such that, for $r \in \mathbb{N}^*$,

$$\mathcal{K}_r = \sum_{i=1}^{n} \mathbb{I}_{(r-1)\delta < R_i \leq r\delta} = \sum_{i=1}^{n} \mathbb{I}_{R_i = r}$$
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Remark: $\sum_{r=1}^{m} K_r = n - N_0$
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Remark: $\sum_{r=1}^{m} K_r = n - N_0$

Random number $Q_n$ of non-null increments: if $Q_n$ non empty set,

$$Q_n = \sum_{i \in \mathbb{N}_{2+}} (m - R_i) = \sum_{j=1}^{m-1} (m - j) K_j$$

taking values in $\{1, \ldots, (m - 1)n\}$
An important result

Lemma

1. For any $\alpha \in [0, 1)$,

$$\frac{Q_n}{n^\alpha} \xrightarrow{Pr} \infty$$

as $n \to \infty$. 

Let $\alpha (m, \tau) = \frac{1}{m} \sum_{j=0}^{m-1} F_{S_j} (\tau / m)$. We have

$$
\frac{Q_n}{n^\alpha (m, \tau)} \xrightarrow{Pr} 0
$$

as $n \to \infty$. 


Lemma

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\frac{Q_n}{n^{\alpha}} \xrightarrow{Pr \ \ n \to \infty} \infty
\]

2. Let $\alpha(m, \tau) = \frac{1}{m} \sum_{j=0}^{m-1} F_S(j\tau/m)$. We have

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\frac{Q_n}{n} \xrightarrow{Pr \ \ n \to \infty} m\alpha(m, \tau)
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Lemma

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2. Let $\alpha(m, \tau) = \frac{1}{m} \sum_{j=0}^{m-1} F_S(j\tau/m)$. We have

$$\frac{Q_n}{n} \xrightarrow{Pr_{n \to \infty}} m\alpha(m, \tau)$$

3. $\mathbb{E}[Q_n^{-1} | Q_n > 0] \xrightarrow{n \to \infty} 0$
Estimation of the distribution of $S$ (1/2)

- Survival function of $S$: $F_S(t; \theta) = \mathbb{P}[S \leq t]$
Estimation of the distribution of $S$ (1/2)

- Survival function of $S$: $\bar{F}_S(t; \theta) = \mathbb{P}[S \leq t]$
- Log-likelihood function:

$$
\ell(\theta|\text{data}) = N_0 \log \bar{F}_S(\tau; \theta) + \sum_{r=1}^{m} K_r \log \left( \bar{F}_S((r-1)\delta; \theta) - \bar{F}_S(r\delta; \theta) \right)
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Estimation of the distribution of $S$ (1/2)

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- Maximum likelihood estimator:

$$\hat{\theta}_n = \arg\max_{\theta \in \Theta} \ell(\theta|\text{data}).$$

No closed-form expression in general
Estimation of the distribution of $S$ (2/2)

Asymptotic normality for $\hat{\theta}_n$ as $n \to \infty$?
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Estimation of the distribution of $S$ (2/2)

Asymptotic normality for $\hat{\theta}_n$ as $n \to \infty$? Yes. . .

- MLE = root of the equation:

$$0 = N_0 \frac{\partial_\theta \overline{F}_S(\tau; \hat{\theta}_n)}{\overline{F}_S(\tau; \hat{\theta}_n)} + \sum_{r=1}^{m} K_r \frac{\partial_\theta \overline{F}_S((r - 1)\delta; \hat{\theta}_n) - \partial_\theta \overline{F}_S(r\delta; \hat{\theta}_n)}{\overline{F}_S((r - 1)\delta; \hat{\theta}_n) - \overline{F}_S(r\delta; \hat{\theta}_n)}$$
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- Convergence of $(K_1, \ldots, K_m)$ to a Gaussian distribution
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- Convergence of $(K_1, \ldots, K_m)$ to a Gaussian distribution

- $\delta$-method for implicitly defined random variables (Benichou and Gail, 89)
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- Convergence of $(K_1, \ldots, K_m)$ to a Gaussian distribution

- $\delta$-method for implicitly defined random variables (Benichou and Gail, 89)

- Closed expression for the Fisher information
Example: exponential distribution

- Closed expression for the MLE:

\[ \hat{\lambda}_n = \frac{1}{\delta} \log \left( \frac{N_0 \tau + \delta \sum_{r=1}^{m} rK_r}{N_0 \tau + \delta \sum_{r=1}^{m} (r - 1) K_r} \right) \]
Example: exponential distribution

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- Asymptotic variance:

\[
\rho^2 = \frac{(e^{\lambda \delta} - 1)^2}{\delta^2 e^{\lambda \delta} (1 - e^{-\lambda \tau})}
\]
Example: exponential distribution

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- Asymptotic variance:

\[ \rho^2 = \frac{(e^{\lambda \delta} - 1)^2}{\delta^2 e^{\lambda \delta} (1 - e^{-\lambda \tau})} \]

Remark: \( \rho^2 \xrightarrow{\delta \to 0} \frac{\lambda^2}{1 - e^{-\lambda \tau}} \)
Estimation of $\mu$ and $\sigma^2$ (1/2)

- Natural estimator of $\mu$:

$$\hat{\mu}_n = \frac{\sum_{i \in \mathcal{N}_{2+}}^{m-R_i} \sum_{j=1}^{\Delta X_{i,j}}}{\delta \sum_{i \in \mathcal{N}_{2+}} (m - R_i)} = \frac{1}{\delta Q_n} \sum_{h=1}^{Q_n} Z_h,$$

where $Z_1, \ldots, Z_{Q_n}$ are the increments between two non-null degradation measures: random number of iid Gaussian random variables with mean $\mu \delta$ and variance $\sigma^2 \delta$. 
Estimation of $\mu$ and $\sigma^2$ (1/2)

- Natural estimator of $\mu$:

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where $Z_1, \ldots, Z_{Q_n}$ are the increments between two non-null degradation measures: random number of iid Gaussian random variables with mean $\mu \delta$ and variance $\sigma^2 \delta$

- Natural estimator of $\sigma^2$:

$$\hat{\sigma}_n^2 = \frac{1}{\delta(Q_n - 1)} \sum_{h=1}^{Q_n} (Z_h - \delta \hat{\mu}_n)^2.$$
Proposition

1. \( \hat{\mu}_n \) is asymptotically normal:

\[
\sqrt{Q_n} (\hat{\mu}_n - \mu) \xrightarrow{d} N \left( 0, \frac{\sigma^2}{\delta} \right)
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Proposition

1. \( \hat{\mu}_n \) is asymptotically normal:

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where \( \alpha(m, \tau) \) is given in the Lemma
Estimation of $\mu$ and $\sigma^2$ (2/2)

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where $\alpha(m, \tau)$ is given in the Lemma

2. $\hat{\sigma}_n^2$ is asymptotically normal:

$$\sqrt{Q_n} (\hat{\sigma}_n^2 - \sigma^2) \xrightarrow{d} N \left( 0, 2\sigma^4 \right)$$
1. Introduction and model

2. Parameters estimation

3. Time-to-failure estimation

4. Application to a dataset

5. Bibliography
Mean time-to-failure estimation

- Mean time-to-failure:

\[ MTTF = \mathbb{E}[S] + \frac{c}{\mu} \]
Mean time-to-failure estimation

- Mean time-to-failure:

\[
MTTF = \mathbb{E}[S] + \frac{c}{\mu}
\]

- Plug-in estimator for MTTF:

\[
\overline{MTTF}_n = \int_0^\infty F_S(u; \hat{\theta}_n) du + \frac{c}{\hat{\mu}}
\]
Mean time-to-failure estimation

- Mean time-to-failure:
  \[ \text{MTTF} = \mathbb{E}[S] + \frac{c}{\mu} \]

- Plug-in estimator for MTTF:
  \[ \widehat{\text{MTTF}}_n = \int_0^\infty \bar{F}_S(u; \hat{\theta}_n) du + \frac{c}{\hat{\mu}} \]

- Asymptotic normality?
Mean time-to-failure estimation

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Mean time-to-failure estimation

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- Plug-in estimator for MTTF:

\[ \overline{MTTF}_n = \int_0^\infty F_S(u; \hat{\theta}_n) \, du + \frac{c}{\hat{\mu}} \]

- Asymptotic normality? Yes! Asymptotic variance:

\[ l(\theta)^{-1} \left( \int_0^\infty \partial_\theta F_S(u; \theta) \, du \right)^2 + \frac{c^2 \sigma^2}{\mu^4 \tau \alpha(m, \tau)} \]
Guo et al. data

Black lines: observed degradation paths
Red dashed line: critical level
Blue dashed line: MTTF estimation
### Fitted parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimation</th>
<th>95% confidence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.023</td>
<td>[0.013, 0.032]</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.108</td>
<td>[0.097, 0.119]</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>0.041</td>
<td>[0.033, 0.048]</td>
</tr>
<tr>
<td>$MTTF$</td>
<td>88.332</td>
<td>[69.438, 107.227]</td>
</tr>
</tbody>
</table>
Estimated survival function
Estimated hazard function
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